A New Insight Into an Old Calculus Mystery: Δx, Δy, dx, dy and the Nature of the Infinitesimal

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Introduction

In this paper, we set forth what seems to be a new insight into the nature of the derivative $f'(x)$ in elementary calculus — an insight that we believe answers at least some of the questions that students often have.

The Definition of the Derivative

The standard textbook definition of the derivative[1], for a continuous function $y = f(x)$ — a curve — having a derivative at x, is:

(1)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

Students are taught, via a diagram such as that in "Fig. 1" on page 4 that $f'(x)$ is the slope of the tangent line at the point $f(x)$ on the curve.

Questions That Bother Some Students

Among the questions that bother students in the first and second semesters of calculus are the following:

What is the difference between Δx and Δy, and dx and dy?[2] Sometimes it is said that these are infinitesimals, that is, numbers that are "arbitrarily small but not 0". What kind of numbers are those? It doesn't seem possible that they are very small positive real numbers, because no matter how small a positive real number we name, there is always a smaller one.

The Contradictory Answers That Are Given in Calculus Textbooks

"...the derivative...is the quotient of the differentials *dy* and *dx*...[3]

"Leibniz's notation [*dy*/*dx*, etc.] suggests that the limit [that defines *dy*/*dx*] is a quotient, whereas the limit ... is *not* a quotient. Leibniz's symbol ... must be taken in its entirety."[4]

"...in the notation...*dy*/*dx*, the symbols *dy* and *dx* have no meaning by themselves. The symbol *dy*/*dx* should be thought of as a single entity, just like the numeral 8, which we do not think of as formed of two 0's."[5]

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1. First set forth by Bolzano in 1817.
2. Some, but by no means all, calculus textbooks set $dy = \Delta x$ and $dy = \Delta y$.
3. Courant, R., *Differential and Integral Calculus*, John Wiley & Sons, Inc., 1970, p. 107.
4. *Kline, Morris, Calculus: An Intuitive and Physical Approach*, Dover Publications, Inc., Mineola, N.Y., 1998, p. 26.
5. Stein, Sherman, *Calculus and Analytical Geometry*, McGraw-Hill Book Company, N.Y., 1973 , p. 90

"...in many calculations and formal transformations, we can deal with the symbols *dy* and *dx* in exactly the same way as if they were ordinary numbers."[1]

A Brief Look at the History

The nature of *dy/dx* baffled some of the best mathematicians for at least the first 150 years after the calculus was discovered..

"As to the ultimate meanings of *dy*, *dx* and *dy*/*dx*, Leibniz remained vague. He spoke of *dx* as the difference in *x* values between two infinitely near points and of the tangent as the line joining such points... The infinitely small *dx* and *dy* were sometimes described as vanishing or incipient quantities, as opposed to quantities already formed. These indefinitely small quantities were not zero, but were smaller than any finite quantity."[2]

"[Euler] denied the concept of an infinitesimal, a quantity less than any assignable magnitude and yet not 0. In his *Institutiones* of 1755 he argued,

> There is no doubt that every quantity can be diminished to such an extent that it vanishes completely and disappears. But an infinitely small quantity is nothing other than a vanishing quantity and therefore the thing itself equals 0. It is in harmony also with that definition of infinitely small things, by which the things are said to be less than any assignable quantity; it certainly would have to be nothing; for unless it is equal to 0, an equal quantity can be assigned to it, which is contrary to hypothesis.

"Since Euler banished differentials he had to explain how *dy*/*dx*, which was 0/0 for him, could equality a definite number. He does this as follows: Since for any number n, $n \cdot 0 = 0$, then $n = 0/0$. The derivative is just a convenient way of determining 0/0..."[3]

"[D'Alambert] held that no such thing as an infinitesimal [for example, *dy* or *dx*] existed in its own right.

"A quantity is something or nothing: if it is something, it has not yet vanished; if it is nothing, it has literally vanished. The supposition that there is an intermediate state between these two is a chimera."[4]

A New Insight

Consider the following diagram.

1. Courant, op. cit., p. 101.
2. Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p. 385.
3. Kline, ibid., p. 429
4. D'Alambert, Jean le Rond, *Mélanges des Littérature, d'histoire, et de philosophie*, 4th ed., 5 vols. Amsterdam, 1767, quoted in Boyer, Carl B., *The History of the Calculus and Its Conceptual Development*, Dover Publications, Inc., N.Y., 1959, p. 248.

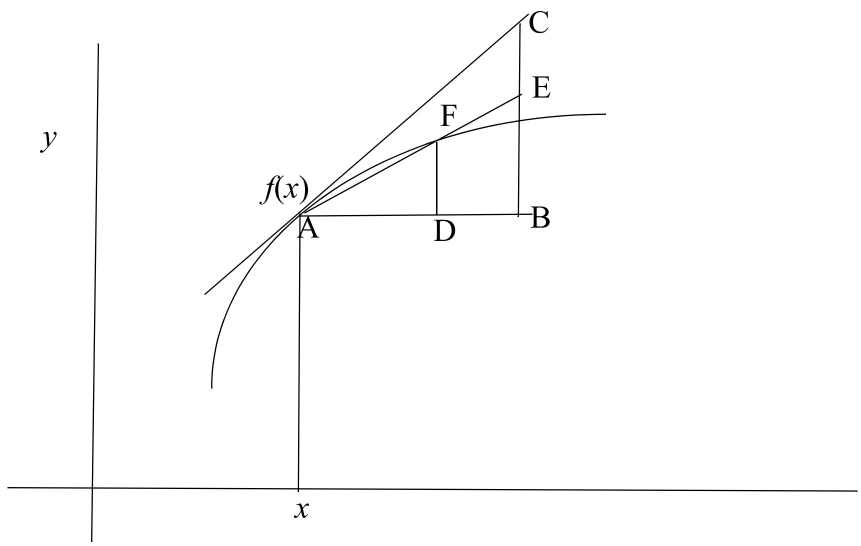

Fig. 1

For a given curve $y = f(x)$ and a given x, we fix points A, B, and C . A is at the point $f(x)$.

Angle BAC = the slope of the tangent to the curve at point A. The tangent of $BAC = BC/AB = f'(x) = dy/dx$.

$\Delta y = DF$; $\Delta x = AD$.

The tangent of angle $DAF = \Delta y/\Delta x$ **=** $\boldsymbol{DF/AD = BE/AB}$. **Neither** $\boldsymbol{BE}$ **nor** $\boldsymbol{AB}$ **becomes arbitrarily small as** $\boldsymbol{\Delta x}$ **and** $\boldsymbol{\Delta y}$ **approach zero. And yet** $\boldsymbol{\Delta y/\Delta x = DF/AD = BE/AB}$ **always.** (E moves vertically up the line BC throughout this process, of course.)

As $\Delta x = AD$ approaches zero, the point F moves to the left along the curve $y = f(x)$ and $\Delta y = DF$ grows smaller. The angle DAF increases *continuously* until it eventually equals the angle BAC. The point E moves vertically upward *continuously* until BE eventually equals the line segment BC. These two equalities occur when $\Delta x = 0$.

We need not wrack our brains trying to figure out the value of $\Delta y/\Delta x$ when both Δy and $\Delta x = 0$. The value is simply BC/AB.

So there is nothing mysterious here *provided we fix our attention on the continuous movement of the line AFE* as $\Delta x = AD$ approaches zero. AFE pivots continuously about the point A as the angle DAF increases. Eventually, AFE is at the same angle as the angle of the tangent, BAC. *All we have done is pivot a line (AFE) about a point (A) until the line is coincident with another line (AC). That is all!*

We will welcome the reader's comments.

Criticism of the Insight

The Insight is based on the assumption that we know what the tangent line AC is. We certainly know that *it exists*, but we do not know (without knowing dy/dx) when the line AFE is coincident with AC.

Two Possible Variations of the Insight

If we draw a straight line through the point (x, y) so that the part of the line that is to the right of the point is below the curve $y = f(x)$, then we can consider rotating the line counterclockwise about the point until it is at an angle such that all points of the curve except (x, y) are on one side of the line, then the line will be tangent to the curve at (x, y).

If the mathematics can be found to track this process, and indicate when the line is at the tangent angle, then that might give us another way of computing the derivative.

The tangent to the curve at the point (x, y) on the curve is the perpendicular to the perpendicular to the curve at that point. Thus, e.g., the tangent to any point on a circle is the perpendicular to the radius at that point.

A Fact About Tangents That Might Be of Potential Use

Let θ be an acute angle in a right triangle. Then by definition of tangent, tan (θ) = (length of opposite side of triangle)/(length of adjacent side of triangle).

However, the value of the tangent remains the same for all larger (or smaller) right triangles having θ as one of the acute angles. In other words, no matter how long we make the opposite side, the length of the adjacent side will always be such that the value of the tangent for the resulting triangle, will be the same.

This fact seems to be of use in proving that, if X, Y are the x- and y-intercepts, respectively, of a function $y = f(x)$, then at the point (x, y) on the curve, $dy/dx = (Y - y)/(X - x)$.

The Fundamental Theorem of the Calculus Made Clear

Although most calculus students understand that the Fundamental Theorem of the Calculus states that integration and finding the derivative are inverse operations, their understanding of the proof varies. In any case, few students, in my experience, can demonstrate an intuitive grasp of why the Theorem is true. I encourage students having difficulty with this matter to see “Appendix A — The Derivative and the Area Under a Curve”, in chapter 2, “Mathematics in the Schools” in *How to Improve Your Math Grades*, by William Curtis, on occampress.com.

Appendix A — The Infinitesimal in the 20th Century

"Paul du Bois-Reymond [1831-1889], Otto Stolz [1842-1905], and Felix Klein [1849-1925] [thought] that a consistent theory based on infinitesimals was possible. In fact, Klein identified the very axiom of real numbers, the Archimedean axiom[1], that would have to be abandoned to obtain such a theory. Skolem [1887-1963] himself in 1934 began the introduction of new numbers — hyperintegers — which were different from the ordinary real numbers, and he established some of their properties. The culmination of a series of papers by several mathematicians was the creation of a new theory which legitimizes infinitesimals. The most important contributor was Abraham Robinson (1918-1974).

"The new system, called non-standard analysis, introduces hyperreal numbers, which include the old real numbers and infinitesimals. An infinitesimal is defined practically as Leibniz did; that is, a positive infinitesimal is a number less than any ordinary real number but greater than zero and, similarly, a negative infinitesimal is greater than any negative real number but less than zero. These infinitesimals are fixed numbers, not variables in Leibniz's sense nor variables which approach zero, which is the sense in which Cauchy sometimes used the term. Moreover, non-standard analysis introduces new infinite numbers, which are the reciprocals of infinitesimals but not the transfinite numbers of Cantor. Every finite hyperreal number r is of the form $x + \alpha$ where x is an ordinary real number and α is an infinitesimal.

"With the notion of the infinitesimal, one can speak of two hyperreal numbers being infinitely close. This means merely that their difference is an infinitesimal. Every hyperreal number is also infinitely close to an (ordinary) real number, the difference being infinitesimal. We can operate with the hyperreals just as we operate with ordinary real numbers.

"With this new hyperreal number system, we can introduce functions whose values may be ordinary or hyperreal numbers..." — Kline, Morris, *Mathematics: The Loss of Certainty*, Oxford University Press, N.Y., 1980, pp. 274-75.

1. This "asserts that, given any real number a, there is a whole number n such that na is larger than any other given real number b." — ibid., p. 274.