Appendix: Global-View- and Ease-of-Understanding-Hierarchies

The Importance of a Hierarchical View of a Subject

I sometimes get the impression in talking to students that, for them, mathematics is a wilderness no different from, say, the tax laws, except there are more numbers and abstract terms. "You guys just keep making this stuff up as you go along!" an exasperated jr. high school student I was tutoring told me. I think that, for many students, precisely what is missing is any sense of hierarchy, of what each subject "looks like from above", i.e., globally, of what is subordinate to what, of what is more important than what, of how this problem relates to that one. Everything seems to be equally important. In short, what is missing is any sense of the lay of the land.

Every math student knows it is possible to pass a math course without having the slightest idea what the subject looks like "from above". The student memorizes definitions, formulas, theorems, algorithms, and applies them. But we seem to be nearing the day when much of this kind of intellectual labor will be performed by the computer..

If you talk to a mathematician about learning mathematics, he will almost certainly say that you can't learn mathematics without doing exercises. I am not sure I can accept this at face value any more. First of all, I would want to develop in any students of mine the ability to discriminate between the more important exercises (at any given stage of their learning) and the less important. The obtaining of a global view of a subject or a chapter and temporarily forgetting about details is itself an exercise, and an important one. The structuring of a typical prose proof as described in the chapter, "Proofs", is an exercise, as is writing down the fundamental intuition underlying part or all of a proof. Indeed, I believe that expressing informally the basic ideas and fundamental intuitions which underlie part or all of a subject, is often a far better exercise than those found at the end of chapters.

No one questions the value of computer graphics in making data or mathematical curves and surfaces easier to understand: why shouldn't we use graphics (computer or otherwise) to make the structure of mathematical subjects easier to understand?

Why Not a Hierarchical View of the Whole of Mathematics?

Given the rate at which mathematical knowledge is expanding, I think it is a scandal that there are no "maps" of the whole of mathematics, and of individual mathematical subjects. By a "map" I mean a diagram showing the relationship of various mathematical subjects.

Why shouldn't freshman students in any subject that makes extensive use of mathematics — certainly engineering, physics, and mathematics itself — be required to take a course that provides a bird's-eye view of the whole of mathematics? How can we possibly ask students to labor over their problems without having the Big Picture, some idea of the entire territory in which they are laboring? If it is not possible for anyone to learn all or most of mathematics in detail — and it hasn't been since 1900 — then why shouldn't mathematics professors regard their primary task as one of imparting knowledge of "where the tools (concepts) are" rather than knowledge of the "tools (concepts)" themselves? So the course (hence book) I have in mind would say to the student: "If you have to solve a problem about this ..., then some of the subjects where you should look for answers are ...; if you have to solve a problem about that... then some of the subjects where you should look for answers are ...; if you have to solve a problem..." An old saying among computer programmers sums up the idea: "You go to school to pick up pointers", the word "pointers" being a programming term for addresses in computer memory where desired data or programs are stored.

Why isn't the structure of mathematics among the very *first* courses that any freshman mathematics student takes? Why don't mathematics courses routinely *begin* with a clear, graphic presentation of the structure of the subject, and likewise for each part of the subject?

"The undergraduate needs a guidebook to the topography of the immense and expanding world of mathematics." — Temple, George, *100 Years of Mathematics*, Springer-Verlag, New York, 1981, p. 1.

The course (book) I have in mind would be a significant extension of the best of the popularizations of modern mathematics that were published in the 20th century, and that, I hope, are continuing to be published. The treatment would be informal, with ample illustrations and a minimum of equations and mathematical symbols, and, wherever possible, indications of the practical applications of the topic discussed (but "practical application" here means in other branches of mathematics as well as in the sciences). I believe that most of the topics listed in the "Systematic List of Articles" in the 1977 edition of the *Encyclopedic Dictionary of Mathematics* (pp. 1529 - 1533) could be included. See next section.

What Does It Mean to Say, "It Is Impossible to Know All of Mathematics"?

Henri Poincaré (1854-1912) is usually regarded as "the last man to have had a universal knowledge of mathematics and its applications."[1] Let us ask what it means to have "a universal knowledge of mathematics".

For most mathematicians, it almost certainly means to know not only the major concepts in each mathematical subject, but also the most important lemmas and theorems, and their proofs, and to be able to solve difficult problems in each subject.

But now let us ask if knowing all of mathematics is really such an either/or proposition – if it really boils down to: either you know everything about each subject, or you do not know all of mathematics. The *Encyclopedic Dictionary of Mathematics*[2] has a section titled, "Systematic List of Articles". The articles are grouped under the following categories:

I Logic and Foundations,
II Set Theory, General Topology, and Category Theory,
III Algebra,
IV Group Theory,
V Number Theory,
VI Euclidean and Projective Geometry,
VII Differential Geometry,
VIII Algebraic Geometry,
IX Algebraic Topology,
X Analysis,
XI Functions of One and Several Complex Variables,
XII Functional Analysis,
XIII Differential, Integral, and Functional Equations,
XIV Special Functions,
XV Numerical Analysis and Computer Science,
XVI Probability,
XVII Statistics,
XVIII Mathematical Programming, Operations Research, and Information Theory,
XIX Mechanics and Theoretical Physics,
XX History of Mathematics.

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1. Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p. 1170.
2. The MIT Press, Cambridge, Mass., 1977.

Are we seriously to believe that there isn't a single professional mathematician with a penchant and talent for writing popularizations of mathematics — say, someone like Ian Stewart — who could write brief descriptions of each of the above mathematical categories, covering all 20 of them in, say, 20 word-processor pages of 12-pt type? (The author would be given access to an illustrator skilled in creating mathematical diagrams.) Needless to say, a thorough index would be included.

Suppose we gave such an author an additional, say, 10 pages for each category, and specifically asked him or her to write for a student or mathematician who was looking for a branch of mathematics that might help in the solving of a specific problem. In other words, we would ask the author to emphasize concepts. In what sense would such an author, or a reader who understood the 200 pages the author had written, know "all" of mathematics, and in what sense would he or she not?

We should point out that the formal notation in which a subject, or brief description of a subject, is typically presented, is usually a block to understanding of underlying ideas (concepts), even though that notation is essential for stating lemmas and theorems and doing proofs. Unfortunately, the skill of being able to describe the major concepts of a subject with a minimum of notation, seems to be very rare among mathematicians.

Why Is It So Difficult to Obtain a Global View of a Subject?

It is amazing how difficult it is to get a global view of a subject from the typical textbook. I truly believe that textbook authors think the only way to obtain such a view is for the student to go through the entire book and work all, or most, of the problems. A global view, they seem to believe, is part of what constitutes an "understanding" of the subject, and an understanding can only be arrived at by many hours of hard study. And you, the student, after years spent in classrooms, probably believe this also. Your years in the classroom have taught you to accept without questioning that a global view is something that comes at the *end.* But that is crazy! A global view can, and should, be presented at the very outset! Why was the subject developed in the first place? What are the major entities with which the subject deals? What are the subsets of these entities? ("Types of" the entity in an Environment). What are some of the relationships between the entities? These and similar questions can be quickly answered with informal language and diagrams.

A Global View (Partial) of Algebraic Topology

If you want convincing evidence of the value of a global view of a subject, try teaching yourself the rudiments of algebraic topology from a textbook. (The best test I know of *any* textbook is to try to teach yourself the rudiments of a subject from it.) There are a number of texts available: I worked primarily from James R. Munkres's *Elements of Algebraic Topology*[1] and Allen Hatcher's *Algebraic Topology*[2]. Detailed criticisms of the texts can be found in Appendix C in the chapter, "Mathematics in the Schools".

A Top-Down Presentation of the Subject

A good way to obtain a global view of a subject is to organize the presentation of the subject via a sequence of questions (programmers will recognize this as a form of "top-down" presentation):

"If the reader only has five minutes, what are the most important things we can tell him?"

"If the reader has five additional minutes, what are the most important additional things we can tell him?"

"If the reader has five additional minutes..."

Etc.

Observe that in no way are we preventing the reader from finding formal definitions and proofs. References to these in the current textbook, or in other textbooks, can always been given in the index.

Here is a possible top-down presentation of the subject of algebraic topology. It would appear starting on the first page of the textbook.

Algebraic topology is the study of a variety of types of algebraic group that can be used to determine if two topological spaces are not homeomorphic. Specifically, if, given two topological spaces, just one of the groups in one space is not isomorphic to the corresponding group in the other space, then the spaces are not homeomorphic. (Unfortunataely, we cannot in general say that if all the corresponding groups are isomorphic, then the spaces are homeomorphic.)

The main types of group are:

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1. Addison-Wesley Publishing Company, Menlo Park, California, 1984.
2. Cambridge University Press, Cambridge, England, 2002.

homotopy groups (here the groups are based on closed paths that can be shrunk to a point in the topological space);
homology groups (here the groups are based on triangulations of the topological space); and
cohomology groups (these are "duals" of homology groups).

Fig. ... [not shown here] is a tree graph showing how *all* the types of homotopy and homology and cohomology groups dealt with in undergraduate texts are related — it shows which types are sub-types of which other types.

All the groups are abelian, and in many cases free abelian, meaning that they have a basis (a set of elements in terms of which all other elements in the group can be uniquely expressed). A typical element of an abelian group is a sum, e.g., $n_1g_1 + n_2g_2$, where n_1 and n_2 are typically, but not necessarily, integers, and g_1, g_2 are elements of the group. The terms n_1 and n_2 are called *coefficients*. We devote a fair amount of effort to investigating various types of coefficients with the goal of expanding our ability to compare the groups for different topological spaces.

We also spend a fair amount of effort investigating maps, mainly, homomorphisms and isomorphisms, between various groups, including the main types of groups.

(End of top-down presentation)

An important rule that textbook authors should follow can be informally expressed as, "Nothing in the book that doesn't come under a heading on the first page!" where by "first page" I mean the top-level view of the subject, such as the one I have just presented. This is simply the most basic aspect of a structure: in a structure there is a place for everything and everything has its place. If you know the structure, then you can always answer the question for any topic in the subject, "What is this part of?"

Why Is So Little Attention Paid to Global Views?

Why textbook authors, and professors in the classroom, pay so little attention to global views is, I think, first, because of the enormous prestige of difficulty in the academic mathematics community, and second because these authors and professors are specialists who have been convinced by their training that if you want to tell the time, then you need to know how the watch is built.

This prejudice was strongly reinforced by an approach to mathematics textbooks that goes by the name "Bourbaki", which is short for "Nicolas Bourbaki", which

"is the collective pseudonym under which a group of (mainly French) 20th-century mathematicians wrote a series of books presenting an exposition of modern advanced mathematics, beginning in 1935. With the goal of founding all of mathematics on set theory, the group strove for utmost rigour and generality ... .

"The emphasis on rigour may be seen as a reaction to the work of Jules-Henri Poincaré, who stressed the importance of free-flowing mathematical intuition, at a cost of completeness in presentation."[1]

It is worthy of note that Poincaré was one of the last mathematicians who is credited with having had a grasp of the whole of mathematics.

Bourbaki's approach to the presentation of mathematical subjects can well be described as an all-or-nothing approach: either perfection, or nothing. The book you are now reading is an attack against that approach — it argues that perfection is only the last stage of a sequence of stages beginning with a global view of a subject, and then gradually progressing, with increasing logical rigor and detail, to that last stage.

Finally, it is possible that professors are afraid that, if it is possible for the student to quickly acquire an idea of what the subject is about, what its key concepts are, and applications, in just a matter of minutes, or an hour or two, then the student will start asking why he or she must work all those problems. Indeed, why he or she should take the course.

In passing I should mention that what made me realize how extraordinarily useful a global view of a subject can be, was the attempt to teach myself the rudiments of certain subjects entirely on my own.

Stepping Back: Why Study a Subject in the First Place?

But at this point, we need to ask ourselves a basic question, namely, "Why do we study mathematical subjects?" The most immediate answer is, of course, "To acquire knowledge so that we will be awarded a degree which in turn will enable us to get a job doing something we enjoy, or at least that we don't hate as much as some other jobs." The next question is, "In what kinds of job will we use what we have learned?"

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1. "Nicolas Bourbaki", *Wikipedia*, May 15, 2008

And the answer is, "In jobs in which we have to teach the subject to others, and/or in jobs where we have to use it to solve problems, either in industry, or in mathematical or physics research."

There is an implication in these answers that we have to learn what we intend to use, and that we can only use what we have learned. But in the modern world, that implication needs to be questioned. First, because we often forget much of what we learned, and hence need a rapid means of looking up what we need (this was one of the main inspirations for the Environment concept). What we need is a way to rapidly find the subject or subjects we must use in order to solve the problems we are confronted with. In many cases, we may already know what these subjects are, from our education. But in many cases we will not know. And that is what a global overview provides, and why one is necessary for *each* subject. I argue that the first task of a mathematics education should be to tell the student where he or she can find techniques for solving various classes of problems. I argue that you should only work problems, do proofs, when you have a clear idea of where the problem or proof exists in the structure of the subject as a whole.

A Mathematics Classic from a Hierarchical Point-of-View

Gauss's *Disquisitiones Arithmeticae* ("Arithmetical Investigations") is a mathematics classic and is generally regarded as the beginning of modern number theory. It was written in 1777-1798, when Gauss was 20-21. (Young mathematicians who believe that competent mathematicians never stoop to publishing their own work might be interested to know that the French Academy rejected the book in 1800, and so Gauss published it on his own in 1801.)

The book collects numerous results by previous mathematicians, including Fermat, Euler, and Legendre, and then adds many additional results of Gauss. The book is 460 pages long and has been regarded as "difficult" although the sections on congruence are easily understandable by undergraduate math majors. But there are about 300 pages on the subject of forms, and this material is, at the least, less familiar to undergraduates. (A form is an expression in which the degree of each term is the same. Thus $ax^2 + 2bxy + y^2$ is a second-degree (quadratic) form.) Let us ask how these 300 pages might be approached by someone whose time is limited. Anyone who has understood the main ideas of this book will (I hope!) not say words to the effect, "Well, you just have to find the time, and then start on the first page of the forms section and work your way through."

We can accomplish a great deal from Gauss's table of contents alone — with, of course, our Environment template as a guide. We begin our forms Environment with a page headed "form". On this page will be "Definition of form", then "Representations of a form", then "Common operations on forms", then "Types of form" — with, of course, references to where in the Environment the details on each can be found, and page references there to Gauss's book.

Let us see what we can learn about forms by working solely from Gauss's table of contents. We see that, among the common operations are: representing numbers by forms, determining equivalence of forms, tranforming forms into other forms, and others. We see that the types of form include opposite forms, neighboring forms, ambiguous forms, forms with a positive nonsquare determinant, forms with square determinant, and others.

We will then add sections dealing with each of these items to our Environment, although in the case of types of form, we might initially just give a brief sketch of the definition of each type of form in the list of types.

I hope you will agree that, using the Environment approach, within less than half an hour, you can get a grasp of the whole subject of forms as it is presented in Gauss's book, and without working through a single equation. Obviously, you can delve into the text to get more details. Furthermore, if you read about forms in other books or papers, you can easily add additional information into your form Environment. No need to have a separate set of notes for each book or paper on the subject that you read.

Ease-of-Understanding Hierarchies

Another kind of hierarchy for a subject is based not on the entity structure of the subject but on relative ease-of-understanding. You have no doubt flipped through textbooks of subjects you knew little or nothing about, and said to yourself, "Well, this looks understandable, but that is thoroughly intimidating...I think I could probably make some headway with this other, maybe with that other, but..." A professor who had taught the course for several years could probably come up with an informal ranking of the concepts of the course based on the relative difficulty students had had with them. The same also applies to proofs. For example, I am sure that most first semester calculus students can understand the idea of an area as a sequence of thin rectangles, and a volume as a sequence of slices, much more easily than they can initially understand the definition of the derivative, or the proof of the fundamental theorem of the calculus.

In algebraic topology, I am sure that most students can understand the *idea* of the triangulation of a geometric object, and the generalization of a triangle (tetrahedron, etc.), and the idea that there is a sequence of n-dimensional triangulations, starting at $n = 1$, and that these are related in a nice way, in that an n-dimensional triangulation is the boundary of an $(n + 1)$-dimensional triangulation, and that if a certain kind of n-dimensional group for each n of one topological space X is not isomorphic to the same kind of n-dimensional group of another topological space Y then the spaces are not homeomorphic — I am sure that each of these informal descriptions is easier to understand than the formal details of a triangulation or of how the boundary function works.

So we can imagine a course (and/or a book) that takes successive passes through the subject in the order of increasing difficulty and detail.