CHAPTER 1

Why Is Mathematics Difficult?

(from William Curtis's *How to Improve Your Math Grades,* on occampress.com)[1]

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1. This line has been added to the first page of the chapter because Google does not reveal this information when it makes the chapter accessible following a user's search.

Why is Mathematics Difficult?

In my experience, there are at least six answers to this question.

1. The Fear Factor

The first is, because the feelings of inferiority and outright fear that many, probably most, students feel when they confront mathematics, severely inhibit students' natural intelligence and creativity. It is as though every mathematical subject, and every concept within a subject, is surrounded by a kind of "force field" that radiates, "Not for you!", "You aren't smart enough!". The origin of this force field may be early experiences in a family in which, say, a father had always been good at mathematics, and had made it clear he expected his children to likewise be good at the subject. In the case of women, the origin might be subtle messages sent by teachers throughout the primary and secondary school years — perhaps without conscious intention — that technical subjects are too hard for girls. Or, it might be the atmosphere that surrounds mathematics and indeed all technical subjects in the nation's most prestigious schools, in which the question is not, Can you learn it?, but Can you learn it the way it is taught and at the pace that those in charge demand?, Are you engineering or scientific or mathematics-professor material, yes or no? In short, Are you a winner or a loser?

In the industrially-developed countries, mathematical ability is a ticket to personal *value*. If you have mathematical ability, there is no doubt that you have a right to be walking the face of the earth. Furthermore, you will always be able to get a decent job, if only as a computer programmer or high school mathematics teacher. Those who do not have mathematical ability have no choice but to go into worthless subjects, namely, the liberal arts, and there waste their lives trying to convince the world that the alternative "truths" in these subjects are "just as good" as the truths defined by mathematics and science. So speaks the culture.

But if you have ever had the opportunity to study — to poke around in — a mathematical subject on your own — when you could take all the time you wanted, when no grades were to be earned, when you were not surrounded by competing students who might be brighter than you, when you could start wherever you wanted to in the subject and were not required to do all the exercises and get all or most of them right, when you could go as deeply or as shallowly as you wanted in any concept, when you could use popularizations to help you, or place an ad to find someone willing to explain things at your pace, knowing that if they didn't do a good job, you could simply pay them off and go look for someone else — in short, if you have ever had the opportunity to study all or part of a subject *on your own terms*, then perhaps you found (it took me many years) that mathematical subjects and concepts do not *inherently* come with an intimidating force field! They are just — there. The subject is just —

there, with its many interesting facts and its still-unanswered questions and unsolved problems inviting your investigation and thought.

When you study something, work on something, that "doesn't count" (e.g., a math subject you are studying purely out of your own interest), your innate intelligence and creativity have a chance to come to the fore. If you can't do something one way, you try another; and if everything fails, you ask someone for help, or give up for the time being. Your judgements during problem-solving — "Well, I got the first exercise, but this one is totally beyond me. I think I can make a beginning on this other. Let's see, maybe if I used Theorem ..." — these judgements do not have built-in shame factors, complete with head-shakings by Princeton professors ("No, he clearly has no ability. Poor guy. Yet he insists on struggling..."). They are simply your judgements, subject to revision as you proceed.

I say again: mathematics does not inherently come with an intimidating force field, it is not inherently owned by professors, it is not inherently a means for separating winners from losers, it is simply a collection of discoveries inviting your interest.Which is not to say that you shouldn't be worried, even anxiety-ridden, when you take a math course. If your future depends on your getting a good grade in the course, it would be strange if you weren't worried! My only point in this section is that you should be clear that this worry and anxiety is not *inherently* a part of mathematics.

2. Absence of the "Big Picture" for Each Subject

The second answer to the question, Why is mathematics difficult?, is that the student is almost never given, at the *beginning* of a course or a textbook, a one- or two-page bird's-eye view — a Big Picture — of the subject.

The traditional classroom course, and textbook, presents a subject in a *linear* format: in order to understand the second lecture (or the second page in the textbook) you need to have understood the first. In order to understand the third lecture (or the third page in the textbook), you need to have understood the second,... etc.

And yet the Big Picture is enormously helpful in understanding a mathematical subject. It shows the Forest, as opposed to merely describing each of the Trees in order.

Big Pictures are available, although you will probably never, in a classroom course or textbook, be told about them. For example, the numerous popularizations, e.g., by authors such as Ian Stewart, are a source of informal Big Pictures.

If you have any doubts about the value of Big Pictures, try to gain an understanding of the overview of partial differential equations (P.D.E.s) from the article on this subject on pages 455-483 in the *Princeton Companion to Mathematics*[1]. The man who wrote this article should have no trouble with the prospect of presenting a table of logarithms in prose.

What the article *should* contain is a set of tree graphs that show the structure of the subject. Certainly the first of these should have a root labeled "Partial Differential Equations (P.D.E.s), types of". The branches below that root should be the principle types of P.D.E.s, and then for each node at the end of each branch, a set of branches giving the types of P.D.E.s in that node's P.D.E., etc.

Furthermore, each node should refer to a section in the article with that type of P.D.E. as title. Subsections should include:

Brief description or definition of that type of P.D.E.;
Properties of that type of P.D.E.;
Procedures for solving that type of P.D.E.;

A source of Big Pictures, though the articles are more detailed and more formal than others, is the *Encyclopedic Dictionary of Mathematics*[1].

But the presentations in the above sources lack two crucial things that you will have to create for yourself. One is a list of the basic *entities* that are dealt with in the subject, along with the *operations* (tasks, "jobs") that are performed on those entities. Thus, e.g.:

• the basic entities dealt with in the calculus are continuous functions, two of the basic operations performed on continuous functions are finding the derivative and integrating.

• the basic entities dealt with in group theory are groups, some of the basic operations performed on groups are:

determining if two groups are the "same" (isomorphic);
determining the type of a given group;
making new groups out of given groups;
determining if a given group has a given type of subgroup (e.g., normal subgroup);
creating a representation of a group;

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1. Princeton University Press, Princeton, N.J., 2008.
1. ed. Iyanaga and Kawada, The MIT Press, Cambridge, Mass., 1977; later edition published in the early nineties

Each presentation of a subject must contain a table or tree graph showing how the types of the basic entities are related. How are all the principal types of continuous functions related? How are the principal types of groups and subgroups related? How are all the principal types of algebras related? (I am referring here to the algebraic structures, not the academic courses.)

If you doubt the value of such a table or tree, take a look at the trees describing the relationship between various types of topological spaces in the articles dealing with topology in the *Encyclopedic Dictionary of Mathematics.*

But we can go farther, *and should*. Why is there no readily-available graphical tree showing the structure of *all* of modern mathematics? As a start, the structure implied by the American Mathematical Society Classification Numbers could be used (these are the numbers that are typically given on the first page of a formal academic paper). Needless to say, accompanying the tree should be an alphabetical index of subjects and, following each subject, the classification number of the subject. At the start of the index would be an explanation of how to go from a classification number to a location in the tree.

How is it possible that, given the enormous size of the subject of mathematics, mathematicians did not create and continually update such a tree many decades ago?

A more extensive discussion of the Big Picture will be found in "Appendix: Global-View and Ease-of-Understanding Hierarchies" at the back of this book.

3. The Difficulty of Looking Up Things Fast...

...In Subjects in General

To arrive at a third answer to the question, Why is mathematics difficult?, let's look at the process you probably often go through in trying to solve a math problem. Let's assume it's a homework problem, and let's assume you are asked to prove something. I am beginning with the example of a proof because for most students this is the hardest type of math problem, although every math problem can be regarded as a proof problem, since, if you show your work, you are, in effect, providing a proof that your solution is correct!

Let me remark in passing, because it is important to keep in mind as you study mathematics, that there are many problems which have more than one solution. For example, if the problem is to find the solution(s) to the equation

$$x + y = 4,$$

one solution is $x = 2$, $y = 2$, another is $x = 3$, $y = 1$, etc. In fact, the equation has an infinite number of solutions. The equation

$$x^2 = 4$$

has two solutions, namely, +2 and –2. The equation

$$3x = 12$$

has one solution, namely $x = 4$.

There are many equations which have *no* solutions if the solutions are limited to a certain class of number. For example, there are no real number solutions to the equation

$$x^2 = -1.$$

So you have this problem of finding a proof. Let's assume that, although you wanted to start early on the homework assignment, because you suspected it would be a difficult one, somehow you didn't. Now it's already 9 p.m., and you have a physics assignment to do after this one.

You read the problem. (I won't give a specific proof problem here because I don't want to get bogged down in the details of a specific subject. But don't worry: we will look at specific proofs in specific subjects later in this book.) You pretty much understand the problem, but when you start to try to solve it, you realize that you're not sure of the precise definition of a couple of the terms and symbols. Better check on those right now. You start to look through the chapter the problem is in — but wait, wasn't one of the symbols defined earlier in the book? You try the previous chapter. (Time is passing!) You can't find it. You check the index. But where should you look for symbols in the index? You already know that there isn't a separate index of symbols such as textbooks sometimes (rarely!) have. You make a random search through the entire index. No, the author doesn't seem to have included symbols. How about your class notes? You start searching through them, but soon grow discouraged, since it suddenly occurs to you that several times the professor said that "the formal definition is in the text."

Well, you decide to go with the definitions as you remember them. OK, set up the problem. It's easy to see that you are being asked to prove a theorem of the form **if p then q**. You know some of the techniques for doing this kind of proof, e.g., you know that you will have a proof if you can prove the contrapositive of the theorem, i.e., if you can prove **if not q then not p**. Or you can just assume **p** and then work your way to **q** by using theorems already proved in the text and/or by using theorems and lemmas you already know from previous courses. You remember some theorems in the assigned chapters that seem relevant to **p**. Some you remember accurately, others you would just like to check on. You start searching through the chapters.

But it isn't easy to find the theorems, because immediately following each one is, naturally, the proof, and, sometimes, a discussion of the proof. As you scan through the chapters, you realize that you have forgotten many of the proofs — well, not so much "forgotten" as that you are not sure of exactly how they go: you have forgotten the main idea behind some of the proofs. With a feeling of panic, you know that right now you would not be able to reproduce these proofs on an exam. Time is passing, but maybe you should stop and review at least the proofs which the professor said were especially important. You pick one and start reviewing it. There are two paragraphs between the statement of the theorem and the proof proper, in which the author explains how the proof is related to other proofs. The proof itself (written out as a succession of paragraphs) cannot simply be *read*; you find that you must grab a piece of paper and rework it through yourself.

But soon you realize all this is taking too long! You'll have to review these proofs later — if worse comes to worst, just before the exam.

Back to the problem. Eventually, you are able to find a couple of theorems which seem clearly to be of use. But you're not sure if these are *all* the theorems that might apply. Also, you are worried that, maybe, some concepts you need are buried in the proofs of these theorems, or maybe in the proofs of other theorems... Time is passing. You try to get a start in proving the contrapositive. Soon you have six, ten, twelve steps written down and you are beginning to lose sight of exactly how far you've gotten so far. Everything seems to be so complicated, everything seems to be so difficult to *hold together* in your mind. Time is passing...

Let's take a look at what made this struggle so difficult. The anger you no doubt felt was justified. After all, you are not stupid. The mere fact that you have been accepted as a student in the school where you are studying is a testament to that.

It is obvious that one of the main problems is that it is very difficult to *look things up* in the typical math textbook. And yet, why shouldn't this be very easy? What is gained, from the standpoint of understanding mathematics, by making it difficult for you to look up definitions of symbols and terms, not to mention theorems, lemmas, proofs and even explanations of difficult concepts? The answer I usually receive when I put this question to math professors is that the student shouldn't rely on looking things up, he should *learn* the material. My immediate reply is, but why would it prevent the student from learning the material if he or she could look things up very rapidly? The answer to this is usually along the lines of "experience shows that students do not learn if you make things that easy for them", at which point I leave.

Let me put it as clearly and forcefully as I can: *The dreadfully inadequate indexes in virtually all textbooks is a disgrace* — one that reveals the dim, not-getting-it, all-or-nothing mentality of the professional mathematician when it comes to learning mathematics in the modern world.

You may be thinking, "Rapid look-up is too ordinary, too — obvious. What I need is some new *tricks*. Otherwise I'll never improve my math grades." But consider this: suppose two students of equivalent mathematical background were isolated in separate rooms and given a problem to solve. They had all relevant textbooks at their disposal, but were not allowed to talk to anyone. Suppose that student A were able to solve the problem quickly, largely because he happened to remember several theorems which were directly applicable. Suppose student B did not remember these theorems, but on searching through the textbooks, found them, and then went on to solve the problem in much the same way as student A. Which student has the greater mathematical ability? Which student should get the higher grade?

The example points out two abilities which are seldom separated in discussions about mathematical education. One is the ability to memorize mathematical facts (theorems, lemmas, concepts), and the other is pattern-recognition ability — meaning, here, the ability to see how a given theorem or concept can be applied to a given problem. Even it were shown that outstanding mathematical ability is *always* accompanied by outstanding mathematical memory, that would not change the fact that the two abilities can be separated, and the possibility that there are ways for almost any student to improve each ability.

But let us return to our example and ask a simple question. Furthermore, let us pretend we have been hired as efficiency experts for problem solvers. The question is, how could we have improved the speed at which student B found the theorems and lemmas he needed? We are not concerned, at this point, with improving student B's own memory; we are only concerned with the speed with which he can find things in the "extended memory" which constitutes all the textbooks at his disposal. To answer this question, we need to consider the typical mathematics textbook. This we will do in the section, "The Typical Mathematics Textbook" on page 17.

...In the Calculus in Particular

One of the things that makes calculus so difficult is the enormous number of facts you need to have at your fingertips in order to solve problems rapidly. (Just consider integration problems.) Yet most of these facts can be made look-up-able. For example, suppose — weeks or months or years after your last calculus course — you needed to know the derivative with respect to x of x^x. You kick yourself for not having made this a permanent part of your memory, along with all other calculus facts, but you need to know the answer now. How long *should* it take you to find it? Ans.: less than 10 seconds: you should be able to look up "x^x, derivative of" in an index of symbols *and* terms (which is what all indexes in mathematics textbooks should be), or under "derivative, types of, ... x^x".

Or suppose you want a summary of facts about the derivative of integrals, including integrals of multi-variable functions, and about the integration of derivatives including partial derivatives. How long *should* it take you to find the summary? Ans. less than 10 seconds: you should be able to look up "derivative, of integrals,..." and "integral, of derivatives, including partial derivatives..." in the index. Of course I am assuming that the textbook author has had the common decency to provide such a summary because he knew from classroom experience that students find it difficult to keep this pair of related matters straight in their minds.

I do not know of any studies of the *structure* of mathematical subjects[1], but I am sure that if and when such a study were carried out on the calculus, one of the findings would be how extraordinarily "flat" the calculus is, meaning, how much that students have to memorize in order to solve problems rapidly. (The tree representing the structure of the calculus typically has many branches from each node.) Subjects that are less "flat" would be those in which students would have to memorize relatively few facts because they could realistically derive what they needed within the time allotments of homework and exams. (The tree representing the structure of the subject typically has relatively few branches from each node.)

Imagine a Computer Program Like This...

Imagine a computer program that would give correct responses to inputs like:

"What is the definition of [term]?" What does [symbol] stand for?"

"Please give me a list of all definitions, theorems and lemmas that contain the terms *u* and *v*."

"Please give me a list of all theorems and lemmas that contain the term *s* in each antecedent."

1. If the reader knows of any, I urge him or her to contact me with the references.

“Please give me a list of all the commonly performed tasks on the entity *w*.”

“Please give me a tree showing the relationships of all types of the entity *h* that occur in this subject.”

“Please give me a list of all the well-known properties of the entity *t*.”

Etc.

Such a program would *not* be artificial intelligence but instead would be an elementary data base of a sort that was available already in the 1970s.

And yet it would permit you to spend most of your time *on the mathematics*, instead of most of your time memorizing the facts that you need in order to do the mathematics. Your goal as a student is to have on hand those facts. One way of achieving that goal is the way that have always followed in math courses, namely, by memorization. Another way would be via the use of a data base such as I have described.

It will take only one such data base to convince students — and even a few math professors! — how much more *efficient* the doing of mathematics is with such data bases.

This whole book is, in essence, a way of implementing, on paper, such a data base.

4. The Prestige of Difficulty Among Mathematicians

A mathematician who had spent many years at one of the nation’s leading research laboratories, once told me that it was customary for mathematicians in his department to dismiss with contempt any new paper that was easy to understand.

Several mathematicians have remarked to me that they have no respect for Benoit Mandelbrot’s fractal geometry — truly one of the most important discoveries of the late 20th century, with applications in physics, biology, chemistry and many other subjects — because the proofs are too simple.

I have known mathematicians who clearly loved the subject because it was incomprehensible to the vast majority of mankind, and who were proud of the fact that their specialty was incomprehensible even to other mathematicians.

If you look at the CV (curriculum vitae, or resumé) of a professional mathematician, you will see paper after paper listed, each with a title that only a specialist can understand. Yet the CV will be read (looked at) by others besides those in the mathematician's field. I am not criticizing the listing of the titles of the papers. I am criticizing what I believe underlies the present form of the CV, namely, the smug satisfaction of the mathematician over the fact that very few of the readers of the CV (including other mathematicians, in this age of specialists) will have any way of judging what each paper is about. A CV doesn't say, "Here is what I worked on, and why I worked on it, and some of the results I achieved," it says, "Unless you are at my level of genius, don't even *think* about understanding what I have accomplished!"

(I must confess that when I look at a list of the faculty members at a major university, and read the specialties of each, I can't help thinking, "All these specialties are '*the same*'! They are different names for the same thing!" The reason I have such an outlandish thought will become clear when you read Chapter 4 of this book, "How to Build an Environment". But in brief: each mathematical subject sets forth various entities, and each entity can be presented via a single template, which is:

name of entity;
definition(s) of entity;
history of entity;
ways of representing entity;
common operations (tasks) we perform on the entity;
properties of the entity;
types of the entity[1];
theorems pertaining to the entity;
closely-related entities.

Once you have begun to think of mathematical subjects in this way, you will see why I say, "All mathematical subjects are 'the same' ". And you will understand why I view with a certain wry amusement, the impression that these faculty lists convey, namely, that of a group of geniuses each of whom was obviously destined from birth to be a genius in just the subjects he or she specializes in, and no others (for they are all different). The rest of us will be lucky, through months of hard work, to gain just a fragment of the unimaginable depth of knowledge of one of these extraordinary individuals.

How Mathematicians Could Make Proofs Much Easier to Understand

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1. Obviously, "types" and "properties" are the same thing, but it is sometimes helpful to have the two sets side-by-side.

The prestige of difficulty is further evidenced by mathematicians' clinging to a protocol for writing proofs, and of checking proofs (their own and those of other mathematicians), that has not changed for centuries. In this protocol, a proof is a succession of paragraphs. The person hoping to understand the proof must understand the entire argument as it develops — all or nothing. Furthermore, the proof of each lemma that is original in the paper (as opposed to being well-known) must be checked before the main argument is attempted. All or nothing.

And yet, in the 1970s, confronted with the growing complexity of computer programs, computer scientists developed a method of programming that vastly simplified the task of writing programs and checking them, and this method, called "structured programming" (or "top-down programming"), can be directly applied to the writing and checking of theorems and lemmas in mathematics. In this method (which is described in detail in the chapter, "Proofs"), a complex proof has a main part consisting of a few steps. The main part, hence the proof as a whole, is valid if the steps constitute a valid logical argument. Because the main part consists of only a few major steps, the writer can see clearly what his overall argument is, and so can the reader. But the main part can only constitute a valid logical argument if each of the steps is true, and so we then proceed to each step in succession. The proof of each can be broken down in the same way as the main part of the proof was. So at any given time, the writer can see clearly what his argument is for the proof of that step, and so can the reader.

But despite the fact that "structured proof" was known already by the 1980s, it is almost never used by professional mathematicians, either in the papers they write, or in the way they check other mathematicians' proofs. If you tell a mathematician, concerning a paper you have recently completed, "The proofs of the lemmas have been checked and deemed correct by several mathematicians, so *initially*, you can assume all the lemmas are true, and just concentrate on the main part of the proof," your words will usually be ignored. The mathematician believes there is only one way to review a paper, and that is to check every statement. If the mathematician doesn't have time to do that, he will decline to review the paper.

To those few who have grown to recognize the extraordinary improvement in efficiency that is afforded by the structured approach to reviewing mathematics papers, the stubborn refusal of mathematicians to adopt this approach is almost beyond endurance. When these few hear, for example, that "only two or three" mathematicians are able to understand a particularly difficult proof, they ask how that can be possible, when any proof, even one that is several hundred pages long, can be put into structured form, with the top level consisting of only a few steps. Of course, it may be necessary for many readers to look up the meanings of many obscure technical terms, but that is simply a look-up process. Most important, the top level, and the proof of each step, consists of only a few steps. Furthermore, the structure of the entire proof can probably be described in a few pages. It is never a case of "all or nothing".

So why this stubborn refusal on the part of mathematicians to adopt this better method? It is hard not to conclude that the reason is that keeping things the way they are makes sure that mathematics remains difficult and therefore prestigious.

How Mathematicians Could Make Textbooks Much Easier to Understand

Generations of students have taken it for granted that, if they don't understand mathematics, it is their fault, just as decades of computer users have taken it for granted that, if they are unable to make the software do what it is supposed to do, it is their fault. ("If only I were smarter! If only I were an engineer!") Which might well be called, "God's gift to software manufacturers", because it means that manufacturers can spend next to no time or money on making their product easily usable by its class of intended users, knowing that the users will only blame themselves. If only the automobile industry were as lucky: if the car doesn't run, it's the driver's fault!

(I should mention in passing that I believe that a major step forward in making software easier to use is contained in Peter Schorer's *How to Create Zero-Search-Time Computer Documentation* (www.zsthelp.com). Sadly, the documentation profession has completely ignored this book, believing that ever more sophisticated documentation software to implement the same old approach to documentation, constitutes progress. It doesn't.)

There is no doubt that mathematics is a more difficult subject than most others. But there are degrees of difficulty — or perhaps I should say, there are different *kinds* of difficulty. At the very least, there is conceptual difficulty, and there is deductive difficulty (finding the right sequence of statements to accomplish a proof). For example, the difficulty of the *concepts* in elementary calculus is far less than the difficulty of solving many types of calculus problems. But concepts, notation, and deductions are presented as a single whole, with no attempt made to draw the student's attention to the distinction between the three. In vector calculus, the *concepts* underlying Gauss's Theorem, Green's Theorem, and Stokes' Theorem are much less difficult than is the new notation, and deductions employing it. Yet no attempt is made to draw the student's attention to the distinction.

Most textbooks are begun with the following assumption, perhaps not even mentally verbalized: "Given that whatever I write must be written for use in a classroom [in other words, must give me a reason to continue to collect a paycheck] what kind of book can I write?" However, ensuring job security is not the best place to begin! It almost guarantees that the fundamental question, ***the student's most***

***important question, "Why is this difficult?*[1] *"*, will not be addressed. A much better place to begin is, "To what degree can I *do away with the need for a teacher* in this subject? How much of this subject can I make look-up-able? How can I increase the speed at which students become able to solve any class of problem in this subject?"

Question: suppose that tomorrow, a pill were developed that made mathematics easy to understand by most people. Would mathematicians be: (a) happy, or (b) sad? My opinion is that there aren't enough psychiatrists in any country to handle the droves of profoundly depressed mathematicians who would throng their offices if such a pill were developed.

But in fact the structured, or top-down, format I described above for proofs, can also be applied to textbooks and, indeed, to popularizations and overviews.

Here, the mathematician begins by writing down the minimum knowledge he will assume of all readers of what follows. He then presents a description of the entire subject that the reader can read and understand in, say, five minutes. (Call this level 1 of the text.)

He then gives an elaboration of what he has just written, this elaboration likewise being understandable in, say, five more minutes of reading. (Call this level 2 of he text.)

He continues this way until he has covered all the material he wants to cover. This, of course, eventually includes definitions and proofs.

Note that the first k levels, where $k \geq 1$, always are a description, to various degrees of detail, *of the entire subject*.

Some of the Best of the Best Do Not Hold Difficulty in High Regard

> "An old French mathematician said, 'A mathematical theory is not to be considered complete until you have made it so clear that you can *explain it to the first man you meet on the street*." This clearness and ease of comprehension, here insisted upon for a mathematical theory, I should still more demand for a mathematical problem if it is to be perfect; for what is clear and easily comprehended attracts, the complicated repels us." — David Hilbert, probably the greatest mathematician of the first half of the 20th century. The French mathematician he refers

1. I have never — not once in all my years of schooling — ever heard a professor mention this question, much less urge students to ask it whenever they are having difficulties. And so I am inclined to call it The Forbidden Question.

to was Joseph Diaz Gergonne (1771-1859), who made his statement in a letter of 1825.

5. Justification for Each Statement in Textbooks Often Not Given

The text in a textbook normally consists of a sequence of statements. Occasionally, there is a phrase like, "By Lemma ... we know that ..." but these are the exception rather than the rule. The student often reads a statement, has no idea why it is true, or has forgotten, and assumes that this is further proof that he or she has no business studying mathematics[1]. The truth is, *every statement* in a textbook whose justification lies within the subject covered by the textbook[2], *should have the justification given immediately following it*. The justification might be a reference to a specific lemma or theorem (with page no.) or to a specific exercise (with its no. and page no.), or to a proof that was given as part of the text (with page no.).

The consistent failure to give these justifications can be regarded as nothing less than a deliberate attempt by the authors to enhance the prestige of their subject ("incredibly difficult!") and to promote the job security of professors, who supply the justifications during the class hours.

I have heard it said that authors omit justifications because they want to exercise their students' minds. But that is nonsense: the place to exercise students minds is in the exercises. Otherwise, why not leave out parts of sentences and paragraphs? That will exercise students' minds even further (if any students are willing to put up with this).

6. Complete Absence of Training in How to Budget One's Time, How to Approach Problems

You would think that, in a subject as difficult as mathematics, there would be a major, ongoing effort on the part of teachers and professors to teach students some ways to deal with that difficulty. Yet that effort is almost never made.

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1. He or she might well start imagining that a silent paraphrase of lines from a Monty Python sketch accompanies such statements: "Do you understand why this is true? Eh? Are you one of us? Nudge-nudge, wink-wink, say no more, say no more?")
2. An author has a perfect right to expect each reader of his textbook to be familiar with certain subjects. Of course, he must state what these subjects are in the Introduction to his book. He is not required to give justifications for statements that are part of these subjects, although common decency and consideration for students demand that he will give a quick reminder of the content of certain difficult topics in one of these subjects.

What the Student Is Encouraged to Believe

All that the student learns — gathers — is that:

• If the student doesn't get it, this is due to his or her lack of ability. In any case, he or she must never even think of questioning the quality of instruction, in particular, the quality of the presentation in the textbook. The textbook is the subject and the subject is the textbook.

• To succeed, the student must either get it very quickly, or else work long hours into the dead of night. That's it. End of story. The mathematics community has clearly never heard of the slogan that was developed in the computer industry many years ago, "Work Smarter, Not Harder!".

• The student who does more problems is better than the student who does fewer problems. The rare student who believes that a written procedure for solving *classes* of problems is much more valuable than the endless grinding out of problems in the hope that the procedure will slowly dawn on him or her— such a student hasn't a prayer of being taken seriously. Learning by doing is the only way.

And yet one of the most valuable things the student can learn is how to approach very difficult tasks. Below, in "What the Student Should Be Taught" on page 16, are some good rules that should be taught and retaught in mathematics courses.

There are two more reasons why mathematics is difficult. One is the lack of adequate illustrations in textbooks — especially in subjects that are not "geometrical". We will have more to say about this in the section, "On the Shameful Lack of Adequate Illustrations in Most Textbooks", in the chapter "Mathematics in the University".

The second reason is that mathematics courses say nothing about how to do mathematics — how to learn concepts, how to approach problems. We discuss this in the next section.

What the Student Should Be Taught

• To list, for each course, what needs to be done by a given date and time, and then allocate an amount of time for each course's assignments, and then, in doing each assignment, to work first on the problems that seem easiest, and save the most difficult ones for last.

• If the student's work will be read by a human grader, to explain clearly what he or she knows, e.g., "We are being asked to prove [statement]. We know that ... and ... so it seems that a good place to begin is with Lemma ... If that fails, then we can try proving the contrapositive ..." The point is to impress the grader with the student's intelligence, so that the grader will think, "Even though this student was unable to solve the problem it is clear that he or she has read the material and can think and therefore ... is one of *Us*!"

• To *add to the index* — to write, in the index, the terms that the student attempted to look up, but that were omitted by the author, and the page number where the student found that the definition is given. Reducing future look-up times will be far more valuable than the student realizes at first.

• To use structured proof (see the chapter, "Proofs") both for understanding textbook proofs, and for coming up with your own proofs in homework and exam problems.

• Never to hesitate to ask that all-important question, *Why is this difficult?* In many cases, the answer will be, *Because the professor and/or textbook are not very good at explaining this*. However, the student should be *very hesitant* about bringing this to the professor's attention!

• *Always* to show enthusiasm for the course in every interaction with the professor or grader(s). *Never* to criticize classroom lectures or the textbook.

The Typical Mathematics Textbook

When studying a technical book — any kind, be it a textbook in mathematics, or in engineering, physics, chemistry or any other science — you should always ask yourself, "What does the organization of this book imply about how it is intended to be used?" A textbook is a tool for use in schools, and is organized accordingly[1]. Like the

1. Not only organized, but also written, accordingly. If you have any doubts about how much is omitted from a typical textbook, try to teach yourself, without any help, the contents of just the first chapter or two of a typical textbook in a subject for which you have the necessary background but about which you know little. The typical textbook is developed from lecture notes and ensures the continued need for a professor.

typical instruction manual for a piece of equipment, it presents its material in the order of how the subject *is constructed*, i.e., of how the subject is "made". It "develops" the subject for an audience of students who learn at the pace and direction of a teacher. It need not be a particularly good teaching device in itself, since there is always a teacher around to explain what the student doesn't understand. Yet the question, "How is this subject made?" — i.e., "What is the logical construction of this subject?" — is only one of many questions you can ask about a subject.

The assumption, in a typical textbook, is that if you want to tell the time, you need to know how the watch is made. This is a natural point of view from which to write the book if you, as a professor of mathematics or any other technical subject, know the subject already, just as it is a natural point of view for the engineers who design electronic equipment. *The typical textbook is the record of someone's teaching of a subject, never of someone's learning or using a subject.*

But the point of view represented by the typical textbook is *not* necessarily the best point of view for you as a student who wants to use the book to solve problems! If you think about how you *use* a textbook, you will notice that your usage does not in general correspond to the organization of information in the textbook. During and after the course, you do not always read in one direction only, i.e., from front to back; instead, you *refer* to the book: you look up things (or try to), you refresh your memory because you have forgotten details or even major points, you search for certain kinds of information, you use the book for brief periods of time, go on to other things, then come back to it. You by no means have the time to derive the answer to every question you are confronted with in the course of your problem solving; you often try to look up the answer (it may be a formula or theorem or lemma or answer to a previous exercise), using the index and table of contents.

It is simply not true that you have to know how the subject is made in order to solve problems in it. If you have ever used a pocket calculator, or a computer, or any home appliance, or if you drive a car, you know that it is not always necessary to know how something is built in order to use it. And similarly, it is often possible to solve problems in the middle of the book without having read, still less solved, all the problems in the first half of the book.

The important question from the point of view of problem solving is not, "How is this subject constructed?" but instead "What kind of a book (or computer program, or expert human assistant) do I wish I had in order to solve problems in this subject?"

Let us for the moment narrow this question to: *"What kind of a book (or computer program, or expert human assistant) do I wish I had in order to prove statements in this subject?"* If you think about this for a while, chances are that you will come up with an answer along the lines of, "I wish I could look up lemmas and theorems by the words they contain. For example, in a modern algebra course, I wish I could look up, say, all the lemmas and theorems having the word "commutative" in the **then** part of the lemma or theorem statement. I wish I could say (to the book or computer program or human assistant I was using) things like, "Give me all the lemmas and theorems containing any one of the following terms (or their synonyms) in the **if** part of the lemma or theorem statement:..." Or, "Give me all the lemmas and theorems containing the following terms (or their synonyms) anywhere in the lemma or theorem statement"...". Of course, you will *certainly* include, "Tell me the definition of the term *x*," and "Tell me the what the symbol *y* stands for."

If you want to assess your skill at doing proofs, *this* is the point from which you should begin: from the point at which you can summon statements of lemmas and theorems as described in the previous paragraph. *Not* from the point of having to memorize an entire course's worth of lemmas and theorems so that you can look up the ones you want in your memory, possibly helped by shuffling through pages of text and exercises.

In the typical textbook, your job of doing proofs is made more difficult by the fact that sometimes the crucial lemma or theorem you need is stated in an exercise in a previous chapter. Furthermore, authors differ in their opinions as to which lemmas and theorems should be treated in the text proper, and which should be relegated to the exercises. All of which may well incline you to believe that there are mysterious reasons why a lemma or theorem is treated as an exercise in one book, and as an item in the text of another. Not so! A true statement is a true statement, regardless whether it is called a lemma, or a theorem, or an exercise.

The consistent failure of authors to provide justifications (with page no. reference) *for each and every statement* in a textbook whose justification is given somewhere in the text, is one of the basic reasons why mathematics is difficult.

In passing, let me mention that computer databases capable of allowing the kind of retrieval of lemma or theorem statements that was described above, were already in existence in the 1960s. So please don't imagine that such a retrieval capability requires "artificial intelligence"! What you might ask yourself, however, is why the mathematics community still hasn't created such databases and made them available to students. Is it because the professors believe that the students should be forced to learn the material before using it? But such databases in no way prevent students from learning and memorizing!

In this book, you will learn a technique for reducing the amount of time you need to spend memorizing and searching for lemma and theorem statements. It is not as good as a computer database, but it is much better than a mere textbook provides.

I am not, of course, saying, that the traditional textbook should be done away with, since there are times when it is valuable to have a logical presentation of the subject, just as there are times when it is valuable to know how a piece of equipment is constructed. What I *am* saying is that the traditional textbook is not the best tool for helping you to solve typical classroom and homework and exam problems.

The Principal Classes of Homework and Exam Problems

The question of how a textbook should be organized for problem-solving in turn forces us to realize that there are different classes of sub-problems which a student typically has to solve in the course of solving classroom and textbook problems. These classes include:

• "What does the term (or symbol) *u* mean?" If *u* represents a difficult concept, "What are some guides to improving my intuitive understanding of *u*?"

• "How should I go about solving a problem of type *v*? What are some good ways of approaching this type of problem?"

A sub-class of this class is, "What is the formula for *w*?"

A much more important sub-class is, "How do I prove the statement *s*?"

• "What does the theorem or lemma called *t* assert?" E.g., in the calculus, what does Rolle's Theorem assert?

• "What theorems or lemmas are closely related to theorem or lemma *t*?"

• "How can I break the proof of theorem *t* down into pieces I can understand?"

• "What theorem or lemma or axiom enables the author to go from this assertion to that one in this proof?"

• "What are the basic operations we perform on the entity *r*?" If *r* is a number, then these basic operations may include addition, subtraction, multiplication, division, finding the "size" of the number (absolute value in the case of the real numbers). If *r* is a set of objects with certain properties, e.g., a topological space, then the set of basic operations may include: determining if the set is "equivalent" to another set (e.g., in topology, homeomorphic); determining the "building blocks" of the set; making new sets of the same kind out of the given set; defining certain important kinds of mappings (functions) between the sets; determining if a given mapping between the sets has certain properties.

And last, but certainly not least,

• "What is the Big Picture of this subject? What does the Forest look like, apart from the Trees?"

At this point, I hope I have started to convince you that the difficulty of finding information *fast* is a major source of the difficulty you often face in solving mathematics problems. You may feel that, well, yes, it has been *a* source of the difficulty you have faced, but you still can't believe that it has all that much to do with the struggles you have endured in this subject. What about the difficulty of understanding mathematical concepts themselves?

Why Are Mathematical Concepts Difficult to Understand?

In this book, the term "mathematical concept" means just about anything with a mathematical name. (In the chapter, "How to Build an Environment", the term "mathematical entity" is sometimes used instead of "mathematical concept", for reasons that will become clear.) For example, some of the mathematical concepts we learn in high school are: constant, variable, polynomial, factor, factoring, equation, solving an equation, logarithm, sine, cosine, tangent, etc., point, line, triangle, square, and other geometric figures, area, perimeter of a geometric figure, etc., and many others.

Among the mathematical concepts we learn in our first years of college mathematics are: set, operation, limit, function, and, specifically, continuous function, derivative, integral, theorem, proof, countable infinity, uncountable infinity, algebra, linear algebra, vector space, group, ring, field, and many others.

Now *one* thing that makes the understanding of these concepts difficult is that they are defined in terms of other concepts. Thus, e.g., a vector space is defined in terms of the concepts of vector, set, function, abelian group, field, and others. How does the typical mathematics textbook, and mathematics course, deal with this fact? It attempts to teach the concepts in *logical order*, i.e., it assumes that, e.g., when you begin your study of vector spaces, you will already know — through having remembered what you learned in previous courses — the meaning of each of the concepts in terms of which a vector space is defined. And, indeed, one of the things that makes mathematics such a frightening subject to many students, is the grandiose manner with which these assumptions are set forth in the list of prerequisites for the course: "Student is assumed to know...and to have a strong background in...as well as having completed the following courses or their equivalents..." And you, reading that at the end of summer vacation, immediately feel the self-doubt, perhaps self-contempt, which schools thrive on, as you realize that you're not sure if you "know" these things after all.

Once again, ease of looking things up *fast* seems to be beckoning to us.

Keep reading.