CHAPTER 1

Why Is Mathematics Difficult?

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In my experience, there are two answers to this question.

The Fear Factor

The first is, because the feelings of inferiority and outright fear that many, probably most, students feel when they confront mathematics, severely inhibit students' natural intelligence and creativity. It is as though every mathematical subject, and every concept within a subject, is surrounded by a kind of "force field" that radiates, "Not for you!", "You aren't smart enough!". The origin of this force field may be early experiences in a family in which, say, a father had always been good at mathematics, and had made it clear he expected his children to likewise be good at the subject. In the case of women, the origin might be subtle messages sent by teachers throughout the primary and secondary school years — perhaps without conscious intention — that technical subjects are too hard for girls. Or, it might be the atmosphere that surrounds mathematics and indeed all technical subjects in the nation's most prestigious schools, in which the question is not, Can you learn it?, but Can you learn it the way it is taught and at the pace that those in charge demand?, Are you engineering or scientific or mathematics-professor material, yes or no? In short, Are you a winner or a loser?

In the industrially-developed countries, mathematical ability is a ticket to personal *value*. If you have mathematical ability, there is no doubt that you have a right to be walking the face of the earth. Furthermore, you will always be able to get a decent job, if only as a computer programmer or high school mathematics teacher. Those who do not have mathematical ability have no choice but to go into worthless subjects, namely, the liberal arts, and there waste their lives trying to convince the world that the alternative "truths" in these subjects are "just as good" as the truths defined by mathematics and science. So speaks the culture.

But if you have ever had the opportunity to study — to poke around in — a mathematical subject on your own — when you could take all the time you wanted, when no grades were to be earned, when you were not surrounded by competing students who might be brighter than you, when you could start wherever you wanted to in the subject and were not required to do all the exercises and get all or most of them right, when you could go as deeply or as shallowly as you wanted in any concept, when you could use popularizations to help you, or place an ad to find someone willing to explain things at your pace, knowing that if they didn't do a good job, you could simply pay them off and go look for someone else — in short, if you have ever had the opportunity to study all or part of a subject *on your own terms*, then perhaps you found (it took me many years) that mathematical subjects and concepts do not *inherently* come with an intimidating force field! They are just — there. The subject is just —

there, with its many interesting facts and its still-unanswered questions and unsolved problems inviting your investigation and thought.

When you study something, work on something, that “doesn’t count” (e.g., a math subject you are studying purely out of your own interest), your innate intelligence and creativity have a chance to come to the fore. If you can’t do something one way, you try another; and if everything fails, you ask someone for help, or give up for the time being. Your judgements during problem-solving — “Well, I got the first exercise, but this one is totally beyond me. I think I can make a beginning on this other. Let’s see, maybe if I used Theorem ...” — these judgements do not have built-in shame factors, complete with head-shakings by Princeton professors (“No, he clearly has no ability. Poor guy. Yet he insists on struggling...”). They are simply your judgements, subject to revision as you proceed.

I say again: mathematics does not inherently come with an intimidating force field, it is not inherently owned by professors, it is not inherently a means for separating winners from losers, it is simply a collection of discoveries inviting your interest.Which is not to say that you shouldn’t be worried, even anxiety-ridden, when you take a math course. If your future depends on your getting a good grade in the course, it would be strange if you weren’t worried! My only point in this section is that you should be clear that this worry and anxiety is not *inherently* a part of mathematics.

The Difficulty of Looking Up Things Fast

To arrive at a second answer to the question, Why is mathematics difficult?, let’s look at the process you probably often go through in trying to solve a math problem. Let’s assume it’s a homework problem, and let’s assume you are asked to prove something. I am beginning with the example of a proof because for most students this is the hardest type of math problem, although every math problem can be regarded as a proof problem, since, if you show your work, you are, in effect, providing a proof that your solution is correct!

Let me remark in passing, because it is important to keep in mind as you study mathematics, that there are many problems which have more than one solution. For example, if the problem is to find the solution(s) to the equation

$$x + y = 4,$$

one solution is $x = 2$, $y = 2$, another is $x = 3$, $y = 1$, etc. In fact, the equation has an infinite number of solutions. The equation

$$x^2 = 4$$

has two solutions, namely, +2 and –2. The equation

$$3x = 12$$

has one solution, namely $x = 4$.

There are many equations which have *no* solutions if the solutions are limited to a certain class of number. For example, there are no real number solutions to the equation

$$x^2 = -1.$$

So you have this problem of finding a proof. Let's assume that, although you wanted to start early on the homework assignment, because you suspected it would be a difficult one, somehow you didn't. Now it's already 9 p.m., and you have a physics assignment to do after this one.

You read the problem. (I won't give a specific proof problem here because I don't want to get bogged down in the details of a specific subject. But don't worry: we will look at specific proofs in specific subjects later in this book.) You pretty much understand the problem, but when you start to try to solve it, you realize that you're not sure of the precise definition of a couple of the terms and symbols. Better check on those right now. You start to look through the chapter the problem is in — but wait, wasn't one of the symbols defined earlier in the book? You try the previous chapter. (Time is passing!) You can't find it. You check the index. But where should you look for symbols in the index? You already know that there isn't a separate index of symbols such as some textbooks have. You make a random search through the entire index. No, the author doesn't seem to have included symbols. How about your class notes? You start searching through them, but soon grow discouraged, since it suddenly occurs to you that several times the professor said that "the formal definition is in the text."

Well, you decide to go with the definitions as you remember them. OK, set up the problem. It's easy to see that you are being asked to prove a theorem of the form **if p then q**. You know some of the techniques for doing this kind of proof, e.g., you know that you will have a proof if you can prove the contrapositive of the theorem, i.e., if you can prove **if not q then not p**. Or you can just assume **p** and then work your way to **q** by using theorems already proved in the text and/or by using theorems and lemmas you already know from previous courses. You remember some theorems in the assigned chapters that seem relevant to **p**. Some you remember accurately, others you would just like to check on. You start searching through the chapters.

But it isn't easy to find the theorems, because immediately following each one is, naturally, the proof, and, sometimes, a discussion of the proof. As you scan through the chapters, you realize that you have forgotten many of the proofs — well, not so much "forgotten" as that you are not sure of exactly how they go: you have forgotten the main idea behind some of the proofs. With a feeling of panic, you know that right now you would not be able to reproduce these proofs on an exam. Time is passing, but maybe you should stop and review at least the proofs which the professor said were especially important. You pick one and start reviewing it. There are two paragraphs between the statement of the theorem and the proof proper, in which the author explains how the proof is related to other proofs. The proof itself (written out as a succession of paragraphs) cannot simply be *read*; you find that you must grab a piece of paper and rework it through yourself.

But soon you realize all this is taking too long! You'll have to review these proofs later — if worse comes to worst, just before the exam.

Back to the problem. Eventually, you are able to find a couple of theorems which seem clearly to be of use. But you're not sure if these are *all* the theorems that might apply. Also, you are worried that, maybe, some concepts you need are buried in the proofs of these theorems, or maybe in the proofs of other theorems... Time is passing. You try to get a start in proving the contrapositive. Soon you have six, ten, twelve steps written down and you are beginning to lose sight of exactly how far you've gotten so far. Everything seems to be so complicated, everything seems to be so difficult to *hold together* in your mind. Time is passing...

Let's take a look at what made this struggle so difficult. The anger you no doubt felt was justified. After all, you are not stupid. The mere fact that you have been accepted as a student in the school where you are studying is a testament to that.

It is obvious that one of the main problems is that it is very difficult to *look things up* in the typical math textbook. And yet, why shouldn't this be very easy? What is gained, from the standpoint of understanding mathematics, by making it difficult for you to look up definitions of symbols and terms, not to mention theorems, lemmas, proofs and even explanations of difficult concepts? The answer I usually receive when I put this question to math professors is that the student shouldn't rely on looking things up, he should *learn* the material. My immediate reply is, but why would it prevent the student from learning the material if he or she could look things up very rapidly? The answer to this is usually along the lines of "experience shows that students do not learn if you make things that easy for them", at which point I leave.

You may be thinking, "Rapid look-up is too ordinary, too — obvious. What I need is some new *tricks*. Otherwise I'll never improve my math grades." But consider this:

suppose two students of equivalent mathematical background were isolated in separate rooms and given a problem to solve. They had all relevant textbooks at their disposal, but were not allowed to talk to anyone. Suppose that student A were able to solve the problem quickly, largely because he happened to remember several theorems which were directly applicable. Suppose student B did not remember these theorems, but on searching through the textbooks, found them, and then went on to solve the problem in much the same way as student A. Which student has the greater mathematical ability? Which student should get the higher grade?

The example points out two abilities which are seldom separated in discussions about mathematical education. One is the ability to memorize mathematical facts (theorems, lemmas, concepts), and the other is pattern-recognition ability — meaning, here, the ability to see how a given theorem or concept can be applied to a given problem. Even it were shown that outstanding mathematical ability is *always* accompanied by outstanding mathematical memory, that would not change the fact that the two abilities can be separated, and the possibility that there are ways for almost any student to improve each ability.

But let us return to our example and ask a simple question. Furthermore, let us pretend we have been hired as efficiency experts for problem solvers. The question is, how could we have improved the speed at which student B found the theorems and lemmas he needed? We are not concerned, at this point, with improving student B's own memory; we are only concerned with the speed with which he can find things in the "extended memory" which constitutes all the textbooks at his disposal. To answer this question, we need to consider the typical mathematics textbook.

The Typical Mathematics Textbook

When studying a technical book — any kind, be it a textbook in mathematics, or in engineering, physics, chemistry or any other science — you should always ask yourself, "What does the organization of this book imply about how it is intended to be used?" A textbook is a tool for use in schools, and is organized accordingly[1]. Like the typical instruction manual for a piece of equipment, it presents its material in the order

1. Not only organized, but also written, accordingly. If you have any doubts about how much is omitted from a typical textbook, try to teach yourself, without any help, the contents of just the first chapter or two of a typical textbook in a subject for which you have the necessary background but about which you know little. The typical textbook is developed from lecture notes and ensures the continued need for a professor.

of how the subject *is constructed*, i.e., of how the subject is "made". It "develops" the subject for an audience of students who learn at the pace and direction of a teacher. It need not be a particularly good teaching device in itself, since there is always a teacher around to explain what the student doesn't understand. Yet the question, "How is this subject made?" — i.e., "What is the logical construction of this subject?" — is only one of many questions you can ask about a subject.

The assumption, in a typical textbook, is that if you want to tell the time, you need to know how the watch is made. This is a natural point of view from which to write the book if you, as a professor of mathematics or any other technical subject, know the subject already, just as it is a natural point of view for the engineers who design electronic equipment. *The typical textbook is the record of someone's teaching of a subject, never of someone's learning or using a subject.*

But the point of view represented by the typical textbook is *not* necessarily the best point of view for you as a student who wants to use the book to solve problems! If you think about how you *use* a textbook, you will notice that your usage does not in general correspond to the organization of information in the textbook. During and after the course, you do not always read in one direction only, i.e., from front to back; instead, you *refer* to the book: you look up things (or try to), you refresh your memory because you have forgotten details or even major points, you search for certain kinds of information, you use the book for brief periods of time, go on to other things, then come back to it. You by no means have the time to derive the answer to every question you are confronted with in the course of your problem solving; you often try to look up the answer (it may be a formula or theorem or lemma or answer to a previous exercise), using the index and table of contents.

It is simply not true that you have to know how the subject is made in order to solve problems in it. If you have ever used a pocket calculator, or a computer, or any home appliance, or if you drive a car, you know that it is not always necessary to know how something is built in order to use it. And similarly, it is often possible to solve problems in the middle of the book without having read, still less solved all the problems in, the first half of the book.

The important question from the point of view of problem solving is not, "How is this subject constructed?" but instead "What kind of a book (or computer program, or expert human assistant) do I wish I had in order to solve problems in this subject?"

Let us for the moment narrow this question to: *"What kind of a book (or computer program, or expert human assistant) do I wish I had in order to prove statements in this subject?"* If you think about this for a while, chances are that you will come up with an answer along the lines of, "I wish I could look up lemmas and theorems by the words they contain. For example, in a modern algebra course, I wish I could look up, say, all the lemmas and theorems having the word "commutative" in the **then** part of the lemma or theorem statement. I wish I could say (to the book or computer program or human assistant I was using) things like, "Give me all the lemmas and theorems containing any one of the following terms (or their synonyms) in the **if** part of the lemma or theorem statement:..." Or, "Give me all the lemmas and theorems containing the following terms (or their synonyms) anywhere in the lemma or theorem statement"..."

If you want to assess your skill at doing proofs, *this* is the point from which you should begin: from the point at which you can summon statements of lemmas and theorems as described in the previous paragraph. *Not* from the point of having to memorize an entire course's worth of lemmas and theorems so that you can look up the ones you want in your memory, possibly helped by shuffling through pages of text and exercises.

In the typical textbook, your job of doing proofs is made more difficult by the fact that sometimes the crucial lemma or theorem you need is stated in an exercise in a previous chapter. Furthermore, authors differ in their opinions as to which lemmas and theorems should be treated in the text proper, and which should be relegated to the exercises. All of which may well incline you to believe that there are mysterious reasons why a lemma or theorem is treated as an exercise in one book, and as an item in the text of another. Not so! A true statement is a true statement, regardless whether it is called a lemma, or a theorem, or an exercise.

In passing, let me mention that computer databases capable of allowing the kind of retrieval of lemma or theorem statements that was described above, were already in existence in the 1960s. So please don't imagine that such a retrieval capability requires "artificial intelligence"! What you might ask yourself, however, is why the mathematics community still hasn't created such databases and made them available to students. Is it because the professors believe that the students should be forced to learn the material before using it? But such databases in no way prevent students from learning and memorizing!

In this book, you will learn a technique for reducing the amount of time you need to spend memorizing and searching for lemma and theorem statements. It is not as good as a computer database, but it is much better than a mere textbook provides.

I am not, of course, saying, that the traditional textbook should be done away with, since there are times when it is valuable to have a logical presentation of the subject, just as there are times when it is valuable to know how a piece of equipment is constructed. What I *am* saying is that the traditional textbook is not the best tool for helping you to solve typical classroom and homework and exam problems.

The Principal Classes of Homework and Exam Problems

The question of how a textbook should be organized for problem-solving in turn forces us to realize that there are different classes of sub-problems which a student typically has to solve in the course of solving classroom and textbook problems. These classes include:

• "What does the term (or symbol) *x* mean?" If *x* represents a difficult concept, "What are some guides to improving my intuitive understanding of *x*?"

• "How should I go about solving a problem of type *w*? What are some good ways of approaching this type of problem?"

A sub-class of this class is, "What is the formula for *y*?"

• "What does the theorem or lemma called *z* assert?" E.g., in the calculus, what does Rolle's Theorem assert?

• "What theorems or lemmas are closely related to theorem or lemma *z*?"

• "How can I break the proof of theorem *x* down into pieces I can understand?"

• "What theorem or lemma or axiom enables the author to go from this assertion to that one in this proof?"

• "What are the basic operations we perform on the entity *u*?" If *u* is a number, then these basic operations may include addition, subtraction, multiplication, division, finding the "size" of the number (absolute value in the case of the real numbers). If *u* is a set of objects with certain properties, e.g., a topological space, then the set of basic operations may include: determining if the set is "equivalent" to another set (e.g., in topology, homeomorphic); determining the "building blocks" of the set; making new sets of the same kind out of the given set; defining certain important kinds of mappings (functions) between the sets; determining if a given mapping between the sets has certain properties.

And last, but certainly not least,

• "What is the Big Picture of this subject? What does the forest look like, apart from the trees?"

At this point, I hope I have started to convince you that the difficulty of finding information *fast* is a major source of the difficulty you often face in solving mathematics problems. You may feel that, well, yes, it has been *a* source of the difficulty you have faced, but you still can't believe that it has all that much to do with the struggles you have endured in this subject. What about the difficulty of understanding mathematical concepts themselves?

Why Are Mathematical Concepts Difficult to Understand?

In this book, the term "mathematical concept" means just about anything with a mathematical name. (In the chapter, "How to Build an Environment", the term "mathematical entity" is sometimes used instead of "mathematical concept", for reasons that will become clear.) For example, some of the mathematical concepts we learn in high school are: constant, variable, polynomial, factor, factoring, equation, solving an equation, logarithm, sine, cosine, tangent, etc., point, line, triangle, square, and other geometric figures, area, perimeter of a geometric figure, etc., and many others.

Among the mathematical concepts we learn in our first years of college mathematics are: set, operation, limit, function, and, specifically, continuous function, derivative, integral, theorem, proof, countable infinity, uncountable infinity, algebra, linear algebra, vector space, group, ring, field, and many others.

Now *one* thing that makes the understanding of these concepts difficult is that they are defined in terms of other concepts. Thus, e.g., a vector space is defined in terms of the concepts of vector, set, function, abelian group, field, and others. How does the typical mathematics textbook, and mathematics course, deal with this fact? It attempts to teach the concepts in *logical order*, i.e., it assumes that, e.g., when you begin your study of vector spaces, you will already know — through having remembered what you learned in previous courses — the meaning of each of the concepts in terms of which a vector space is defined. And, indeed, one of the things that makes mathematics such a frightening subject to many students, is the grandiose manner with which these assumptions are set forth in the list of prerequisites for the course: "Student is assumed to know...and to have a strong background in...as well as having completed the following courses or their equivalents..." And you, reading that at the end of summer vacation, immediately feel the self-doubt, perhaps self-contempt, which schools thrive on, as you realize that you're not sure if you "know" these things after all.

Once again, ease of looking things up *fast* seems to be beckoning to us.

There are two more reasons why mathematics is difficult. One is the lack of adequate illustrations in textbooks — especially in subjects that are not "geometrical". We will have more to say about this in the section, "On the Shameful Lack of Adequate Illustrations in Most Textbooks", in the chapter "Mathematics in the Schools".

The second reason is that mathematics courses say nothing about how to do mathematics — how to learn concepts, how to approach problems. As you will see, the method set forth in this book contains *within it* a way to do mathematics.

Keep reading.