Is It Legitimate to Begin a Sentence With

"If Counterexamples Exit, Then..."?

by

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Motivation

Earlier versions of our proposed proof(s) of the $3x + 1$ Conjecture, in the paper, "A Solution to the $3x + 1$ Problem", on occampress.com, received several repeated criticisms. The purpose of this paper is to respond to those criticisms.

Summary of Repeated Criticisms

Most of the criticisms boil down to readers' attempts to dismiss our proof(s) by inappropriate applications of the truth table for implication. By "inappropriate" we mean, "contrary to normal practice." For example, the statement, or its equivalent, "If counterexamples exist, then there is a minimum counterexample", has no doubt been uttered many times, and accepted as obviously true, by $3x + 1$ researchers.

If someone were to say instead, in response to the statement, "But if in fact, counterexamples do not exist, then the antecedent of the statement is false. But since 'false implies anything' , the statement is ambiguous, because if the antecedent is false, the statement is true, but if the antecedent is true and the consequent is false, then the statement is false," we — and, we are sure, the vast majority of 3x + 1 researchers, and mathematicians in general — would reply that such a response is irrelevant — inappropriate, perhaps even absurd — because the purpose of the statement is to state a fact, and, indeed a fact from which one might try to prove the $3x + 1$ Conjecture.

In normal practice, an implication p implies q, is used to convey information, which means that p and q are both assumed true. Consider, for example, Fermat's Little Theorem, which states that if p is a prime positive integer, and a is a positive integer, then $a^{p-1} \equiv 1 \bmod p$. In normal practice, this Theorem is not meant to be an opportunity to apply the implication truth-table to show that the stattement is ambiguous. It is meant to convey the information that, if p is a prime positive integer, then if you raise another positive integer a to the $p - 1$ power, you will find that the result is congruent to 1 mod p.

The specific criticisms that have been made are as follows:

• The word "counterexample" is not legitimate in a lemma or in a proof connected with the $3x + 1$ Problem

• It is illegitimate to have "If counterexamples exist..." and "If counterexamples do not exist..." in a proof;

• The phrase "Whether or not counterexamples exist..." is meaningless or at best ambiguous; therefore any proof that contains this phrase is fallacious.

• A sentence beginning, "If counterexamples exist..." is ambiguous;

• The phrase "If counterexamples exist" can only be used in a proof by contradiction

• The sentence, "The set X is the same regardless whether counterexamples exist or not" is meaningless.

• No comparison of the two cases, counterexamples exist and counterexamples do not exist, is legitimate because we live in only one universe and in that universe only one of the cases holds. Our only problem is that we do not know which universe we are living in, the one in which counterexamples exist or the one in which counterexamples do not exist.

We will respond to each of these criticisms below. We invite reader comments — especially comments that are supported by authoritative sources.

Criticism: The Word "Counterexample" Is Not Legitimate in a Lemma Or In a Proof Connected With The 3*x* + 1 Problem

Only one reader has made this objection. The reader asserted that one can only show directly, without proof by contradiction, that each odd, positive integers maps to 1, or does not map to 1.

Reply: We believe that most researchers working on the $3x + 1$ Problem would regard this objection as extreme.

Criticism: It Is Illegitimate To Have "If Counterexamples Exist..." And "If Counterexamples Do Not Exist..." in a Proof

This in essence a claim that it is illegitimate to *compare* the two cases, counterexamples exist, and counterexamples do not exist, because such a comparison implies that counterexamples exist and counterexamples do not exist, which, of course, is absurd.

Reply 1

Those readers who objected to our making the comparison in essence argued that the following equivalence was true. Here, p denotes *counterexamples do not exist*.

((*if p then q*) and (*if not-p then r*)) *is-equivalent-to* (*p and not-p*).

If we make the legitimate assumption that q is true (else why would we bother?) then, as the reader can check via the truth table for the claimed equivalence, the left-hand-side is true, but the right-hand side is false. So the equivalence does not hold, and the readers were wrong.

Reply 2

The two phrases are routinely used, without objection, in discourse concerning the $3x + 1$ Problem — for example, in statements along the lines: "If counterexamples exist, then the range of the $3x + 1$ function is a proper subset of the odd, positive integers, whereas if counterexamples do not exist, the range is the entire set of odd, positve integers."

Reply 3

Our Lemma 8.7 in the paper, "Are We Near a Solution to the 3x + 1 Problem?" on occampress.com uses the two phrases in its statement, and thus far we know of no objections from readers.

Reply 4

Several lemmas in the same paper begin with the phrase "If counterexamples exist, then ..." and have an implicit complement, "If counterexamples do not exist, then ..." An example of such a lemma is Lemma 5.0 So far we know of no objections from readers.

Criticism: The Phrase "Whether Or Not Counterexamples Exist..." Is Meaningless Or At Best Ambiguous; Therefore Any Proof That Contains This Phrase Is Fallacious

Reply

This phrase is equivalent to the two phrases in the previous Objection. We can only point to examples in which the phrase is neither meaningless nor ambiguous. Among them:

"Whether or not odd perfect numbers exist, 6 and 28 are even perfect numbers."
"Whether or not the Riemann Conjecture is true, Fermat's Last Theorem is true."
"Whether or not counterexamples to the $3x + 1$ Conjecture exist, 13 maps to 1."

These are not meaningless. Nor are they merely trivially true because they are equivalent to *(if p then q) and (if not-p then q)*, which, if we assume that q is true, is a true statement by the associated truth table). They are instead *statements of fact* of the form *whether or not p, q*. They assert that the truth or falsity of p has no effect on the truth of q. The reader can no doubt add numerous other examples involving statements of fact.

Criticism: A Sentence Beginning, "If Counterexamples Exist..." Is Ambiguous

The claim is that the sentence is ambiguous because we do not in fact know at present if counterexamples exist or not.

Reply 1

The claim has been restricted to sentences in our earlier proposed proof of the $3x + 1$ Conjecture, which seems strange, because readers have voiced no objections to a similar sentence in the statement of Lemma 5.0, which is referenced in our earlier proof, and which states:

If a counterexample exists, then for all i ≥ 2, each i-level tuple-set contains an infinity of i-level counterexample tuples and an infinity of i-level non-counterexample tuples.

Nor have there been any objections to our proofs by contradiction, which in effect begin with the sentence, "If counterexamples exist, then ... "

So, at the least, the readers seem to have been inconsistent in their claim.

Reply 2

Readers making the claim have also applied it (in email communications) to implications such as

If counterexamples exist, then there are odd, positive integers that do not map to 1 via the 3x + 1 function.

These readers seem not to be aware of what logicians call “Definitional Implication”. In this type of implication, the antecedent is assumed to be true. The consequent then provides information *about the state of affairs that exists if the antecedent is true*. For example, consider:

If unicorns exist, then there are white horses with a goat's beard and a large, pointed, spiraling horn projecting from their forehead.

This implication is true because the consequent is the definition of “unicorn”. The implication is not rendered ambiguous by the fact that we do not know for certain if unicorns exist. In fact, the implication is true even if we decide that unicorns do not exist. The truth follows from the definition of “unicorn”.

But Definitional Implication is not merely implication in which both the antecedent and consequent are true. For example, “If counterexamples exist, then 2 + 2 = 4” is not an example of Definitional Implication, because the consequent gives no information about the antecedent.

Reply 3

Other readers have applied the claim to implications such as:

If counterexamples exist, then we may not be able to discover this fact using the computer because the length of infinite cycles might be too long to detect under the limitations of computation time or storage space, or because, even if there are no cycles, there is no way of detecting, with a computer, if the iterations of the 3x + 1 function on the argument x will never yield a 1.

Such an implication may be called an Informational Implication. In this type of implication, the antecedent is assumed true, and the implication is true if the consequent provides true information about the subject of the antecedent. But this true information is not necessarily a direct consequence of the definition of the antecedent, as is the case in Definitional Implication.

Here again, it is not valid to call the implication ambiguous on the grounds that we do not know if the antecedent is true. For, consider the equivalent of a statement that was certainly made in the literature prior to Wiles' proof of Fermat's Last Theorem (FLT), namely,

If a counterexample to FLT exists, then it will not be possible to discover that counterexample by computer test, because the numbers involved are too large.

Again: it is not valid to call the implication ambiguous on the grounds that no one knew if the antecedent was true. The journal editors would certainly not have published the statement if they thought it was ambiguous.

Furthermore, the statement remains true even after Wiles' proof, provided we make appropriate indications of the time of its utterance, for example,

If a counterexample to FLT had existed, then it would not have been possible to discover that counterexample by computer test, because the numbers involved were too large.

Both Definitional Implication and Informational Implication (the latter is our term) are examples of Formal Implication, "in which a certain formal connection between antecedent and consequent is an indispensable condition of the meaningfulness and truth of the implication."[1]

Criticism: The Phrase "If Counterexamples Exist" Can Only Be Used In a Proof By Contradiction

Reply

No reader that we know of has objected to Lemma 5.0 in the above-cited paper, which begins with a sentence that is the equivalent of "If counterexamples exist" , namely, "Assume a counterexample exists." No reader that we know of has objected to Lemma 8.7 in the above-cited paper, which uses the phrase explicitly.

In any case, it would seem that these objections are nullified if the lemmas or other statements they appear in, are part of a proof by contradiction, which must begin with, "Assume counterexamples exist."

Criticism: "The Set *X* Is the Same Regardless Whether Counterexamples Exist or Not" Is Meaningless

This criticism was originally applied to the initial wording of Lemma 8.8, which was:

The set J of odd, positive integers maps to 1, regardless whether counterexamples exist or not.

The criticism was retracted by most readers when we changed the wording to

Exactly one set J of odd, positive integers maps to 1, regardless whether counterexamples exist or not,

and then added:

In other words:

If counterexamples exist, then the set of odd, positive integers that map to 1 is J.

If counterexamples do not exist, then the set of odd, positive integers that map to 1 is J.

However the criticism has persisted in other contexts. We do not know why, but we nevertheless believe the criticism is wrong in those contexts. For example, let "Conjecture *Y*" denote any conjecture that is unresolved at the time of reading the following. Then it seems to us perfectly clear that

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1. Tarski, Alfred, *Introduction to Logic and to the Metholdology of Deductive Sciences*, Oxford University Press, N.Y., 1970, p. 26

The set P of positive primes is the same regardless whether counterexamples to Conjecture Y exist or not,

simply means

There is exactly one set P of positive primes, regardless whether counterexamples to Conjecture Y exist or not.

In other words:

If counterexamples to Conjecture Y exist, then the set of positive primes is P = {2, 3, 5, 7, ...}.

If counterexamples to Conjecture Y do not exist, then the set of positive primes is P = {2, 3, 5, 7, ...}.

Criticism: No Comparison of the Two Cases, Counterexamples Exist and Counterexamples Do Not Exist, Is Legitimate Because We Live in Only One Universe

This criticism is sometimes summed up as, "It is what it is."

Reply

This criticism would in fact apply to a function $f(x)$ in which for each odd, positive integer x, a value was selected at random from a very large number of odd, positive integers. In this case, it would be legitimate to say of each value, "it is what it is".

To compare the function f with another one, g, over the same domain, with values selected by the same random process, would be pointless, because even if $f(x) = g(\mathrm{x})$ for several different arguments x, we could not draw any inferences about the values yielded by other x.

The situation with the $3x + 1$ function is entirely different. For example, if just one odd, positive integer is a counterexample to the $3x + 1$ Conjecture, then so are an infinite number of odd, positive integers.

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