# The Structure of the 3*x* + 1 Function:

# An Introduction

by

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# Introduction

## Statement of Problem

For $x$ a positive odd integer, set

$$C(x) = \frac{3x+1}{2^{ord_2(3x+1)}}$$

where $ord_2(3x + 1)$ is the largest exponent of 2 such that the denominator evenly divides the numerator. Thus, e.g., C(17) = 13, C(13) = 5, C(5) = 1. The $3x + 1$ Problem, also known as the $3n + 1$ Problem, the Syracuse Problem, the Collatz Problem, Ulam's Problem, Kakutani's Problem, and Hasse's Algorithm, asks if all repeated iterations of $C$ eventually terminate at 1. The conjecture that they do is hereafter called the $3x + 1$ Conjecture. We call $C$ the $3x + 1$ function; note that $C(x)$ is by definition odd.

Other equivalent formulations of the 3x + 1 problem are given in the literature; we base our formulation on the $C$ function (following Crandall) because, as we shall see, it brings out certain structures that are not otherwise evident.

## Purpose of This Paper

This paper presents two remarkably simple structures underlying the $3x + 1$ function, namely, tuple-sets, which describe the structure of the function "in the forward direction", and recursive "spiral"s, which describe the structure of the inverse function — the structure "in the backward direction". The paper also presents several possible strategies for solving the $3x + 1$ Problem that are based on these structures. These are described under "Possible Strategies for Proving the 3x + 1 Conjecture Using Tuple-sets" on page 18, and "Possible Strategies for Proving the 3x + 1 Conjecture Using "Spiral"s" on page 34.

## An Important Fact to Keep in Mind

This paper sets forth numerous results concerning tuple-sets and recursive "spiral"s. However, in all but a few cases, these results apply equally whether a counterexample to the $3x + 1$ Conjecture exists or not. (See details under "Preliminary Discussion of Strategies" on page 18.) That does not necessarily mean that these results are "useless" as far as a proof of the Conjecture is concerned, but it does mean that the reader should be cautious about assuming that each new result necessarily brings us closer to a proof. Of course, the $3x + 1$ function is interesting in its own right, and in this regard each result that adds to our understanding of this function, has value.

## On the Style of This Paper

It will be obvious at a glance that this is not a formal paper intended for publication in a journal. Early experience showed that most prospective readers are hard-pressed for time, and typically can only afford to spend a few minutes browsing the text. These persons seldom have the time even to search the text for the definition of a symbol. Furthermore, they have a variety of backgrounds: undergraduate and graduate mathematics, computer science, and electrical engineering majors; and academics and industrial professionals in these subjects. We have therefore done his best to enable the reader to acquire at least a superficial understanding of the underlying concepts and logical arguments as rapidly as possible. As a result, several rules governing the correct style of formal mathematical papers have been broken. For example, whereas in a formal paper one would denote the frequently mentioned sequence {1, 5, 21, 85, 341, ... } (the set of numbers mapping to 1 in one iteration of the $3x + 1$ function) by a symbol, say, $S_1$, and the sequence of intervals defined by $S_1$ by another symbol, say, $I_1$, we have instead frequently written out {1, 5, 21, 85, 341, ... } itself, or referred to it by its formal name (in this paper), *the base sequence relative to 1*.

Numbering of lemmas and figures is the same in this paper as in previous versions of the paper — hence not necessarily consecutive, due to the addition or deletion of lemmas and figures in various revisions.

In the interest of conciseness most proofs of lemmas are omitted in the present paper. For a fuller discussion, including proofs, see “The Structure of the $3x + 1$ Function”, available on the web site www.occampress.com.

Some of the results supporting the strategies already exist in the literature. We have tried to indicate these wherever possible. However, as far as we know, the value, as far as suggesting strategies for a solution to the Problem are concerned, of the “graphical” presentations of the two structures described in this paper has not been recognized.

This paper is a work in progress, and thus may contain errors. We will appreciate readers notifying him of any they find.

*The reader is encouraged to use the “Table of Symbols and Terms” on page 97 in order to save time in locating definitions.*

*The reader is also encouraged to contact us at peteschorer@cs.com for explanations of any parts of this paper that the reader finds difficult.*

## In Memoriam

Many of the lemmas in this paper, and in the paper, “The Structure of the $3x + 1$ Function”, which is accessible on the web site, www.occampress.com, were proved by Michael O’Neill. It was with great sadness that we learned that O’Neill died in November, 2003, after a brief illness. He made a major contribution to this research, and he is sorely missed.

## Collaborator Sought

We are seeking a qualified collaborator to help develop the ideas in this paper.

## Section 1: Tuple-Sets

In the first part of this paper, we describe a structure called "tuple-sets" that underlies iterations of the $3x + 1$ function — in other words, that describes the function in the "forward" direction. The structure called "recursive 'spiral's" is presented in the second part of this paper, and describes the inverse of the $3x + 1$ function — in other words, describes the function in the "backward" direction.

The "spatial", "geometric", "graphical" nature of both structures is important for the strategies it suggests.

We begin with some definitions.

### Definitions

**Iteration**

An *iteration* takes an odd, positive integer, $x$, to another odd, positive integer, $y$, via one application of the $3x + 1$ function.

**Trajectory**

A *trajectory* (sometimes called an *orbit*) is a sequence of one or more successive iterations of $C$, i.e., if the sequence is finite,

$$(C^k(x))_{k \geq 0} = (x, C(x), C^2(x), \ldots C^k(x))$$

or, if the sequence is infinite,

$$(C^\infty(x)) = (x, C(x), C^2(x), \ldots)$$

The last element of the finite sequence need not be 1 and it need not be an infinity of successive 1's in the case of an infinite sequence.

(See definition of *tuple*, below.)

**Power of 2**

By a power of 2 we mean a positive integer power of 2.

**Exponent**

If $C(x) = y$, with $y = (3x + 1)/2^a$, we say that *$x$ maps under iteration to $y$* (or *$x$ maps directly to $y$*) *via the exponent $a$*, and that *$a$ is the exponent associated with $x$*. We will sometimes speak of $a$ as *mapping directly to $y$*. The sequence $\{a_2, a_3, ..., a_i\}$, where $a_2, a_3, ..., a_i$ are the exponents associated with $x$, $C(x)$, ..., $C^{(i-2)}(x)$ respectively, is called an ***admissible vector*** in [3]. We define the function $e(x)$ to be the exponent associated with $x$. We will sometimes refer to $y$ as a *range element*. It is easily shown (Lemma 0.2) that $y$ cannot be a multiple-of-3. Any element $x$ of the domain of the $3x + 1$ function, whether multiple-of-3 or not, we will sometimes refer to as a *domain element*.

**Tuple**

A tuple is a trajectory, finite or infinite. A finite tuple is denoted $<x, y, y',..., y'^{\cdot\cdot}{}'>$. An infinite tuple is denoted $<x, y, y', ... >$

**Tuple-sets**

(The reader might find it helpful to refer to Fig. 1 while reading the following.)

Let $A = \{a_2, a_3, ..., a_i\}$ be a finite sequence of positive integers (i.e., exponents), where $i \geq 2$. The *tuple-set* $T_A$ consists of all and only the following *tuple*s:

all tuples $<x>$ such that $x$ does not map to any number via $a_2$;

all tuples $<x, y>$ such that $x$ maps to $y$ via $a_2$ (i.e., $e(x) = a_2$) but $y$ does not map to any number via $a_3$;

all tuples $<x, y, y'>$ such that $x$ maps to $y$ via $a_2$ (i.e., $e(x) = a_2$) and $y$ maps to $y'$ via $a_3$ (i.e., $e(y) = a_3$) , but $y'$ does not map to any number via $a_4$;

...

all tuples $<x, y, y',...,y'^{\cdot\cdot}{}', y'^{\cdots}{}'>$ such that $x$ maps to $y$ via $a_2$ (i.e., $e(x) = a_2$) and $y$ maps to $y'$ via $a_3$ (i.e., $e(y) = a_3$) and ... and the $(i$-1)th element $y'^{\cdot\cdot}{}'$maps to $y'^{\cdots}{}'$ via the exponent $a_i$ (i.e., $e(y'^{\cdot\cdot}{}') = a_i$).

In the case of maximum length tuples $t$ only, we say that *t is defined by* the exponent sequence $A$. Similarly, given any tuple $t$ of $i$ elements, $i \geq 2$, we say that *t produces,* or *defines,* or *generates* the sequence $A$ if $t$ is defined by the exponent sequence $A$. Finally, we say that the tuple-set $T_A$ *is defined by* the sequence $A$.

Thus, in Fig. 1, where $A = \{a_2, a_3, a_4\} = \{1, 1, 2\}$, the tuple-set $T_A$ includes:

the tuple $<1>$, because $e(1) \neq a_2$;
the tuple $<3, 5>$ , because $e(3) = a_2 = 1$, but $e(5) = 4 \neq a_3 = 1$;
the tuple $<15, 23, 35>$, because $e(15) = a_2 = 1$, and $e(23) = a_3 = 1$, but $e(35) = 1 \neq a_4 = 2$.

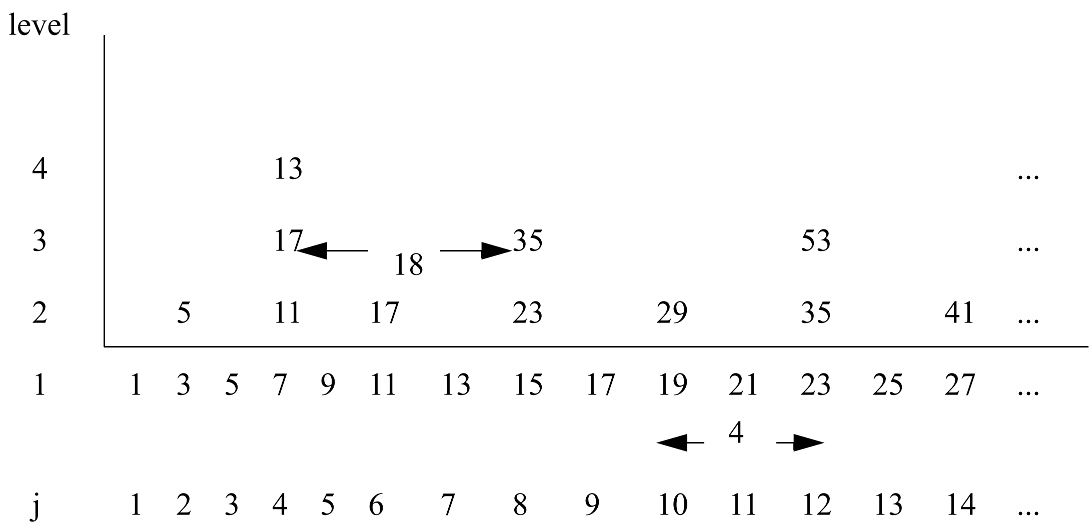


**Fig. 1. Part of the tuple-set $T_A$ associated with the sequence $A = \{1, 1, 2\}$**

Fig. 1 shows part of a tuple-set, namely, the tuple-set $T_A$ associated with the sequence $A = \{1, 1, 2\}$.

The 2nd element of the 8th tuple, $t_{8(2)}$, is 23 because 23 is the range element mapped to by the 1st element, 15, in one iteration ($a_2 = 1$).

The 4th element of the 4th tuple, $t_{4(4)}$, is 13 because 13 is the range element mapped to by the 3rd element, 17, in one iteration ($a_4 = 2$).

There is no 2nd element of the 1st tuple because there is no range element mapped to by 1 such that $a_2 = 1$.

There is no 2nd element of the 5th tuple because there is no range element mapped to by 9 such that $a_2 = 1$.

As stated above, tuples in a tuple-set are ordered according to their first elements. Thus, there is always a unique first (leftmost) tuple in every tuple-set. We adopt the convention of orienting tuples vertically on the page.

**Level in a Tuple-set**

A *level j* in a tuple-set is defined as follows. If $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, is a finite sequence of exponents, the subscript $j$ in $a_j$, $2 \leq j \leq i$, denotes the *level j* in $T_A$. As specified under the definition of tuple-set, we begin numbering our levels with 2 so that level 1 is then the level containing the set of all possible tuple first elements $\{1, 3, 5, 7, ... \}$ in any $T_A$, that is, the set of odd, positive integers.

If a tuple has an element at level $j$, but none at level $j + 1$, we will refer to the tuple as a *j-tuple*, or a *j-level tuple*. If the tuple also has an element at level $j + 1$, we will sometimes refer to the tuple as a ($\geq j$)-tuple. The longest tuple in any tuple-set defined by an exponent sequence of length $i$ - 1 is an $i$-level tuple.

In the case that $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, we will refer to $T_A$ as an *i-level tuple-set* and we will refer to $A$ as an *i-level exponent sequence*. An $i$-level exponent sequence consists of ($i$ - 1) exponents. Clearly, every range element mapped to by a given $i$-level exponent sequence occurs in level $i$ of the corresponding tuple-set.

**Tuples Consecutive at Level j**

Tuples *consecutive at level j*, $j \geq 2$, are defined as follows. Let $t_k$, $t_m$ be ($\geq j$)-tuples in some $T_A$. If there is no ($\geq j$)-tuple between $t_k$ and $t_m$, we say that $t_k$ and $t_m$ are *tuples consecutive at level j*. Here, "between" means relative to the natural linear ordering of tuples based on their first elements.

Thus, for example, in Fig. 1, tuples 4 and 8 are consecutive at level 3.

**Ordering of Tuples in a Tuple-set**

See under "Remarks About the Distance Functions" on page 12.

**Row**

Let $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, be a sequence of exponents, and let $T_A$ be the corresponding tuple-set. Then a *level-j row*, $R_j$, where $1 \leq j \leq i$, in $T_A$ is the set of all $j$th tuple-elements in tuples consecutive at level $j$. We shall see that, as a result of the distance functions defined in lemmas 1.0 and 1.1, that each row is a congruence class — specifically, a reduced residue class mod $2 \bullet 3^{i-1}$, $i \geq 2$. We shall also see that the $3x + 1$ function can — and perhaps should! — be defined as a function *on these congruence classes*, rather than merely on odd, positive integers. This definition holds even if we include negative elements of each congruence class.

Among the questions that it is natural to ask regarding the top ($i$-level) row $R_i$ in an $i$-level tuple-set are:

To which ($i$+1)-level rows in the set of ($i$+1)-level tuple-sets does $R_i$ map under the set of all exponents 1, 2, 3, 4, ... ?

How is the set of all exponents partitioned in this mapping? (A total of $2 \bullet 3^{i-1}$ of rows $R_i$ over all $i$-level tuple-sets maps to a total of $2 \bullet 3^{(i+1)-1}$rows $R_{i+1}$ over all $i$+1 level tuple sets; the infinite number of exponents is partitioned into a finite set of classes.)

If $y$ is an element in a row $R_i$ that maps to an element of a row $R_{i+1}$ under the exponent $a_{i+1}$, what is the next larger element in row $R_i$ that maps to the row $R_{i+1}$?

Answers to these and other questions are given in the following lemmas, which are stated in "Appendix A — Statements of Lemmas": Lemmas 7.25, 7.27, 7.3, 7.31, 7.32, 7.35, 7.36, 7,38, 7.4.

**Extensions of Tuples and of Tuple-sets**

Let $T_A$ be a tuple-set defined by the sequence of exponents $A = \{a_2, a_3, ..., a_i\}$. Then any tuple-set $T_{A'}$ defined by a sequence of exponents $A' = \{a_2, a_3, ..., a_i, a_{i+1}\}$ is called an *extension* of $T_A$. We define extensions of tuples in a similar manner. Thus, a ($\geq i$)-tuple in $T_{A'}$ is an extension of an $i$-tuple in $T_A$.

If $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, is a sequence of exponents, then we define an *initial sub-sequence of the exponent sequence A* as the sequence $\{a_2, a_3, ..., a_j\}$, where $2 \leq j \leq i$. Thus, for example, $\{a_2\}$ is an initial sub-sequence of $A$, and so is $\{a_2, a_3, a_4\}$, but, for example, $\{a_3, a_4\}$ is not. We define an *inititial sub-sequence of a tuple $t_k$* similarly.

With the concept of extensions of tuples and tuple-sets established, we can see that every $j$-tuple, $2 \leq j \leq i$, defined by an initial sub-sequence $\{a_2, a_3, ..., a_j\}$ of $A$ is in the tuple-set $T_A$.

**Non-terminating Tuple (n-t-v-1, n-t-v-c)**

As stated under “Trajectory” on page 4, a trajectory (tuple) may be finite or infinite. We will use the term *non-counterexample tuple* to denote a finite tuple whose elements map to 1, and the term *counterexample tuple* to denote a finite tuple whose elements are counterexamples. We will sometimes use the term *n-t-v-1* (non-terminating-tuple-via-1) to denote an infinite tuple whose elements map to 1, and the term *n-t-v-c* (non-terminating-tuple-via-c (*c* for *counterexample*)) to denote an infinite tuple whose elements are counterexamples.

It is possible that a tuple contains a repetition of one of its elements. (The tuple <1, 1, 1, ...> is a trivial example, and the only known example at time of writing.) Clearly, any such tuple is infinite. If the repeated element is not 1, then the tuple contains solely counterexample elements. Results concerning cycles are given in “Appendix A — Statements of Lemmas” on page 49.

## Graphical View of a Tuple-set

At this point, it will be helpful if we get an abstract view of the various-length tuples in a tuple-set. Let $T_A$ be any tuple-set, with $A = \{a_2, a_3, ..., a_i\}$. Then, as shown in Fig. 3.05, there is an infinity of tuples consecutive at level *i* and, indeed, at all levels $1 \leq j \leq i$. Between each pair of *i*-level tuples there is a finite set of tuples consecutive at level *i* - 1. Between each pair of these is a finite set of tuples consecutive at level *i* - 2, etc., down to level 1. The distance (numerical difference) between elements of tuples at each level will be specified in Lemmas 1.0 and 1.1.

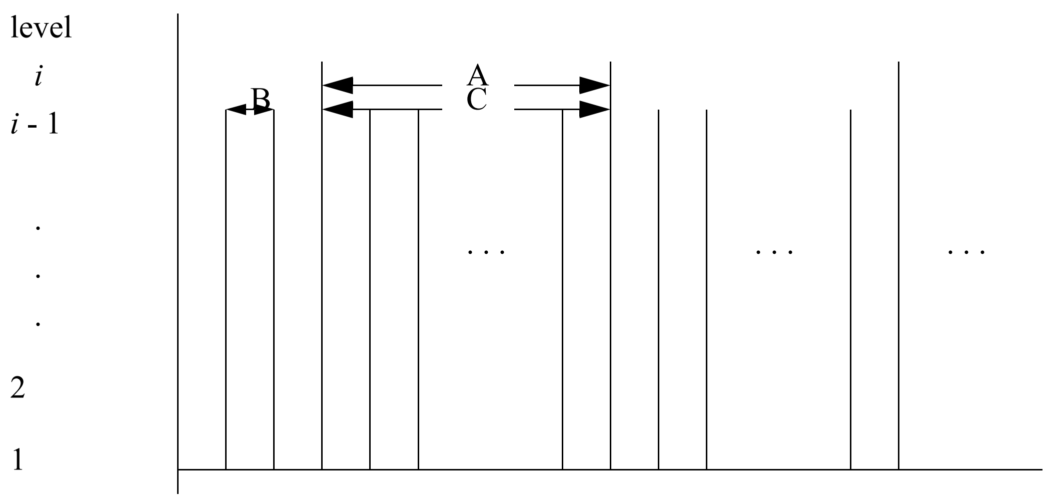


**Fig. 3.05. Graphical view of tuples in a tuple-set.**

A, the distance (numerical difference) at level *i* between elements of tuples consecutive at level *i*, which is $2 \bullet 3^{i-1}$. (Lemma 1.0 (a))

B, the distance (numerical difference) at level *i* - 1 between elements of tuples consecutive at level *i* - 1, which is $2 \bullet 3^{i-2}$. (Lemma 3.0 (a))

C, the distance (numerical difference) at level *i* - 1 between elements of tuples consecutive at level *i*, which is

$$lcm(2 \cdot 2^{a_i}, 2 \cdot 3^{i-2}) = 2 \cdot 2^{a_i} \cdot 3^{i-2}$$

where *lcm* is the least common multiple. (Lemma 1.1)

The reader may find the following intuitive description of a tuple-set to be helpful.

• Every $i$-level tuple-set $T_A$ , $A = \{a_2, a_3, a_4, ..., a_i\}$, can be viewed as a “picket fence”, infinitely long to the right. (Pickets correspond to tuples.) There is:

an infinity of tuples of length 1;
an infinity of tuples of length 2;
an infinity of tuples of length 3;
...
an infinity of tuples of length $i$;

• Furthermore, if counterexamples exist, there is (by Lemma 10.0):

an infinity of counterexample tuples and an infinity of non-counterexample tuples in the infinity of tuples of length 1;
an infinity of counterexample tuples and an infinity of non-counterexample tuples in the infinity of tuples of length 2;
an infinity of counterexample tuples and an infinity of non-counterexample tuples in the infinity of tuples of length 3;
...

an infinity of counterexample tuples and an infinity of non-counterexample tuples in the infinity of tuples of length $i$;

• Finally, there is, for each exponent $a_{i+1}$ (i.e., for each non-negative integer):

an infinity of counterexample tuples in $T_A$ that are extended by $a_{i+1}$ and
an infinity of non-counterexample tuples in $T_A$ that are extended by $a_{i+1}$. (Lemma 2.0)

## Graphical Views of the Set of All Tuple-sets

The reader may also find it helpful to have a graphical view of the set of *all* tuple-sets, particularly when the reader reviews the lemmas below concerning rows and extensions of tuple-sets.

Probably the best graphical view is that of an infinitary tree, as shown in Fig. 3.07, because the set of exponents by which the $i$-level row in any $i$-level tuple-set can be mapped to some $(i + 1)$-level row of some $(i + 1)$-level tuple-set is precisely the set of all possible exponents, namely, $\{1, 2, 3, ...\}$(Lemma 7.25). In the figure, exponents are given next to (some) branches. Each node represents an infinity of tuple-set elements, namely, all $j$-level elements, $j \geq 1$, this infinite set conceived of as running perpendicularly *into* the page.

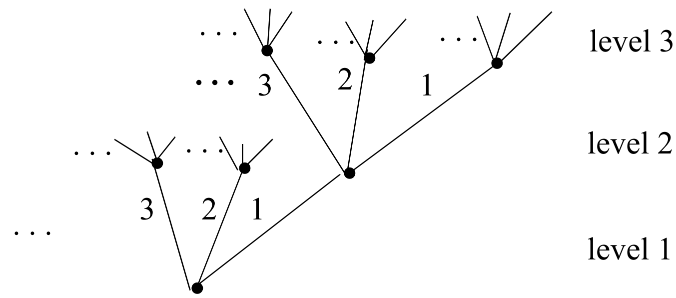


**Fig. 3.07. Graphical view of the set of all tuple-sets**

We can further simplify the graphical representation of the set of all tuple-sets by recognizing that the top row of each $i$-level tuple-set is generated by the top rows of $(i - 1)$-level tuple-sets, $i \geq 3$, and that there are only a finite number of top rows of all $i$-level tuple-sets, $i \geq 2$ (each row is a reduced residue class mod $(2 \cdot 3^{i-1})$). For example, Fig. 3.08 shows the generating relationship between the top rows of all 2-level tuple-sets, and the top rows of all 3-level tuple-sets. Each arrow represents the generating function via all exponents. The arrow points to the row generated. Note that, even though each row is identified by its first element, the contents of rows with the same first element at different levels are not identical, because of the distance function $d(i, i)$ (Lemma 1.0 (a)).

By Lemma 7.25, the same generating relationship between successive top levels holds for all higher levels, so that the infinitary tree of all tuple-sets can, without loss of generality, be reduced to a finitary tree, namely, a $(2 \cdot 3^{i-2})$ -ary tree, $i \geq 2$. ($(2 \cdot 3^{i-2})$ is the number of reduced residue classes mod $(2 \cdot 3^{i-1})$.)

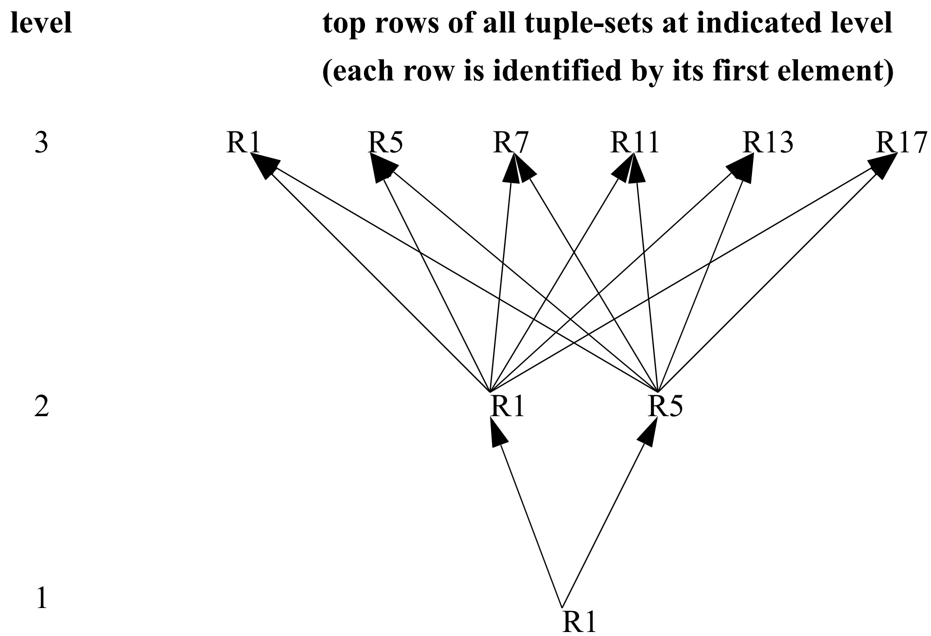


**Fig. 3.08. Generating relationship between top levels of all 2-level tuple-sets and top levels of all 3-level tuple-sets**

## Distance Functions on Tuple-sets

**Lemma 1.0** (a) Let $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, *be a sequence of exponents, and let* $t_k$, $t_m$ *be tuples consecutive at level i in* $T_A$. *Then d(i, i), the distance between* $t_k$ *and* $t_m$ *at level i, is defined to be the absolute value of the difference between the level i elements of* $t_k$ *and* $t_m$, *i.e., is defined to be* $|t_{k(i)} - t_{m(i)}|$, *and is given by*:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

*(b) Let* $t_k$, $t_m$ *be tuples consecutive at level i in* $T_A$. *Then d(1, i), the distance between* $t_k$ *and* $t_m$ *at level* 1, *is defined to be the absolute value of the difference between the level 1 elements of* $t_k$ *and* $t_m$, *i.e., is defined to be* $|t_{k(1)} - t_{m(1)}|$, *and is given by:*

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3})\ldots(2^{a_i})$$

Thus, in Fig. 1, the distance $d(3, 3)$ between $t_{8(3)} = 35$ and $t_{4(3)} = 17$ is $2 \cdot 3^{(3-1)} = 18$. The distance $d(1, 2)$ between $t_{12(1)} = 23$ and $t_{10(1)} = 19$ is $2 \cdot 2^1 = 4$.

### Remarks About the Distance Functions

(1) Strictly speaking, we should include the sequence $A$ of exponents as arguments of $d(1, i)$, $d(i, i)$, but this notation would be cumbersome and, since typically this sequence is known, unnecessary.

(2) The distance functions make clear that, for each finite sequence of exponents, there exists an infinity of tuples produced by that sequence. (The equivalent of this statement is made in [3] (p. 48).) In particular, there exists an infinity of tuples consecutive at level $i$ for all $i \geq 2$.

(3) In each i-level tuple-set, $i \geq 2$, there exists:

an infinite sequence of 1-level tuples, and
an infinite sequence of 2-level tuples, and
an infinite sequence of 3-level tuples, and
..., and
an infinite sequence of $i$-level tuples.

The relation between the tuples at any one level relative to those of another level, is outlined in Fig. 3.05, under "Graphical View of a Tuple-set" on page 8, and can be deduced from the table "Distances between elements of tuples tk, tm, consecutive at level i" on page 13.

(4) Lemma 1.0 (a) makes clear that no two $i$-level tuples in a given $i$-level tuple-set have the same last element. In fact, the values of the last elements of $i$-level tuples in an $i$-level tuple-set always increase as one proceeds along the sequence of $i$-level tuples.

(5) The formula for $d(1, i)$ implies that it is possible for pairs of tuples consecutive at level $i$ in one tuple-set to be the same distance apart, at level 1, as pairs of tuples consecutive at level 1 in another tuple-set. For example, this would occur between tuples consecutive at level 2 in $T_A$ when $A = \{2\}$ ($d(1, 2) = 2 \cdot 2^2 = 8$) and between tuples consecutive at level 3 in $T_{A'}$ when $A' = \{1, 1\}$($d(1, 3) = 2 \cdot 2^1 2^1 = 8$).

(6) The distance, at level $j$, $2 \leq j < i$, between elements of tuples consecutive at level $i$, is given in Lemma 1.1.

(7) It is straightforward to prove that the distance functions carry over into the odd, negative integers as well. (The proof is contained in the proof of Lemma 2.0.)

**Lemma 1.1**. *Let $T_A$ be a tuple-set defined by a sequence $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$. Then the distance $d(j, i)$ between elements at level $j$, $1 \leq j \leq i$, of tuples $t_k$, $t_m$ consecutive at level $i$ is given by the following table:*

**Table 1: Distances between elements of tuples $t_k$, $t_m$, consecutive at level i**

| Level | Distances between elements of $t_k$, $t_m$ at level |
|---|---|
| $i$ | $2 \cdot 3^{i-1}$ |
| $i-1$ | $2 \cdot 3^{i-2} \cdot 2^{a_i}$ |
| $i-2$ | $2 \cdot 3^{i-3} \cdot 2^{a_{i-1}}2^{a_i}$ |
| $i-3$ | $2 \cdot 3^{i-4} \cdot 2^{a_{i-2}}2^{a_{i-1}}2^{a_i}$ |
| ... | ... |
| 2 | $2 \cdot 3 \cdot 2^{a_3} \ldots 2^{a_{i-1}}2^{a_i}$ |
| 1 | $2 \cdot 2^{a_2}2^{a_3} \ldots 2^{a_{i-1}}\ 2^{a_i}$ |

## Summary of Properties of Tuple-sets

We now provide a table that summarizes our results on tuple-sets and rows in a tuple-set. (Recall that a row is simply the set of elements at a given level in a given tuple-set.) We break the properties of rows into three parts: those concerning top rows, those concerning middle rows, and those concerning the bottom (i.e., first) rows. The phrase "extension of a top row $R_i$" means the same thing as "the top row $R_{i+1}$ mapped to by a top row $R_i$".

The table entry for each property whose value is known includes a reference to definitions or lemma(s) that establish the value.

*Note*: some table-rows may have the same content as other rows, though under different properties. This redundancy is deliberate, the purpose being to aid understanding and to make the looking up of properties easier.

Statements of all referenced lemmas are given in the Appendix.

**Table 2: Some important properties of tuple-sets**

| Property | Value of property | Reference |
|---|---|---|
| Sequence of exponents, $A$, that define a tuple-set $T_A$ | $A = \{a_2, a_3, ..., a_i\}$, $a_i \geq 1$. | Definition of tuple-set |

**Table 2: Some important properties of tuple-sets**

| Property | Value of property | Reference |
|---|---|---|
| Structure of tuple-sets (not of tuples within tuple-sets) | Infinitary tree, equivalent to a $2 \cdot 3^{i-2}$-ary tree. Thus, in the latter, finitary, tree:<br><br>**level 2** has $2 \cdot 3^{2-2} = 2$ nodes (the 2 top rows of all 2-level tuple-sets), mapped to by 2 equivalence classes of exponents;<br><br>**level 3** has $2 \cdot 3^{3-2} = 6$ nodes (the 6 top rows of all 3-level tuple-sets), mapped to by 6 equivalence classes of exponents;<br><br>**level 4** has $2 \cdot 3^{4-2} = 18$ nodes (the 18 top rows of all 4-level tuple-sets), mapped to by 18 equivalence classes of exponents;<br><br>etc. | Lemma 7.3 |
| $2 \cdot 3^{i-1}$ | Distance between elements of tuples successive at level $i$ in an $i$-level tuple-set | Lemma 1.0 |
| $2 \cdot 3^{i-2}$ | Number of top rows of all $i$-level tuple-sets; also<br><br>Number of exponent equivalence classes (and the maximum exponent), from which exponents mapping to the top row of any $i$-level tuple-set, from the top rows of all $i$- 1 level tuple-sets, must be selected. | Lemmas 3.055, 3.057<br><br>Lemma 7.3 |

**Table 3: Some important properties of the top (i.e., level *i*) row of an *i*-level tuple-set**

| Property | Value of property | Reference |
|---|---|---|
| Distance $d(i, i)$ between successive elements of a top row, i.e., between $i$-level elements of tuples consecutive at level $i$ | $d(i, i) = 2 \bullet 3^{i - 1}$ | Lemma 1.0 (a) |
| Total number of different top rows over the set of all $i$-level tuple-sets | $\phi(2 \bullet 3^{i - 1}) = 2 \bullet 3^{i - 2}$ = the number of reduced residue classes mod $2 \bullet 3^{i - 1}$ | Lemmas 3.055, 3.057 |
| Distance between successive exponents in an exponent equivalence class mapping from an $(i$-1)-level top row to an $i$-level top row. All members of a class map to the same level-$i$ top row from the same $(i - 1)$ - level top row. | $2 \bullet 3^{i - 2}$ | Lemma 7.3 |
| Total number of exponent equivalence classes mapping a level- $(i - 1)$ top row to all level-$i$ top rows | $2 \bullet 3^{i - 2}$ | Lemma 7.3 |
| Smallest exponent mapping to any given top row of an $i$ -level tuple-set from any top row of an $(i - 1)$-level tuple-set | $\leq 4$ | Lemma 7.35 |
| Upper bound on exponents mapping from any given top row of an $(i - 1)$-level tuple-set to the top row of any $i$-level tuple-set | $2 \bullet 3^{i - 2}$ (All larger exponents are elements of equivalence classes having smaller minimum elements) | Lemma 7.3 |

**Table 3: Some important properties of the top (i.e., level *i*) row of an *i*-level tuple-set**

| Property | Value of property | Reference |
|---|---|---|
| Beginning of sequence of exponents mapping to any given $i$-level top row from *any* ($i$- 1)-level top rows | *For an i-level top row mapped to by odd exponents*: 1,3, *, or 1, *, 5, or *, 3, 5. *For an i-level top row mapped to by even exponents*: 2, 4, *, or 2, *, 4, or *, 4, 6, where * denotes a "missing" exponent due to absence of a multiple-of-3 in the $i$-level top row. The * recurs after every two non-* exponents. | Lemma 15.0 |
| Sequence of exponents mapping from any given ($i$ - 1)-level top row to *all* $i$ -level top rows | 1, 2, 3, ..., $2 \bullet 3^{i-2}$, with each exponent mapping to a unique $i$-level top row. A larger exponent $a'_i$ then maps to the same row as one of the above exponents $a_i$ does if $a'_i \equiv a_i \mod 2 \bullet 3^{i-2}$. | Lemma 7.3 |
| Minimum element in a top row | Minimum residue in a reduced residue class mod $2 \bullet 3^{i-1}$ | Lemmas 3.055, 3.057 |
| Formula for the minimum element of the top row of an $i$-level tuple-set, given only the sequence of exponents defining the tuple-set | See Lemma 7.38 | Lemma 7.38 |
| Formula for the minimum element of the top row of an ($i$+ 1)-level tuple-set mapped to by the top row of an $i$-level tuple-set via an exponent $a_{i+1}$ | See Lemma 7.36 | Lemma 7.36 |
| Distance between successive elements of (sub-row of) top row of an $i$-level tuple-set that generates a top row of an ($i$ + 1)-level tuple-set via the exponent $a_{i+1}$ | $lcm(2 \cdot 3^{i-1}, 2 \cdot 2^{a_{i+1}})$, where *lcm* denotes least common multiple | Lemma 1.1 |

**Table 3: Some important properties of the top (i.e., level *i*) row of an *i*-level tuple-set**

| Property | Value of property | Reference |
|---|---|---|
| Successive elements of (sub-row of) top row of i-level tuple-set map to successive elements elements of top row of (i + 1)-level tuple-set? | Yes. | Lemma 7.40 |
| Set of elements in all top rows of all *i*-level tuple-sets | Set of range elements, i.e., set of odd, positive integers not multiples of 3 | Lemma 3.28 |
| Relationship between top rows of all *i*-level tuple-sets and top rows of all ($i$ + 1)-level tuple-sets | (1) *Each* top row in an *i*-level tuple-set generates, via all exponents $a_{i+1}$, the top rows of *all* ($i$ + 1)-level tuple-sets.<br><br>(2) For *each* ($i$ + 1)-level top row, if it is desired to generate the row via all possible exponents, then *all* *i*-level top rows are required . | (1) Lemma 7.25<br>(2) Lemma 7.27. |

**Table 4: Some important properties of the middle (i.e., levels 1 < *j* < *i*) row of an i-level tuple-set**

| | | |
|---|---|---|
| Distance, $d(j, i)$ between elements at level $j$ of successive tuples consecutive at level $i$ | $d(j, i) =$<br><br>$lcm(2 \cdot 3^{j-1}, 2 \cdot 2^{a_{j+1}} \cdot 2^{a_{j+2}} \ldots \cdot 2^{a_i})$<br><br>where *lcm* is the least common multiple. | Lemma 1.1 |
| For each $i$ and each $j$, minimum elements of level $j$ rows over all $i$-level tuple-sets | General formula not yet known; must be determined empirically for each given tuple-set | |
| For each $j$, set of elements in all $j$-level rows of all $i$-level tuple-sets | Set of range elements, i.e., set of odd, positive integers not multiples of 3 | Lemma 3.28 |

**Table 5: Some important properties of the bottom (i.e., level 1) row of an *i*-level tuple-set**

| Property | Value of property | Reference |
|---|---|---|
| Distance, $d(1, i)$, between successive tuple elements at level 1 of tuples consecutive at level $i$ | $d(1, i) =$ $2 \cdot 2^{a_2} \cdot 2^{a_3} \cdot \ldots \cdot 2^{a_i}$ | Lemma 1.0 (b) |
| Set of elements in bottom row of all $i$-level tuple-sets | Set of odd, positive integers | Lemma 3.28 |

## Tuple-sets and Finite Stopping Times

In the literature on the $3x + 1$ Problem, the term *stopping time* is defined as the smallest $k$ such that $C^{(k)}(n) < n$, in other words, the smallest number of iterations of the $3x + 1$ function on $n$ such that a value smaller than $n$ is produced. (Note: in the literature this definition of stopping time usually is made relative to a different definition of the $3x + 1$ function than ours, namely, one in which a division by 2 is considered an iteration of the function.

Tuple-sets make it easy to show the equivalent of a well-known result, namely (and here stated informally) that "most" exponent sequences have finite stopping times. For, consider, first, that $C(x) = y$ and $y > x$ iff $ord_2(3x + 1) = 1$. Thus, e.g., $C(7) = 11$ and $ord_2(3 \bullet 7 + 1) = 1$. Now consider any exponent sequence of length $(i - 1)$, $i \geq 2$. Then there are $2^{(i-1)} - 1$ ways that an exponent equal to 1 can appear in such a sequence. For each such way except for that in which there are $(i - 1)$ exponents each equal to 1, there is an infinity of possible exponent sequences of length $(i - 1)$, since an exponent can be any positive integer. Take any such way $w$ containing $j$, $0 < j < (i - 1)$, exponents each equal to 1. Then all but a finite number of sequences corresponding to $w$ will have finite stopping times, for (informally) it is always possible to find a single exponent sufficiently large to "overcome" the above-mentioned increasing effect of the exponents equal to 1, and all larger exponents will likewise overcome this effect.

## Possible Strategies for Proving the 3*x* + 1 Conjecture Using Tuple-sets

### Preliminary Discussion of Strategies

One of the characteristics of the $3x + 1$ function that makes proving the $3x + 1$ Conjecture so difficult, is that virtually every fact we prove about the function applies equally to counterexamples and to non-counterexamples. This is true of virtually every lemma in this paper, including, e.g., such well-known elementary facts as:

*If x maps to y in one iteration of the 3x + 1 function, then:*
*If* $x \equiv 1 \bmod 4$ *then the exponent of* 2 *is* $\geq 2$;
*If* $x \equiv 3 \bmod 4$ *then the exponent of* $2 = 1$;
*If* $y \equiv 1 \bmod 3$ *then the exponent of* 2 *is even*;
*if* $y \equiv 2 \bmod 3$ *then the exponent of 2 is odd*. (Lemmas 5.5, 5.7)

Proving such facts can lead us to believe that we are making progress toward proving the $3x + 1$ Conjecture, when all that we can say for certain is that we are increasing our knowledge of the properties of the $3x + 1$ function.

In attempting to prove the Conjecture using tuple-sets, we must realize that, whether or not there is a counterexample, the set of all tuple-sets will remain unchanged. Every odd, positive integer, whether counterexample or non-counterexample, will occupy exactly the same place in every tuple of which the integer is a member. This is in contrast to the recursive "spiral"s structure described in the second part of this paper. This structure describes the inverse of the $3x + 1$ function. There, the existence of counterexamples to the Conjecture would make a definite difference in the set of all "spiral"s describing the inverse of the function. If the Conjecture is true, then every odd, positive integer has a position in the infinite set of "spiral"s whose base element is 1. If the Conjecture is false, then *some* odd, positive integers, namely, those that map to 1, have positions in the infinite set of "spiral"s whose base element is 1, but in addition, there are other integers, namely, counterexamples, that occupy positions in at least one other infinite set of "spiral"s. Clearly, the integers in the infinite set that map to 1, constitute a set disjoint from the integers (counterexamples) in the other infinite set or sets of "spiral"s. (For a further elaboration on the material of this paragraph, see "Appendix D — A Curious Fact About the Inverse of the 3x + 1 Function" on page 84 and "Appendix E — A Curious Fact About Tuple-sets" on page 90.)

So it is important that we ask the following question:

**Question 1. How Does the Existence of Counterexamples "Make a Difference" in the Set of All Tuple-Sets?**

We can give at least two answers to this question. First, some definitions.

*Definition*. Let $M_i$ denote the set of all minimum residues of reduced residue classes mod $2 \bullet 3^{i-1}$. There are $2 \bullet 3^{i-2}$ such residues as stated above under "Summary of Properties of Tuple-sets" on page 13, and below in Lemma 3.0574 . Thus:

for level $i = 2$, $M_i$ has $2 \bullet 3^{2-2} = 2$ elements, namely $M_i = \{1, 5\}$;
for level $i = 3$, $M_i$ has $2 \bullet 3^{3-2} = 6$ elements, namely $M_i = \{1, 5, 7, 11, 13, 17\}$;
for level $i = 4$, $M_i$ has $2 \bullet 3^{4-2} = 18$ elements, namely $M_i = \{1, 5, 7, 11, 13, 17, 19, 23$, and all other odd, positive integers up to and including 53 that are not multiples of $3\}$.

$M_i$ is the set of last elements of all first $i$-level tuples in all $i$-level tuple-sets (by Lemma 1.0) .

**Lemma 3.057**. *The set of minimum elements of all top rows in all i-level tuple-sets is the set of minimum residues of the set of reduced residue classes mod* $2 \bullet 3^{i-1}$.

**Proof:** follows directly from the fact that a row is a reduced residue class mod $2 \bullet 3^{i-1}$ . . □

**Lemma 3.0574.** *For each* $i \geq 2$, *the number of elements of* $M_i$, *which we will denote* $|M_i|$, *is* $\varphi(2 \bullet 3^{(i-1)}) = 2 \bullet 3^{(i-2)}$, *where* $\varphi$ *is Euler's totient function, i.e., the function that returns the number of numbers less than its argument and relatively prime to its argument.*

**Proof:** The number of numbers less than $2 \cdot 3^{(i-1)}$ and relatively prime to it is given by Euler's totient function $\varphi$, which for powers of two primes $p^n$, $q^m$ is $\varphi(p^n q^m) = (p-1)p^{n-1}(q-1)q^{m-1}$. Applying this formula to $2 \cdot 3^{(i-1)}$, we get $2 \cdot 3^{(i-2)}$. . □

*Definition*. We call the elements of $M_i$, *anchors at i*, and we call the the *i*-level tuples they are the last elements of, *anchor tuples at i*. (We do not give a special name to the *first* element of the first *i*-level tuple in an *i*-level tuple-set. i.e., to the first element of an anchor tuple)

Clearly, because the tuples in each tuple-set are linearly ordered in the natural way by first elements of tuples, there is exactly one *i*-level anchor tuple in each *i*-level tuple-set. Furthermore, by definition of $M_i$, this anchor tuple is the *first i*-level tuple in each *i*-level tuple-set.

Thus, for example:
at level 2, the total number of anchors is $2 \cdot 3^{(2-2)} = 2$. These anchors are 1 and 5. The tuple <1, 1> is the 2-level anchor tuple of the 2-level tuple-set $T_A$, where $A = \{2\}$. The tuple <13, 5> is the 2-level anchor tuple of the 2-level tuple-set $T_A$, where $A = \{3\}$.
at level 3, the total number of anchors is $2 \cdot 3^{(3-2)} = 6$. These anchors are 1, 5, 7, 11, 13, 17. The tuple <13, 5, 1> is the 3-level anchor tuple of the 3-level tuple-set $T_A$, where $A = \{3, 4\}$. The tuple <7, 11, 17> is the 3-level anchor tuple of the 3-level tuple-set $T_A$, where $T_A = \{1, 1\}$.

A helpful tabular representation of anchors and anchor tuples, namely, the "anchor rectangle at *i*" and the "Infinite Anchor Rectangle" is given in "Appendix A1 — Lemmas and Definitions Used in Implementations of the "Pushing Away" and "Missing Sequences" Strategies" on page 61.

**First Answer to Question 1**

We express the answer in the form of a lemma:

**Lemma 10.96.**
**(a)** *If a counterexample exists, then for all* $i \geq i_0$, *where* $i_0$ *is the smallest i such that a counterexample is an anchor at i, the set of anchor tuples at i is partitioned into two disjoint sets: the set* $\{t_c\}$ of *counterexample anchor tuples and the set* $\{t_{nc}\}$ *of non-counterexample anchor tuples. Otherwise, if there are no counterexamples, the set of anchor tuples at i,* $i \geq 2$, *consists exclusively of non-counterexample anchor tuples.*

**(b)** *For each* $i \geq i_0$, *let* $\{A_{nc}\}$ *denote the set of all exponent sequences defined by* $\{t_{nc}\}$ *in part (a), and let* $\{A_c\}$ *denote the set of all exponent sequences defined by* $\{t_c\}$ *in part (a). Then* $\{A_{nc}\} \cap \{A_c\} = \varnothing$.

**Proof: (a)** follows directly from the fact that no tuple can be simultaneously a non-counterexample and a counterexample tuple, and **(b)** from the fact that the set of all anchor tuples at any *i* defines the set of all *i*-level exponent sequences. □

Thinking about how Lemma 10.96 might lead us to a proof of the 3*x* + 1 Conjecture brings us, sooner or later, to the following question:

**Why Are There An Infinite Number of Tuples in Each Tuple-set?**

The answer is: because every sequence of positive integers defines a tuple-set, and because the last element of each tuple maps directly to one and only one odd, positive integer. For, consider the tuple-set $T_A$ defined by the exponent sequence $A = \{a_0, a_1, a_2, ..., a_i\}$. $T_A$ has an extension for each positive integer $a_{i+1}$, otherwise there would exist a sequence of positive integers that did not define a tuple-set. But since the last element of each tuple in $T_A$ maps directly to one and only one odd, positive integer, and since each tuple-set $T_{A'}$, $A' = \{a_0, a_1, a_2, ..., a_i, a_{i+1}\}$, likewise has an extension for each positive integer $a_{i+2}$, it follows that, for *each* $a_i$, there exists an *infinity* of tuples in $T_A$ whose last elements directly map to their respective odd, positive integers *via* $a_i$. (This is what Lemma 2.0 states:

**Lemma 2.0**: *Every i-level tuple-set can be extended by any exponent* $a_{i+1}$*. Or, in other words, for each i-level tuple-set and for each* $a_{i+1}$*, every i-level row — though not every element in every i-level row — maps to a non-empty row in some (*$i$+1*)-level tuple-set.*)

Thus, e.g., in the tuple-set $T_A$, $A = \{2\}$, the last element of the tuple <9, 7>, namely, 7, maps to 11 via the exponent 1. And similarly, in the same tuple-set, the last element of the tuple <25, 19>, namely 19, maps to 29 via the same exponent, 1. (However, 11 then maps to 17 via the exponent 1, whereas 29 maps to 11 via the exponent 3.)

Each $i$-level tuple in an $i$-level tuple-set has an extension via some exponent $a_i$. An infinity of such tuples have an extension via the exponent 1, another infinity have an extension via the exponent 2, another infinity via exponent 3, etc. The details are given in Lemmas 7.25 through 7.4 (see Appendix A).

Now Lemma 10.0 implies that, whether or not a counterexample exists, there is an infinity of non-counterexample tuples in each tuple-set. Or, in other words, the set of all (finite) non-counterexample tuples defines the set of all (finite) exponent sequences, hence the set of all tuple-sets, regardless whether a counterexample exists or not.

The question we must ask ourselves is this: if we remove an infinity of elements (namely, counterexample elements) from the top row of each tuple-set — and not merely an infinity of elements, but an infinity of elements that guarantee that the set of tuples so removed (i.e., the counterexample tuples) do likewise define the set of all (finite) exponent sequences, as Lemma 10.0 requires, hence the set of all tuple-sets — if we remove all these elements, is it possible that the set of non-counterexample tuples (i.e., the tuples that remain) can still define the set of all finite exponent sequences, especially given that there is no redundancy in the set of anchor tuples for each $i$ (i.e., each anchor tuple defines one and only one $i$-level exponent sequence, and all $i$-level exponent sequences are defined by the set of all anchor tuples)?

The answer is not clear. All that we can say at this point is that for each $i \geq i_0$, where $i_0$ is as defined in Lemma 10.96 above, the presence of counterexamples removes an infinite set of exponent sequences from those defined by non-counterexample anchor tuples at $i$ if there are no counterexamples.

**Second Answer to Question 1**

The second answer to Question 1 follows directly from the definition of a counterexample, namely, a number that does not eventually map to 1.

*If a counterexample exists, then for all $i < i_0$, where $i_0$ is as defined in Lemma 10.96, all elements of $M_i$ map to 1, and hence are "connected" to elements of the infinite set of recursive "spiral"s (see second part of this paper) with base element 1. At $i = i_0$, however, there is at least one element of $M_i$ that is not "connected" to the set of elements that map to 1. In particular, this element (a counterexample) is not connected to any element of $M_i$ for any i, $2 \leq i < i_0$. Otherwise, if there are no counterexamples, the set of elements of $M_i$, for all $i \geq 2$,* are *"connected" to elements of the infinite set of recursive "spiral"s with base element 1.*

**Strategies for Proving the 3x + 1 Conjecture That Are Suggested by the Answers to Question 1**

The above answers suggest the following strategies.

We would have a proof of the Conjecture if we could show:

(1) that the assumption of a counterexample implies that one or more non-counterexamples were not mapped to by exponent sequences that an existing result required.

A major problem connected with this strategy is that, at each $i \geq i_0$, where $i_0$ is the smallest level at which a counterexample is an anchor at *i*, there *does* exist a set of non-counterexample range elements that is, in fact, mapped to by every *i*-level exponent sequence, and similarly for counterexample anchors. This is not a contradiction to Lemma 10.96, above, because to obtain each such set of range elements requires that we go outside the set of anchors at *i*. Further details on these sets are given in "Appendix A1 — Lemmas and Definitions Used in Implementations of the "Pushing Away" and "Missing Sequences" Strategies" on page 61.

Or we would have a proof of the Conjecture if we could show:

(2) that a counterexample never becomes an element of an anchor tuple at any level *i* (for this implies that no counterexample exists).

A brief description of this strategy is given in the next sub-section. Several possible proofs of the 3*x* + 1 Conjecture derived from this strategy — which we call the "Pushing Away" Strategy — are given in "Appendix B — Possible Proofs of the 3x + 1 Conjecture Using the "Pushing Away" Strategy" on page 75.

Or we would have a proof of the Conjecture if we could show:

(3) that there is no minimum counterexample.

A discussion of this strategy is given under "Strategy of Proving There Is No Minimum Counterexample" on page 25.

## The "Pushing Away" Strategy in Brief

In the "Pushing Away" Strategy we attempt to show that every tuple containing an assumed counterexample is "pushed away" from tuples whose elements map to 1, i.e., every tuple containing a counterexample must always be the second, or third, or fourth, or ... tuple in any tuple-set, but never the first. Thus counterexample tuples never become anchor tuples, hence counterexample tuples do not exist (by the Corollaries to Lemmas 10.90 and 10.91 (see Appendix A1)).

### How the Pushing Away Strategy Resolves a Seeming Paradox Concerning Tuple-sets

The basic idea underlying the Pushing Away strategies can be used to resolve a seeming paradox concerning the cardinality of tuples and of tuple-sets. Stated informally, the seeming paradox arises as follows. It is easily shown that the cardinality of tuple-sets is countably infinite (Lemma 1.2). But tuple-sets are defined by finite sequences of positive integers. Since we know that every tuple <*x*>, *x* an odd, positive integer, has an infinite sequence of extensions, {<*x*>, <*x*, *y*>, <*x*, *y*, *y'*>, ... }, and that such a sequence defines an infinite sequence of exponent sequences that we shall denote {*A*({}), *A*(<*x*>), A(<*x*, *y*>), A(<*x*, *y*, *y'*>), ... } we can speak of tuple-sets that are, "in the limit", defined by infinite sequences of positive integers. The cardinality of all infinite sequences of positive integers is easily shown to be uncountably infinite. But then what are the contents of this uncountable infinity of tuple-sets, given that only a countable subset contain infinite tuples generated by odd, positive integers? Are "most" of the tuple-sets in this uncountable infinity empty? But if so, how does the tuple-set generated by an infinite sequence of positive integers that is *not* a sequence generated by an odd, positive integer, "know" when to stop containing tuples, given that *every* finite sequence of positive integers generates a tuple-set containing an infinity of tuples, even if the finite sequence is the initial part of an infinite sequence not defined by an odd, positive integer *x*? In other words, the seeming paradox is that a sequence of tuple-sets, each set containing an infinity of tuples, can, "in the limit", be empty (if that is in fact the case).

The resolution of this seeming paradox is as follows. Let *x* be any odd, positive integer. (It is irrelevant here whether *x* is a counterexample or not.) Then {<*x*>, <*x*, *y*>, <*x*, *y*, *y'*>, ... } is an infinite sequence of tuples that gives rise to an infinite sequence of tuple-set extensions. (Each tuple defines an exponent sequence that defines a tuple-set.)

Now it is easily shown (Lemmas 3.0, 4.0) that there exists an $i$ such that some tuple $t$ in the above sequence, $t$ having length $(i - 1)$, is the first $i$-level tuple in its $i$-level tuple-set[1], and that all tuples that are extensions of $t$ remain first $(i + k)$-level tuples in their respective $(i + k)$-level tuple-sets, $k \geq 1$.

But now consider an infinite sequence $s$ of positive integers that is *not* one of the infinite sequences generated by extensions of tuples <*x*>, where *x* is any odd, positive integer. In this case, no first $i$-level tuple in an $i$-level tuple-set remains a first $(i + k)$-level tuple in all $(i + k)$-level tuple-sets generated by $s(i + k)$, where $k \geq 1$ and $s(i + k)$ is the first $(i + k - 1)$ elements of $s$. In other words, if we could observe the sequence of tuple-sets generated by the sequence of exponent sequences $s(2)$, $s(3)$, $s(4)$, ... we would observe that the first $i$-level tuple in each corresponding $i$-level tuple-set does not permanently remain an extension of the same tuple <*x*> ! Informally, the first $i$-level tuples "keep moving to the right", meaning that they keep having higher and higher numbers *x* as their first elements. (This phenomenon is explained in more detail in the next sub-section. Examples of infinite sequences $s$ of positive integers that are *not* one of the infinite sequences generated by extensions of tuples <*x*>, where *x* is any odd, positive integer, are given.) Thus, indeed, "in the limit", the tuple-sets generated by infinite exponent sequences $s$ different from those generated by odd, positive integers *x* are "empty". For, if you specify any *x*

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1. Such a tuple is called an *anchor tuple*. See definition in "Preliminary Discussion of Strategies" on page 18.

you claim is the first element of a tuple in one of these tuple-sets , I can show you a tuple-set defined by some sequence *s*(*i*) in which *x* is not the first element of any tuple.

*However*, in the case of tuple-sets generated by sequences corresponding to those generated by odd, positive *x* (regardless whether *x* ultimately maps to 1 or *x* is a counterexample), "in the limit" the infinite tuple <*x*, *y*, *y'*, ...> is the first *and only* tuple in the corresponding tuple-set, for the distance functions defined in Lemmas 1.0 and 1.1 imply that all other tuples are pushed infinitely far away. Thus the tuple-set defined by the infinite sequence of positive integers defined by <*x*, *y*, *y'*, ...> is not empty.

The fact that the same "pushing away" phenomenon that underlies our above-described Pushing Away strategies for proving that no counterexamples exist, also resolves the seeming paradox concerning (some) tuple-sets defined by infinite sequences of positive integers — a paradox having nothing to do with counterexamples — lends support, at least in our opinion, to the importance of the pushing away phenomenon.

## Some Infinite Exponent Sequences That are Not Generated by Any Odd, Positive Integer, *x*

It is easily shown that the cardinality of all infinite sequences of positive integers is uncountable, whereas the cardinality of the odd, positive integers is countable, and each such integer generates (defines) exactly one infinite sequence of exponents. This is the simplest proof that there are infinite exponent sequences that are not generated by any domain element, *x*. (The reader is encouraged to also read "Some Infinite Inverse Exponent Sequences That are Not Generated by Any Range Element, y" on page 34.)

An obvious next question is, can we give specific examples of infinite exponent sequences that are not generated by any odd, positive integer? The answer is yes.

**Lemma 1.5**. *Each cycle in the odd,* ***negative*** *integers defines an infinite exponent sequence* $\underline{A}$ *such that no odd, positive integer x generates* $\underline{A}$*. Examples of such sequences* $\underline{A}$ *are: A * {1, 1, 1, ... }, A * {1, 2, 1, 2, ..., } and A * {1, 1, 1, 2, 1, 1, 4}, where A is a finite (possibly empty) exponent sequence, and "*" denotes concatenation of sequences.*

**Proof**:

1. The reader can easily verify for himself that the sequences following *A* in the statement of the Lemma, do, in fact, define cycles in the odd, *negative* integers: {1, 1, 1, ... } is generated by -1 (the cycle is <-1, -1, ... >). {1, 2, 1, 2, ... } is generated by -5 (the cycle is <-5, -7, - 5, ...>). {1, 1, 1, 2, 1, 1, 4} is generated by -17 (the cycle is <-17, -25, -37, -55, -41, -61, -91, -17, ...>).

2. As stated under "Remarks About the Distance Functions" on page 12, it is easily shown that the distance functions defined by Lemma 1.0 (a) and (b) extend to the odd, *negative* integers. We will call a tuple-set that includes the odd, negative integers, an *extended tuple-set*.

3. Assume an odd, positive integer *x* exists such that *x* generates the sequence {1, 1, 1, ..., }. Then *x* and -1 are first elements of tuples in the infinite sequence of extended tuple-sets defined by {1}, {1, 1}, {1, 1, 1,}, ...

4. But this means that eventually the distance functions defined by Lemma 1.0 (a) and (b) will be contradicted, e.g., -1 and $x$ will, for all $i$ greater than some minimum $i$, be the first elements of consecutive $i$-level tuples, which contradicts Lemma 1.0 (b).

4. A similar argument applies to the other two cycles.

5. Lemma 2.0 assures us that the argument holds following any arbitrary finite exponent sequence $A$. □

### Strategy of Proving There Is No Minimum Counterexample

Assume a counterexample exists. Consider the minimum counterexample $y$ that is a range element.

Then $y$ has the following properties: (1) for all $z$ resulting from iterations of $y$, $z \geq y$; and (2) for all $x$ mapping directly or indirectly to $y$, $x \geq y$; in particular, this means that $y$ must be mapped to, in one iteration of the $3x + 1$ function, by exponents of even parity. (Lemma 5.0 states that each range element is mapped to by all exponents of one partity only; if $y$ were mapped to, in one iteration of the $3x + 1$ function, by exponents of odd parity, then $y$ would be mapped to by the exponent 1, and that would mean that $y$ were mapped to by an $x < y$, contradicting (2).)

We observe immediately that one $y$ fulfilling conditions (1) and (2) is ... the range element 1(!). Furthermore, since, by Lemma 7.0, every exponent sequence, possibly followed by a buffer exponent, maps to 1, but since, in our case, by our even parity requirement, we are concerned only with exponent sequences whose last element is even, and since 1 is mapped to solely by even exponents, it is tempting to regard the buffer exponent as irrelevant. Unfortunately, a basic property of tuple-sets blocks us. The basic property is that for every finite exponent sequence, there exists a tuple-set containing, if counterexamples exist, an infinity of non-counterexample tuples and an infinity of counterexample tuples (Lemma 10.0). Thus there exists an infinity of tuple-sets whose defining exponent sequences give us, in each tuple, $x$, $y$, $z$ satisfying properties (1) and (2), and an infinity of these tuples are non-counterexample tuples, and an infinity are counterexample tuples.

The reader might be interested in several results regarding the minimum counterexample given in "Appendix G — Results on the Minimum Counterexample" on page 95.

### Testing for Counterexamples

We cannot *necessarily* determine by computer testing, if counterexamples exist. Of course, if such a test reveals a cycle, then we have determined that a counterexample exists. However, if the test program simply keeps running beyond the time our computer resources allow, we cannot know whether the reason is that the original number $n$ being tested is a counterexample, or whether the reason is that it simply takes an inordinate length of time for $n$ to yield 1.

### A Way to Reduce Computation Time in Computer Testing of the 3*x* + 1 Conjecture

The existence of exponent sequences with the less-to-greater property suggests a method for reducing the computation time for testing the $3x + 1$ Conjecture.

If a counterexample exists, then there is a minimum counterexample. Consider any sequence $A$ of exponents having the less-to-greater property. Now since, according to reliable reports, the Conjecture has been tested and found valid for all odd, positive integers through $56 \times 10^{15}$, and since $56 \times 10^{15}$ is greater than $2^{55}$, we know, by Lemma 1.0(b), that all exponent sequences $A$ having the less-to-greater property, and whose sum is ≤ 54, have been tested and have failed to reveal a minimum counterexample. Therefore the only candidates $x$ for minimum counterexamples must lie at distances of $2 \cdot 2^{54} = 2^{55}$. Furthermore, there exists an algorithm for generating all sequences of a given length ($P(i - 1)$), or a given sum, having the less-to-greater property, and so testing can continue, up to the limits of modern computing power, for a minimum counterexample, making use of the distance function established by Lemma 1.0 (b).

## Strategy of Using a Topology Defined on Tuples or Tuple-sets

It is natural to wonder if defining an appropriate topology on tuples or on tuple-sets might lead us to a proof of the $3x + 1$ Conjecture. For example, if we could define a Hausdorff topology on tuples, then show that the assumption of a counterexample to the Conjecture implies that an infinite sequence of tuple-extensions converges to two or more points, we would have a proof of the Conjecture, because that would be a contradiction, since in a Hausdorff space, if an infinite sequence converges, it converges to only one point. Another possibility might be the following: show that the set of numbers mapping to 1 is connected, and that all counterexamples, if any exist, are partitioned into one or more connected sets. Then show that these various sets form a set that is at once connected and disconnected, a contradiction.

### Proposed Topology, TT

We define a *separate* topology TT on the tuples (prefixes of infinite tuples) in the tuple-sets of *each* infinite sequence of tuple-set extensions. Thus, if $T_A$ is a tuple-set, $A = \{a_2, a_3, ..., a_j, ..., a_i\}$, $i \geq 2$, then the topology TT relative to $T_A$ is defined on the tuples in the sequence of tuple-set extensions defined by the exponent sequences $A^*\{a_{i+1}\}$, $A^*\{a_{i+1}\}^*\{a_{i+2}\}$, $A^*\{a_{i+1}\}^*\{a_{i+2}\}^*\{a_{i+3}\}$, ... , where $a_{i+j}$, $j \geq 1$, is any exponent. Each tuple-set is a neighborhood of the tuples it contains.

The reader can verify for him- or herself that the topologies so defined fulfill the requirements of a topology, namely, that it is a collection of the subsets of the set $\{T\}$ of all tuple-sets in the sequence of tuple-set extensions such that:

∅ and $\{T\}$ are in *TT*;
The union of any subcollection of $\{T\}$ is in TT;
The intersection of the elements of any finite subcollection of $\{T\}$ is in TT.[1]

**Lemma 10.8**. *The topology TT is Hausdorff.*

**Proof**:

A topology is Hausdorff if, for every pair of points $p$, $p'$ in the space $X$, there exist disjoint neighborhoods $U_1$, $U_2$ such that $U_1$ is a neighborhood of $p$, and $U_2$ is a neighborhood of $p'$.

By the distance functions established in Lemmas 1.0 and 1.1, the distance between first elements of tuples consecutive at level $i$ in any $i$-level tuple-set, increases with $i$. Therefore, no two infinite tuples can remain indefinitely in the same sequence of tuple-set extensions. Or, in other

1. From Munkres, James R., *Topology: A First Course*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975, p. 76.

words, any two infinite tuples <$x$, ...> and <$x'$, ...>, $x \neq x'$, are eventually in different tuple-sets. □

**Lemma 10.83**. *A metric exists on the topological space TT.*

**Proof**: We define a metric μ on tuples in a tuple-set as follows.
Let $t$, $t'$ be tuples. Then:
If $t = t'$, $\mu(t, t') = 0$.
If $t$, $t'$ are tuples in the unique 1-level tuple-set {<1>, <3>, <5>, ...}, $\mu(t, t') = 1/(|(t(1) - t'(1))|)$, where |...| denotes absolute value, and $t(1)$, $t'(1)$ are the first (and only) elements of the tuples $t$, $t'$ respectively. By definition of the $3x + 1$ function, we know that the denominator in the value of μ in this case is the absolute value of the difference of two odd numbers.

If $t$, $t'$ are $i$-level tuples in an $i$-level tuple-set, $i \geq 2$, $\mu(t, t') = 1/(|t(i) - t'(i)|)$, where |...| denotes absolute value, and $t(i)$, $t'(i)$ are the $i$-level elements of $t$, $t'$, respectively. By Lemma 1.0(a) we know that the denominator is the absolute value of a multiple of $2 \bullet 3^{(i-1)}$.

The proof that μ is in fact a metric follows directly from the definition of a metric, namely, a function $d$ from pairs of elements $(x, y)$ of a nonempty set $X$ to the nonnegative real numbers such that:

$d(x, y) = 0$ if and only if $x = y$.
$d(x, y) = d(y, x)$.
$d(x, y) \leq d(x, z) + d(z, y)$ for all $z$ in $X$. □

**Similarity of μ to a Frequently-Used *p*-adic Metric**
Readers who are acquainted with $p$-adic number theory will recognize a similarity between our metric μ and the frequently-used $p$-adic metric $|x - y|_p$ defined as:

$$|x - y|_p = \frac{1}{p^{ord_p(x-y)}}$$

where $ord_p(x - y)$ is the exponent of the largest power of $p$ that evenly divides $x - y$. Observe that, if $x$, $y$ are both divisible by different powers of $p$, then their difference, $x - y$, is divisible by the *lower* of the two powers [5]. This fact corresponds to the fact, derived directly from our definition of the topology TT, that two tuples $t$, $t'$ are in all tuple-sets defined by exponent sequences that are the same as initial sub-sequences of the exponent sequences for $t$ and $t'$. At the least, $t$ and $t'$ are in the 1-level tuple-set $T_{\{\}}$. Thus, in the above $p$-adic metric, two numbers are $p$-adically "close" (i.e., the $p$-adic distance between them is small) if they are both divisible by a large power of $p$. Similarly, two tuples are "close" in terms of our metric μ if they are both elements of a tuple-set defined by a "long" exponent sequence. (They are even closer if they are separated from each other by a "large" number of other tuples in the tuple-set. Further experience with the metric μ is necessary in order to determine if this additional factor — the actual distance between the tuples in a given tuple-set — is necessary for our purposes.)
Observe that our metric μ differs from the above $p$-adic metric in that, in general, $\mu = 1/(m \bullet 2 \bullet 3^{(i-1)})$, $m \geq 1$.

**Using the Topology TT and the Metric μ to Prove the 3*x* + 1 Conjecture**

We now show how the topology TT and the metric μ might be used to prove the 3*x* + 1 Conjecture. In particular, we describe a possible implementation of the strategy described at the start of this sub-section, namely, that of showing that the assumption of a counterexample implies that the same infinite sequence converges to two points, which is not possible in a metric space.

Let *x* be an odd, positive integer that ultimately maps to 1. By definition of tuple and of tuple-set, each sequence of tuples, {<*x*>, <*x*, *y*>, <*x*, *y*, *y′*>, ... }, establishes a corresponding sequence of exponent sequences, $\{\{\}, A, A', A'', ...\}$, which in turn defines a sequence of tuple-set extensions, $\{T_{\{\}}, T_A, T_{A'}, T_{A''}, ...\}$.

Assume a counterexample exists. Then by Lemma 10.0, each tuple-set extension contains an infinity of tuples (n-t-v-1s) whose elements map to 1, and an infinity of tuples (n-t-v-cs) each containing counterexamples, and this is true regardless how long the exponent sequence $A''^{...}{}'$ defining the tuple-set is.

Then, in a sense that we believe can be made precise, in the limit, each infinite sequence of exponents converges to two points, namely, a point defined by numbers that map to 1, and a point defined by counterexamples.

## Remark About the Above Strategies

Occasionally, a reader will argue that none of the above strategies can be considered valid until we show that it will not also prove that there are no "counterexamples" in the domain of the odd, negative integers. For, if the strategy should in fact prove there are no such "counterexamples", then the strategy could not possibly be correct, since at least one "counterexample" is known in that domain, namely, -17, which gives rise to an infinite loop.

Our reply to the above argument is the following:

The "Statement of Problem" on page 2 makes clear that the domain of the 3*x* + 1 function in this paper is the odd, positive integers. Furthermore, all the proofs of lemmas (and the illustrative examples), and the above strategies of the 3*x* + 1 Conjecture, are carried out in this domain. In particular, the number 1, which is explicitly mentioned in several lemmas, and the other minimum residues of the reduced residue classes mod $2 \bullet 3^{(i - 1)}$, $i \geq 2$, which play such an important role in some of the proofs and, in particular, in the above strategies, are odd, positive integers (as these minimum residues always are in number theory). For example, a first *i*-level tuple in an *i*-level tuple-set is identified by the fact that its last element is such a minimum residue. (What residues will take the place of these minimum residues in the negative-integer domain?)

Having said all that, we will be the first to admit that the behavior, in terms of tuple-sets, of the 3*x* + 1 function on the odd, negative integers, is definitely of interest. In fact, it is easily shown that the distance functions (Lemmas 3.0, 4.0) carry over directly to the odd, negative integers. Thus, e.g., $(3 \bullet (13) + 1)/2^3 = 5$. The distance functions say that the next 2-level tuple in the negative direction should have $13 - 2 \bullet 2^3 = -3$ as first element. And indeed, we find that $(3 \bullet (-3) + 1)/2^3 = -1$, and $-1 + 2 \bullet 3^{(2-1)} = 5$, as the distance functions require.

Nevertheless, either the lemma proofs, and the above strategies, are correct as they stand, or they are not. The question why the strategies show there is no counterexample among the odd, positive integers, and why it is a fact that there is at least one "counterexample" among the odd, negative integers, is a separate issue.

Of course, if the strategies, when taken over the odd, negative and the odd, positive integers, enable us to show that there both is, and is not, a counterexample among the odd, positive integers, then we have discovered something whose importance far exceeds that of the $3x + 1$ Problem, namely, the inconsistency of number theory I

## Turning Tuple-sets "Inside-out"

Natural curiosity compels us to ask if it might be worthwhile to investigate the relationship between the sequences of numbers in tuples, and the sequences of numbers that define tuple-sets. Of course, the number of exponents that define any $i$-level tuple-set $T_A$, $i \geq 2$, is $(i - 1)$, whereas the number of elements in any tuple in $T_A$ is $i$. But, as a start, we might consider the question, Is there anything of interest to be learned in taking any tuple in $T_A$, allowing the sequence of its elements to define another tuple-set $T_{A'}$, picking any tuple in $T_{A'}$ and allowing its elements to define a tuple-set $T_{A''}$, etc.?

## Section 2. Recursive "Spiral"s

In the first part of this paper, we described a structure called "tuple-sets" that underlies iterations of the $3x + 1$ function — in other words, that describes iterations of the function in the "forward" direction. In this part, we describe a structure called "recursive 'spiral's" that describes the inverse of the $3x + 1$ function — in other words, describes iterations of the function in the "backward" direction.

The "spatial", "geometric", "graphical" nature of both structures is important for the strategies it suggests.

We begin with some definitions.

## Definitions

### Recursive "Spiral"

A *recursive "spiral"* is the infinity of odd, positive integers that map to a given range element in one iteration of the $3x + 1$ function, as established in the proof of Lemma 5.0. (See Fig. 4.) Each range element in the infinity of elements in turn sets up a recursive "spiral", etc. Thus the infinite set of all "spiral"s relative to a given range element are a *self-similar* structure ([4], p. 34).

The recursive "spiral" structure has been independently discovered by at least two researchers besides us, although we are not aware of anything in the literature that deals explicitly with this structure.

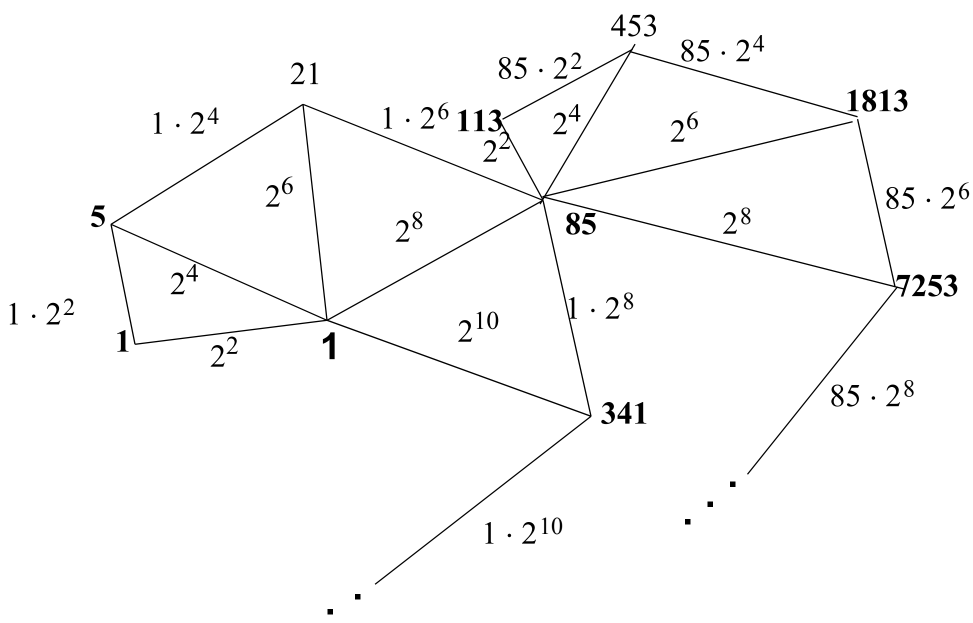


**Fig. 4. Recursive "spirals" structure of computations produced by the $3x + 1$ function.**

Bold-faced numbers are range elements (21 and 453 are multiples of 3, hence not range elements). Partial "spirals" surrounding the base elements 1 and 85 are shown. The line connecting 1813 to 85 is marked with a $2^6$ because $(3 \cdot 1813 + 1)/ 2^6 = 85$. The line connecting 453 to 1813 is marked $85 \bullet 2^4$ because $453 + 85 \bullet 2^4 = 1813$. The exponents of 2 are not always even, of course. The "spiral" of numbers (not shown) mapping to 341 has odd exponents.

### Level *i* and Base Sequence of Elements

Let $y$ be a range element, e.g., 1. We define $y$ to be at *level 0 relative to y*. We now define all $x$ that map to $y$ in a single iteration to be at *level 1 relative to y*. (*Warning*: no suggestion is intended that the term *level* as defined here is the same as the term *level* as defined above for tuple-sets, although, as we shall see, there is a strong relationship between the two terms.) We define all odd, positive integers of level 1 iterations that map to $y$ to be level 1 *elements*. These elements constitute a sequence (a "spiral" in Figure 4). Specifically, they constitute a unique *base sequence* (relative to $y$). Thus, for example, the base sequence relative to 1 is the sequence {1, 5, 21, 85, 341, ... }. We define all $x$ that map to a level 1 element in a single iteration to be at level 2 relative to $y$, and similarly for level 2 elements. And so on for all levels $i$. When $y$ is understood, we will sometimes eliminate the phrase, "relative to $y$". The expressions *level i sequence* and *level i "spiral"* thus mean the same thing.

We define the range element mapped to, in a single iteration, by each element of a "spiral", to be a *center element*, because it is the center of a "spiral", as shown in Figure 4. Often, we will call a center element a *base element*. The infinite set of elements that map to a given base element corresponds, in [3] (p. 21), to a *predecessor set*, although unlike a predecessor set, the infinite set of elements that map to a given base element contains no even numbers.

We say that elements of the base 1 sequence map *directly* to the base element, and that elements of level $i$, $i > 1$, map *indirectly* to the base element.

We define a *path*, relative to any base element $y$, to be a finite sequence of elements of "spirals" at levels $i, i - 1, i - 2, ... 0, i \geq 0$, such that the "spiral" element at level $j$, $1 \leq j \leq i$ maps directly to the element at level $j - 1$ in a single iteration. Thus, e.g., <13, 5, 1> is a path. A path is thus the equivalent of a tuple. Each path defines an exponent sequence, e.g., in the case of our example, the sequence {3, 4}.

Some examples of elements at different levels: If $y = 1$, then 1, 5, 21, 85, 341, ... are level 1 elements relative to 1. They also constitute the base sequence relative to 1, which is the center, or base element. For the center element (or base element) 5 in this base sequence, the level 1 elements are 3, 13, 53, 213, 853, .... These are level 2 elements relative to 1.

We define the set of odd, positive integers lying *between* any two successive elements of a level $i$ sequence to be *interval*s of that sequence. Thus, for example, 7, 9, 11, 13, 15, 17, 19, are the elements of the second interval in the level 1 sequence, 1, 5, 21, 85, 341, ... When necessary, we number the intervals in a given sequence starting with 1.

## Distance Functions on "Spiral"s

The proof of Lemma 5.0 implicitly defines two distance functions on "spirals": one, between any "spiral" element and the base element of the "spiral", and the other between successive elements of a "spiral". We will refer to these simply as *"spiral" distance functions*, specifying which one we mean as required. We give these functions in the next lemma.

**Lemma 11.0.** *(a) The distance between the jth element, $j \geq 1$, of a "spiral", and the base element y of the "spiral", is given by $|(2^k y - 1)/3 - y|$, where k is the jth element in the sequence <1, 3, 5, ..., > or the sequence <2, 4, 6, ..., > as established by y.*

*(b) The distance between successive elements x, x' of a "spiral" is given by 3x + 1, i.e.,* $x' = 4x + 1$*;*

*(c) If x, x' are elements of a "spiral" then* $x, x' \equiv 5 \bmod 8$.

**Proof:**

(a) Follows directly from Lemma 5.0.

(b) By Lemma 5.0 we have

$$\frac{3x+1}{2^j} = y$$

and

$$\frac{3x'+1}{2^{j+2}} = y$$

so that

$$\frac{3x+1}{2^j} = \frac{3x'+1}{2^{j+2}}$$

and hence

$$2^2 x + 1 = x'$$

and thus

$$x' - x = (2^2 x + 1) - x = 3x + 1$$

(c) Follows directly from (b). □

## Summary of Properties of Recursive "Spiral"s

We now provide a table that summarizes our results on recursive "spiral"s.

*Note*: some table-rows may have the same content as other rows, though under different properties. This redundancy is deliberate, the purpose being to aid understanding and to make the looking up of properties easier.

Statements of all referenced lemmas are given in "Appendix A — Statements of Lemmas" on page 49.

**Table 6: Some important properties of recursive "spiral"s**

| Property | Value of property | Reference |
|---|---|---|
| Self-similarity | Each (non-multiple-of-3) element of a "spiral" is the base element of a "spiral" each (non-multiple-of-3) element of which is the base element of...<br><br>Also, for all base points $y$, the infinite set of "spiral"s relative to $y$ is path-similar to every other such infinite set, i.e., all paths, as defined by finite exponent sequences, exist in each such infinite set. | Lemmas 0.2, 5.0, 15.85. |
| Set of elements in a "spiral" | $\{x \mid x = (2^k y - 1)/3\}$, where $y$ is the base element of the "spiral" and all $k$ are either even or odd, depending on y. Thus, the number of elements in a "spiral" is infinite. | Lemma 5.0 |
| Distance between $j$th element of a "spiral" and its base element $y$ | $\lvert(2^k y - 1)/3 - y\rvert$, where $k$ is the $j$th element of $\{1, 3, 5, ... \}$ or $\{2, 4, 6, ...\}$, depending on $y$. | Lemma 11.0 |
| Distance between successive elements $x$, $x'$, of a "spiral" | $3x + 1$ | Lemma 11.0 |
| Number of levels in the infinite set of "spiral"s relative to any given base element | Infinite | Lemma 5.0 |
| In the infinite set of "spiral"s relative to any given base element, number of paths defined by any given exponent sequence $A = \{a_2, a_3, ..., a_i\}$. | Infinite, i.e., there are an infinite number of paths for *each* exponent sequence, as in tuple-sets. | Lemma 7.0 |

**Table 6: Some important properties of recursive "spiral"s**

| Property | Value of property | Reference |
|---|---|---|
| Congruence classes to which base element and "spiral" elements belong | For all $i \geq 2$, and for each base element $y$:<br>(1) $y$ is an element of a reduced residue class mod $2 \cdot 3^{i-1}$;<br>(2) the elements of the base sequence (i.e., of the "spiral" having $y$ as base element) are elements of a sequence $s$ of all reduced residue classes mod $2 \cdot 3^{(i-2)}$, with $s$ being repeated endlessly over all elements of the"spiral". | Lemma 15.85. |

## Some Infinite Inverse Exponent Sequences That are Not Generated by Any Range Element, *y*

It is natural to ask, regarding recursive "spiral"s, the equivalent of the question we asked, regarding tuple-sets, under "Some Infinite Exponent Sequences That are Not Generated by Any Odd, Positive Integer, x" on page 24, namely, are there infinite exponent sequences that are not generated by any infinite path in the infinite set of "spiral"s defined by any range element, y? The answer is yes.

**Lemma 12.5**. *Let* $\underline{A}$ = *...A*A*A*A′ be an infinite sequence of positive integers, where A, A′ are finite sequences, A is not empty, A′ is possibly empty, A is repeated infinitely and successively, and is such that in every path <x, ..., y> defined by A, x is less than y. Then no range element y′ defines* $\underline{A}$. An example of such an $\underline{A}$ is {...1, 1, 1}.

**Proof**:

Each successive *A* (moving from right to left) is produced by successively smaller *x*'s. The existence of $\underline{A}$ therefore implies the existence of an infinitely decreasing sequence of odd, positive integers, which is impossible. □

## Possible Strategies for Proving the 3*x* + 1 Conjecture Using "Spiral"s

### Fractal-Based Strategy

Since an infinite set of recursive "spiral"s relative to a base element constitute a self-similar structure, it is natural to ask if such a structure has a fractal dimension, and if so, what the dimension is. Then, if we know the dimension, we can perhaps use it to prove that only one such set is required to "cover" the odd, positive integers. But we must keep in mind that the 3x - 1 function has a structure very similar to that of the 3x + 1 function, and yet it has known counterexamples, e.g., a cycle that involves 17.

To compute the fractal dimension $d$ of an infinite set of recursive "spiral"s relative to a base element, we must be able to know the scaling ratio of successive approximations to the fractal object that constitutes the limit of the approximations. That is, we need to be able to compute:

$$d = \frac{\log n}{\log s}$$

where (informally),

$n$ is the number of "sides" in the next approximation;

$s$ is the size of each "side" in the next approximation relative to the length of a "side" in the previous approximation.

(See any of the well-known Mandelbrot works for a formal definition.)

If we take the level 1 "spiral" to be the first approximation to the final fractal object that constitutes the infinite set of "spiral"s, then it is natural to take the "line" connecting two successive elements of this "spiral" to be a "side" (see Fig. 4), and the length of this "side" to be that defined by the distance functions, namely, $4x + 1$, where $x$ is the smaller of the two elements.

However, the total length of the first approximation is then clearly infinite, as is the total length of the second approximation, which we take to be the total length of all level 2 "spiral"s. This does not enable us to compute $d$. Another approach is to take only the first $k$ "side"s of the level 1 "spiral", the length of which is finite; and then compute the length of the first $k$ "side"s of the level 2 "spiral"s yielded by the first $k$ elements of the level 1 "spiral".

We must temporarily leave it to the reader to work out the details from this point. We will welcome reader comments.

### Strategy of Proving Existence of a Certain Map Between Tuples and Paths in "Spirals"

Probably the most direct approach to a proof of the $3x + 1$ Conjecture using "spiral"s would be by proving there exists a one-one onto map between tuples in tuple-sets and finite paths in the infinite set of "spiral"s whose base element is 1. Such a proof would prove the $3x + 1$ Conjecture because the set of all tuple-sets represents the set of all finite computations by the $3x + 1$ function.

We must confess that we spent an inordinate amount of time trying to discover such a mapping by trying to figure out where, in the infinite set of "spirals" having base element 1, each tuple "belonged" — in other words, by trying to map tuples onto paths in this set of "spirals". A much better idea *initially* (which, after the fact, is obvious) is to proceed in exactly the opposite direction: to try to discover where, in the set of all tuple-sets, each infinite set of "spiral"s (regardless of its base element) "belongs" or "fits in". If we view the matter in this way, we see immediately that *each element (except a multiple of 3) of each tuple in each tuple-set is the base element of an infinite set of recursive "spirals"!* (Recall that multiples of 3 only occur at level 1 in any tuple-set.) An awesome structure, indeed! Of course, we still retain the converse goal, namely, that of discovering where, in the set of infinite "spirals" whose base element is 1, each tuple in each tuple-set "belongs" or "fits in", always keeping in mind that, if a counterexample exists, this goal will not be achievable, in which case our goal then becomes that of discovering where, in the set of all possible infinite sets of "spiral"s, each tuple in each tuple-set belongs, or fits in.

In any case, we can say, now that we know how recursive "spirals" fit into tuple-sets, that recursive "spiral"s show how tuple-sets are related to each other, and that answers a question that has confronted us since we first discovered tuple-sets.

As an aid in conceiving the structure resulting from the insertion of an infinite set of "spirals" at each element of each tuple in each tuple set defined by a sequence of length ≥ 2, we may imagine each infinite set of "spiral"s as lying in a plane perpendicular to the page, the page containing (some of) the tuples in a tuple-set. The base element (which is a tuple element) of each infinite set of spirals is, in turn, an element of another infinite set of "spiral"s, namely, that established by the tuple element mapped to by the base element tuple element in accordance with the sequence of exponents that define the tuple-set.

But to show where tuples fit into "spirals", we need to "split the nodes", i.e., "split the base elements" in each infinite set of "spirals" having a given base element. That is, we must remember that each element of a tuple in a tuple-set (except the first element) is mapped to by only one element and, in turn, may or may or may not have one or more extensions in that tuple-set. In an infinite set of "spirals", on the other hand, an infinite set of elements maps to each base element (node) (unless the node is a multiple of 3). So in order to bring the form of such an infinite set into closer conformity with the form of tuple-sets, we must "split" each node (base element) *y* into an infinite set of nodes each of which is equal to *y*. An example is shown in the following figures.

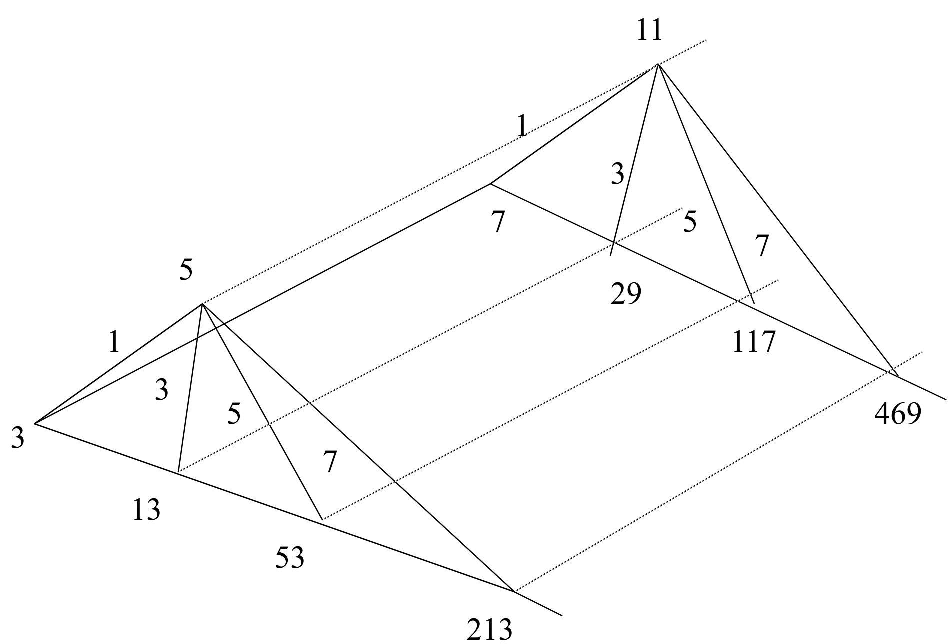


**Fig 7. Example of the merging of tuple-sets and recursive "spiral"s: first stage.**
1, 3, 5, 7, ... are exponents of 2.

The line 5, 11, ... represents the top row of every 2-level tuple-set defined by an odd exponent. The other lines running diagonally into the page to the right represent bottom rows of 2-level tuple-sets.

Thus, e.g., we see the first two tuples in each of the tuple-sets defined by $A$ = {1}, {3}, {5}, {7}. These tuples are, respectively, <3, 5> and <7, 11>; <13, 5> and <29, 11>; <53, 5> and <117, 11>, and <213, 5> and <469, 11>.

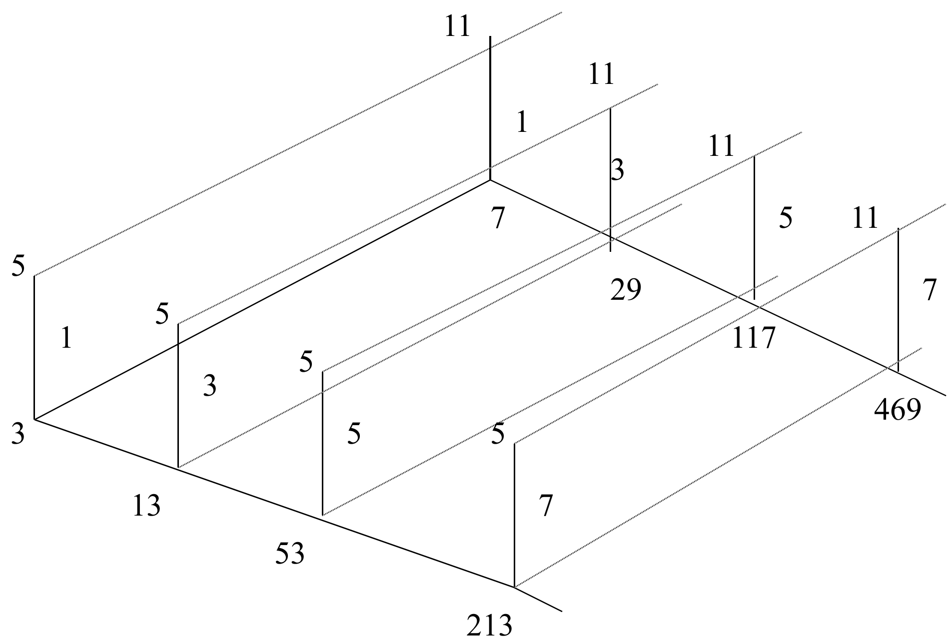


**Fig 7. 5. Example of the merging of tuple-sets and recursive "spiral"s: second st**age, **showing the "splitting of nodes" in recursive "spiral"s (see Fig. 7.).**

## Finding "Locations" of Range Elements in Tuple-sets

We can now attempt to correlate the "locations" (defined below) of a given odd, positive integer $u$ in the set of all tuple-sets, with its "locations" in the set of all recursive "spirals" relative to a given base element, in particular, the base element 1. If this correlation allows us to show that every assumed counterexample has a "location" in the infinite set of recursive "spirals" having base element 1 (which would be a contradiction) then we will have proved the $3x + 1$ Conjecture.

To begin our search for this correlation, let us ask a seemingly meaningless question, namely, "*Where* is the integer $n$ mod $m$?" To show that, from the right point of view, the question is not meaningless, we recall the fundamental fact of elementary congruence theory, namely, that for each non-negative integer $n$, and for each modulus $m$ (also a non-negative integer), there exists an $r$ such that $n \equiv r$ mod $m$, where $r$ is a minimum residue mod $m$. This congruence in turn means that there exists a non-negative integer $k$ such that $n = r + km$.

We can therefore say that, for each modulus $m$, each $n$ has a "location" that is defined by the ordered triple ($r$, $k$, $m$). (This definition is a case of "what" = "where": *what* the value of a variable $n$ is, is a function of *where* it is, i.e., of its location ($r$, $k$, $m$). The benefits of assigning geometric locations to numbers is an old one in mathematics, going back to the beginnings of analytic geometry in the 1600s, and further extended through the use of the complex plane, beginning in the early 1800s, and given new impetus by Minkowsi's *Geometry of Numbers* (1896), which set forth a way of assigning coordinates to the elements of a module.)

Now, the distance functions established in Lemmas 1.0 and 1.1 in effect tell us that each sequence $A$ of exponents, $A = \{a_2, a_3, ..., a_i\}$, establishes a sequence of moduli, namely, the moduli

$$m_i = 2 \cdot 3^{i-1}$$

$$m_{i-1} = lcm(2 \cdot 3^{i-2}, 2^{a_i})$$

$$m_{i-2} = lcm(2 \cdot 3^{i-3}, 2^{a_{i-1}} 2^{a_i})$$

.

.

.

$$m_1 = 2 \cdot 2^{a_2} 2^{a_3} \dots 2^{a_i}$$

where *lcm* denotes the least common multiple.

Let $y$ be a range element in any tuple in the tuple-set $T_A$. Then for each of these moduli, $y$ has an address, ($r_j$, $k_j$, $m_j$), where $1 \le j \le i$. This is the same thing as saying that $y$ is an element of many tuples in $T_A$, which is the same thing as saying that $y$ is an element of many "spiral"'s defined by elements of tuples in $T_A$. (Note: as of yet, we do not know a general formula for computing the $r_j$ except in the case of $j = i$.)

Thus we would like to define a function $F$ which, for any tuple-set $T_A$, and for any range element $y$, will return all the locations of $y$ in $T_A$, i.e., all the tuples containing $y$, and the index in each tuple of $y$, each such location being simultaneously the location of an element in a recursive "spiral". Formally, $F(A, j, y) = (r_j, k_j, m_j)$, where $A = \{a_2, a_3, ..., a_i\}$, $1 \le j \le i$, $y$ is any range element, and $(r_j, k_j, m_j)$ is as defined above. Clearly, any given $y$ has an infinite number of locations, even if $i$ is fixed, because $y$ is mapped to by an infinity of exponents, hence $y$ is an element of a different tuple in each of an infinity of tuple-sets.

Let us consider a few examples of the function $F$. $F(\{a_2\},1,1)$, where $a_2$ is any even exponent, $= (1, 0, 2 \cdot 2^{a_2})$; $F(\{a_2\},2,1) = (1, 0, 2 \cdot 3^{2-1})$. (Note that a value of ($r_j$, 0, $m_j$) means that $r$ is a minimum residue mod $m_j$.)

For $A = \{2, 1, 1\}$, $j = 3$, $y = 29$, we have $F(A, 3, 29) = (11, 1, 2 \bullet 3^{(3 - 1)})$

Now let us ask: What is the unique characteristic of any counterexample? Answer: that it never appears in the infinite set of "spiral"s whose base element is 1. Thus if we can use the function $F$ to show that every assumed counterexample is an element of the infinite set of spirals having base element 1, this contradiction will give us a proof of the $3x + 1$ Conjecture.

We conclude with the observation that each element $y$ in the infinite set of recursive "spiral"s relative to a given base element, also has a "location" if it is in that infinite set — a location that can be specified by the sequence of exponents that lead from the base element to $y$. Note that this sequence is the *reverse* of the sequence that would lead to the base element from $y$ in a tuple-set.

Thus, we can label each of the elements in an infinite set of recursive "spiral"s relative to a base element, by one or more of the "location"s of that element in one or more tuple-sets, and, conversely, we can label any element of a tuple by one or more of the "location"s of that element in the infinite set of recursive "spiral"s relative to one or more base elements.

## Strategy of "Filling-in" of Intervals in the Base Sequence Relative to 1

The following conjecture is clearly equivalent to the $3x + 1$ Conjecture:

**Conjecture 4.** Every interval in the base sequence relative to 1, i.e., in the sequence {1, 5, 21, 85, 341, ..., }, is eventually filled by elements that map to 1.

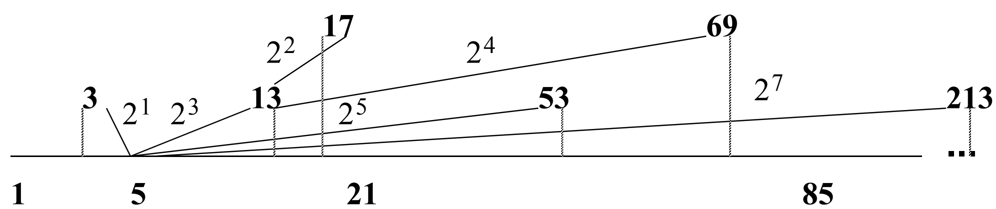


**Fig. 5. Illustration of part of the "filling-in" process.**

Before we discuss possible proofs of this conjecture, we will state several conjectures each of which, if true, implies the truth of Conjecture 4. These conjectures are as follows.

**Conjecture 5.** Let $y$ be any base element, and let $x$, $x'$ be successive elements of the base sequence relative to $y$. Then all elements of the interval between $x$ and $x'$ are filled by "spiral" elements at no higher level than $3x + 2$ relative to $y$. In other words, all elements of the interval map to $y$ in no more than $3x + 2$ iterations of the $3x + 1$ function.

Thus, e.g., let $y = 1$ be our base element. We are asking about the filling in of the intervals in the base sequence relative to 1, i.e., the sequence {1, 5, 21, 85, 341, ...} Conjecture 5 says that each element (odd, positive integer) in the interval between, e.g., $x = 21$ and $x' = 85$ will be found to map to $y = 1$ in no more than $3x + 2 = 3(21) + 2 = 65$ iterations of the function. (Note that this number of iterations is larger than the 41 required for 27 to map to 1, and 27 is considered to require an "unusually large" number of iterations for its size. However, We have not checked all odd, positive integers between 21 and 85.)

**Conjecture 6**. An infinity of intervals of the base sequence {1, 5, 21, 85, 341, ... } are filled in solely by odd, positive integers that map to 1.

Note that Conjecture 6 differs from Conjecture 4 in that the latter states that *all* intervals are filled in by odd, positive integers that map to 1, whereas the former merely specifies an infinity of intervals. The reason the truth of Conjecture 6 implies the truth of the $3x + 1$ Conjecture is that the presence of an infinity of intervals $I$ each of which is filled in solely by odd, positive integers that map to 1 implies that counterexample sequences are forced to "leap over" these $I$, and this is prohibited by Lemma 14.0 (see under "Three Important Lemmas" on page 41).

Observe that a proof of Conjecture 6 need not specify what the first such filled-in interval is (we know, by computer test, that at least the first 26 intervals are filled in solely by odd, positive integers that map to 1[1]), nor does the proof need to specify the distance between any two of the infinity of intervals.

**Conjecture 7.0** If a counterexample exists, then at least one interval in the base sequence relative to 1 is filled in solely by counterexamples.

The reason the truth of this conjecture implies the truth of the $3x + 1$ Conjecture is that such a filled-in interval would force it to be "leaped over" by successive elements of a higher-level sequence that maps to 1, thus contradicting Lemma 14.0 (see under "Three Important Lemmas" on page 41).

**Plausibility Argument for the Truth of Conjecture 7.0**

For each base element, whether counterexample or not, the structure of the infinite set of recursive "spiral"s relative to that base element is "similar" to the structure of the "spiral"s for any other base element — "similar" based on the properties described in Table 6, "Some important properties of recursive "spiral"s," on page 33.

We have heard, from a source we consider reliable, that as of Nov., 1998, the $3x + 1$ Conjecture had been verified for all integers (even and odd) up to about 56 quadrillion[2], i.e., $56 \cdot 10^{15}$.

Now the number of numbers (even and odd) in successive intervals of the base sequence relative to 1 are $2^2, 2^4, 2^6, \ldots$ This means that, **at least the first 26 intervals in the base sequence are known to be filled with odd, positive integers that map to 1**. The proof is:

$$((2^2)^1) + ((2^2)^2) + ((2^2)^3) + \ldots + ((2^2)^{26}) = \frac{(2^2)^{27} - 1}{2^2 - 1} - 1 < 2^{54} < 56 \cdot 10^{15}$$

But since, for *any* base element $y$, the structure of the infinite set of recursive "spiral"s relative to $y$ is "similar" (in the sense stated above) to the structure for any other base element $y'$, it seems plausible that a similarly long sequence of intervals must be completely filled in by integers (even and odd) that map to the smallest counterexample that is a base element. But this is

1. See "Plausibility Argument for the Truth of Conjecture 7.0" on page 40.
2. The web site www.ericr.nl/, which we consider reliable, reported in June, 2006 that the number was then now more than $48.4 \cdot 10^{16}$.

prohibited by Lemma 14.0 (see under "Three Important Lemmas" on page 41), in particular, by this lemma as applied to the base sequences of range elements in the first 26 intervals.

**Three Important Lemmas**

We now state three lemmas that are important, perhaps essential, for proving any of the Conjectures 4.0, 5.0, 6.0, or 7.0.

The first states, informally, that no element of a higher level sequence is "wasted" by being mapped "on top of" one of the base sequence elements. In other words, we never need to worry about "overfilling" an interval. This is Lemma 13.5, whose formal statement is as follows:

**Lemma 13.5**. *For all base sequences, and for all levels $i \geq 2$ relative to a base sequence, no element of a level i sequence is an element of the base sequence. Thus, in particular, no level i element, $i \geq 2$, is an element of the base sequence relative to 1, i.e., of the base sequence {1, 5, 21, 85, 341, ... }.*

The second lemma states, informally, that no interval of the base sequence is "leaped over" by successive elements of a higher level sequence once that sequence gets started. This is Lemma 14.0, whose formal statement is as follows:

**Lemma 14.0.** *For any "spiral" at any level $i \geq 1$, the sequence of elements of the "spiral" map to successive intervals of the base sequence.*

The third lemma states, informally, that an infinity of successive intervals of the base sequence can always be filled with an arbitrary number of elements that map to 1 (up to the number of elements in each interval). This is Lemma 17.0, whose formal statement and proof[1] are as follows. We begin with a lemma.

**Lemma 17.0.** *For all $m \geq 2$, and for all $k \geq 2$, there exists an infinity of consecutive intervals in the base sequence relative to 1, i.e., in the sequence {1, 5, 21, 85, 341, ... }, such that each such interval contains $((m^k - 1)/(m - 1) - 1)$ numbers that map to 1.*

**Proof**: The infinite set of recursive "spiral"s relative to base element 1 can be regarded as an infinitary tree, since every non-multiple-of-3 in the set is mapped to by an infinity of non-multiples-of-3 (Lemmas 5.0, 15.85), and since there can be no cycles in the set because all numbers map to 1. (It is easily shown that only every third element of a recursive "spiral" is a multiple of 3. Thus, e.g., in the recursive "spiral" {7, 29, 117, 469, 1877, 7509, ...} (whose base element is 11), 117 and 7509 are the only multiples of 3 in the first six elements.)

Therefore, for all $m \geq 2$, we can select the first $m$ non-multiples-of-3 in each "spiral" that map to the base element $y$, where "first" here means in the order of increasing values of the $x$ that map to $y$. We thus get an $m$-ary tree. (The reader may find it helpful to refer to Table 7 while reading the following.)

Level $k = 0$ contains solely the number 1, hence no "spiral";

---

1. The proof of this lemma is given here because the lemma is not at present stated and proved in any of our other papers on the $3x + 1$ problem.

Level $k = 1$ contains solely the "spiral" {1, 5, 21, 85, 341, ... }; higher levels of "spiral"s will be filling in intervals in this "spiral", so we do not count it;
Level $k = 2$ contains $m$ "spiral"s. (In Table 7, $m = 3$ "spiral"s.);
Level $k = 3$ contains $m \bullet m$ "spiral"s. (In Table 7, $m \bullet m = 3 \bullet 3 = 9$ spirals.);
etc.

**Table 7: "Spiral"s for $m = 3$, $k = 3$**

| level k | No. of infinite "spiral"s | Infinite "spiral"s | Base element of "spiral"s |
|---|---|---|---|
| 0 | 0 | | |
| 1 | 1 | {1, 5, 21, 85, 341,...} | 1 |
| 2 | 3 | {3, 13, 53, 213, 853, ...}<br>{113, 453, 1813, 7253, ...}<br>{227, 909, 3637, 14549, ...} | 5<br>85<br>341 |
| 3 | 9 | {17, 69, 277, 1,109,...}<br>{35, 141, 565, 2261,...}<br>{1137, 4549, 18197, 72789,...}<br>{75, 301, 1205, 4821, ...}<br>{2417, 9669, 38677, 154709, ...}<br>{4835, 19341, 77365, 309461,...}<br>{151, 605, 2421, 9685, ...}<br>{4849, 19397, 77589, 310357,...}<br>{9699, 38797, 155189, 620757, ...} | 13<br>53<br>853<br>113<br>1813<br>7253<br>227<br>3637<br>14549 |

Thus the number of different "spiral"s is given by $m + m \bullet m + m \bullet m \bullet m + ... + m \bullet m \bullet m \bullet ... \bullet m$ [$k$-1 $m$'s in the last term], which, by a basic fact of elementary algebra = $((m^k - 1)/(m - 1) - 1)$ spirals. Lemmas 13.5 and 14.0 assure us that there will be no duplicate numbers in any two different "spiral"s. □

Thus, as the reader can see from Table 7, for $m = 3$, $k = 3$, we have $(3^3 - 1)/(3 - 1) - 1 = 26/2 - 1 = 12$ "spiral"s, ignoring the "spiral"s for $k = 0$ and 1.

Is this lemma sufficient for a proof of Conjecture 6, and hence of the $3x + 1$ Conjecture? The following facts are relevant to an answer to this question.

First, by Lemma 5.0 we know there is an infinite set of recursive "spiral"s for each counterexample, if any exists. Can there be an infinite number of *disjoint* such sets, each consisting (solely) of counterexamples? Lemma 17.0 implies the answer is no, because otherwise there would be an infinite number of numbers in each of an infinite number of intervals in the base sequence relative to 1, which is impossible.

Second, it is not possible that in any "spiral" in the infinite set of all such "spiral"s relative to any base element, all but a finite number of elements are cycle elements. (If it were, then we might not have an *m*-ary tree for all $m \geq 2$ in the case of an infinite set composed of counterexamples in which there were an infinite number of cycle elements.) However, Conway or Thompson proved already by 1973 that only a finite number of cycles is possible,[1] and therefore at most a finite number of numbers in any "spiral" can be cycle elements.

Given these two facts, it might be possible to prove Conjecture 6, and hence the $3x + 1$ Conjecture, by arguing that since each non-multiple-of-3 in the infinite set of recursive "spiral"s relative to the base element 1 adds a number mapping to 1 to each of an infinite sequence of successive intervals in the base sequence relative to 1, the assumption of a counterexample implies that in at least one interval, a counterexample must be "mapped on top of" by a number mapping to 1, contradicting Lemma 13.5.

Further thoughts on the possibility of proving the 3x + 1 Conjecture using Lemma 17.0 are given in "Appendix F — Further Thoughts on the "Filling-in" Strategy" on page 92.

**A Difficulty With the "Filling-in" Strategy**

It is tempting to try to implement the "Filling-in" Strategy by arguing that since there is an infinite set of range elements *y* in the infinite set of recursive "spiral"s relative to 1 (the proof is simple), and since each range element *y* is itself mapped to by an infinite set of recursive "spiral"s relative to *y*, then we only need the show that it is always possible to add at least one more element to an infinite set of intervals in the base sequence relative to 1. (Recall that the base sequence relative to 1 is {1, 5, 21, 85, 341, ...}; interval 1 in this sequence consists of the number 3; interval 2 consists of the numbers 7, 9, 11, 13, ..., 17, 19; interval 3 consists of the numbers 23, 25, 27, ..., 81, 83; etc.). Thus, in particular, there can be no fixed number of counterexample elements in each of an infinite set of intervals.

The problem with this implementation is the following.

Suppose there exists a non-counterexample range element $y_1$ that is mapped to by a base sequence (i.e., a "spiral") that places a number in interval 2, a number in interval 3, a number in interval 4, etc. (We are not concerned here with determining which non-counterexample range elements, if any, fulfill this condition, or any of the following similar conditions, since the exact elements are not relevant to the point we are trying to make.)

Suppose, further, that there exists a non-counterexample range element $y_2$ that is mapped to by a base sequence (i.e., a "spiral") that places a number in interval 3, a number in interval 4, a number in interval 5, etc.

Suppose, further, that there exists a non-counterexample range element $y_3$ that is mapped to by a base sequence (i.e., a "spiral") that places a number in interval 4, a number in interval 5, a number in interval 6, etc.

Etc.

And suppose, finally, that there are no other non-counterexample range elements.

We see immediately that for any number $n \geq 1$ of non-counterexamples we name, there exists an infinity of successive intervals each of which contains *n* non-counterexamples. And yet we also see that no interval is filled with non-counterexamples. Informally, we say that the problem

1. We do not have the reference. The existence of the proof was mentioned in a lecture by H. Hasse at the University of New Zealand, New Zealand, on 10/26/73.

is that the first elements of the base sequences "move forward" too rapidly. The next few sub-sections contain a discussion of possible ways to overcome this problem.

**The Problem of the Rate of Increase of the Smallest Element at Each Level**

In thinking about a proof of any of the Conjectures 4.0, 5.0, 6.0, 7.0, we confront the problem of the rate of increase of the the smallest element at each level.

If $x$ yields $y$ in one iteration of the $3x + 1$ function via the exponent $a_j = 1$, then $x < y$ — i.e., $y = (3x + 1)/2^1$ implies $x < y$— and $a_j = 1$ is the only exponent when this occurs. This is in our favor, of course. Otherwise, $x$ will be greater than the base element, except in the case when $y = 1$. We need to be sure that $x$ will never be "too large". Let us consider the "spiral" whose base element is 13, namely, the sequence {17, 69, 277, 1109, 4437, 17749, ....}, which produces 13 via the exponents 2, 4, 6, 8, 10, 12, ... respectively. 17 is larger than 13, but not a great deal larger.

**An Upper Bound on the Rate of Increase of the Smallest Element at Each Level**

Let $s_n$, $n \geq 1$, be the smallest element of level $n$ in the infinite set of recursive "spiral"s relative to base element 1. Our goal here is to find an upper bound on the rate of increase of the $s_n$.

Let $y = s_n$ for some $n$. The worst case occurs if for an arbitrary number of levels $n + 1, n + 2, ..., s_{n+1}, s_{n+2}, ...,$ is a multiple of 3, and the mapping from each such $s_{n+k}$, $k \geq 1$, to level $k$ - 1 is via the exponent 2. Then, to begin with, the smallest range element $y'$ (i.e., the element that is not a multiple of 3) at level $n + 1$ is given by

$$y' = \frac{y2^4 - 1}{3}$$

and, as the reader can check, the smallest range element $y'^{\cdots}{}'$($k$ primes) at level $n + k$, $k \geq 1$, is given by

$$y'^{\cdots}{}' = \frac{y2^{4k} - 2^{4(k-1)} - 3 \bullet 2^{4(k-2)} - 3^2 \bullet 2^{4(k-3)} - \ldots - 3^{k-1}}{3^k}$$

Letting $y = 1$, we can compare the value of $y'^{\cdots}{}'$ with the value of the first element $y_k$ of the $k$th interval in the sequence of intervals defined by the base sequence relative to 1, namely, {1, 5, 21, 85, 341, ...}. From our remarks earlier in this subsection, we see that:

$$y_k = \left(1 + \left(\frac{(2^2)^k - 1}{2^2 - 1} - 1\right)\right)$$

Whether or not $y'^{\cdots}{}'$ is too large or not will have to be determined through further investigation. A lemma that might be useful here is the following:

**Lemma 18.0**. *Let S denote the level 1 "spiral" (i.e., the base sequence) relative to 1, {1, 5, 21, 85, 341, ... }. Let x, x′ be successive elements of the sequence S. Then the set of odd, positive integers lying between x and x′, i.e., the set {x + 2, x + 4, ..., x′– 4, x′– 2} we call an interval of the* *sequence S. The intervals of S are numbered 1, 2, 3, ... Thus the odd positive integer 3 is the sole element of interval 1; the odd, positive integers 7, 9, 11, 13, 15, 17, 19, are all the elements of interval 2, etc.*

*Let y be a range element in the interval $k \geq 2$. Then y is mapped to, in one iteration of the 3x + 1 function, by a range element that lies either in interval $k - 1$, k, or $k + 2$.*

**Proof**: It is easily shown that, in any "spiral", if the first multiple-of-3 (no multiple-of-3 is a range element) is the first element of the "spiral", then the next multiple-of-3 is the fourth, and the next the seventh, etc. If the first multiple-of-3 is the second element of the "spiral", then the next multiple-of-3 is the fifth, and the next the eighth, etc. If the first multiple-of-3 is the third element of the "spiral", then the next multiple-of-3 is the sixth, and the next the ninth, etc.

For our purposes, the two worst cases we must consider are

(1) *y* is mapped to by odd exponents;
the first element of the base sequence relative to *y* is not a multiple-of-3;
*y* is the first element of interval *k*.

In this case, *x* is in interval $k - 1$, because $(3x + 1)/(2^1) = y$ implies that *x* is about 2/3 *y*.

(2) *y* is mapped to by even exponents;
the first element of the base sequence relative to *y* is a multiple-of-3;
*y* is the last element of interval *k*.

In this case, *x* is in interval $k + 2$, because $(3x + 1)/(2^4) = y$ implies that *x* is about 16/3 *y*, which places *x* in interval $k + 2$, by Lemma 11.0. □

**Can Tuple-sets Provide a Means of Solving the Rate-of-Increase Problem?**

We conclude our discussion of the rate-of-increase problem by considering whether tuple-sets might provide us with a means of solving the problem.

In our discussion, "Strategy of Proving There Is No Minimum Counterexample" on page 25, we defined the less-to-greater property of an exponent sequence. Recall that an exponent sequence *A* has this property if, for any tuple *t* in a tuple-set $T_A$ defined by such a sequence, the last element of *t* is greater than the first. In an infinite set of recursive "spiral"s relative to some base element *y* — an infinite set that contains the last element of *t* — *t* occurs as a sequence of "spiral" elements. Here, the last element of *t* is at some level *k*, and the first element is at a level $k + i - 1$, where *i* is the number of elements in *t*. So, given a tuple *t* with the less-to-greater property, we can find a sequence of elements in the infinite set of "spiral"s containing *t* and we can say that this sequence has a *greater-to-less* property.

But the existence of such sequences of elements solves the problem of $\mathbf{y}'^{\cdots\prime}$ growing too rapidly! For it means that we can always find a greater-to-less sequence of arbitrary length in the infinite set of recursive "spiral"s relative to the base element 1. (Proof: by Lemma 10.0, every tuple-set contains an infinity of tuples whose elements map to 1, regardless of whether a counterexample exists or not. Therefore, for any *i*-level exponent sequence having the less-to-greater property, we can always find an infinity of tuples that define inverse paths having the greater-to-less property in the infinite set of recursive "spiral"s relative to the base element 1. And we know that for each $i \geq 2$, there exists at least one exponent sequence having the less-to-greater property, namely, the sequence $A = \{1, 1, ..., 1\}$ (($i$ - 1) 1s). )

However, there is a catch. In order to use greater-to-less sequences, we must prove that the last element *y* of the tuple *t* (i.e., the first element of the corresponding greater-to-less sequence of elements in the set of infinite recursive "spiral"s relative to 1) is never so large that the number of elements in *t* (in the inverse path) is insufficient to fill in at least one interval as desired. In other words, suppose that *y* occurs at level *n*, and suppose the inverse path contains *i* elements. If we fill an interval *I* in the base sequence relative to 1 with *m* elements via elements at level $n + 1$, and then with *m* more via elements at level $n + 2$, etc., then we are assured of being able to fill the interval with $m(i - 1)$ elements that map to 1. Is that sufficient to fill the interval completely?

**A Possible Solution to the Rate-of-Increase Problem**

Thirteen, which is a non-counterexample at level 2 (i.e., level 2 relative to base element 1), is mapped to, in one iteration of the $3x + 1$ function, by 17, 69, 277,... , and thus there is a level 3 non-counterexample in intervals 2, 3, 4, ... In fact, as the reader can confirm for himself, there is more than one level 3 non-counterexample in each of these intervals.

Consider the set of all level 3 non-counterexample range elements in their respective intervals. *Each* of these is mapped to by an infinity of level 4 non-counterexample elements. Lemma 18.0 assures us that two successive level 4 non-counterexample range elements are at most two intervals apart.

Now consider, similarly, the set of all level 4 non-counterexample range elements, and the set of all level 5 non-counterexample range elements, and ... Can we show that the required filling-in process follows? In particular, can we show that it follows from the fact that there is more than one level 2 *range* element in an infinity of intervals, and that Lemma 17.0 (under "Three Important Lemmas" on page 41) implies that for *each* of these range elements, and for all $m \geq 2$, and for all $k \geq 2$, there exists an infinity of consecutive intervals such that each interval contains $((m^k - 1)/(m - 1) - 1)$ non-counterexample elements?

(Informally, as we progress through intervals 1, 2, 3, ..., we progress through a succession of level 2 range elements, *each of which* is the root of an *m*-ary tree of range elements that *each* "place" an infinity of range elements in an infinity of subsequent intervals. Thus, the rate-of-increase problem might be solved by the fact that we can increase the number of non-counterexample elements in an infinity of successive intervals simply by allowing *m* and *k* for the level 2 non-counterexample range elements in any *fixed, finite succession* of intervals 1, 2, 3, ..., *n*, ) to increase without bound.)

## Strategy of "Filling-in" of Residue Classes

The top rows of all 2-level tuple-sets are {1, 7, 13, 19, ... } and {5, 11, 17, 23, ... } (Lemma 3.057). The elements of the first row are mapped to by all even exponents, and the elements of the second by all odd exponents. These facts, and Lemma 15.0, suggest a "filling-in" strategy for tuple-sets that is analogous to the one described above for recursive "spiral"s. For, since, by Lemma 15.0, the parity of exponents mapping to any base element of a "spiral" alternates, this means that any "spiral" in the infinite set of "spirals" whose base element is 1, "fills in" an infinite number of "locations" in the above two rows. For example, the base sequence relative to 1 is {1, 5, 21, 85, 341, ... }. So 1 and 85 "fill in" the locations 1 and 85 in {1, 7, 13, 19, ... }. 5 and 341 fill in the locations 5 and 341 in {5, 11, 17, 23, ... }. Elements of every higher-level "spiral" fill in additional locations in these two rows. Each "spiral" fills in an infinite number of locations in each of the two rows. Lemma 15.85 generalizes Lemma 15.0 to higher-level rows. Thus, informally, we can think of the elements of each "spiral" as "winding" endlessly through ever increasing elements of a sequence of reduced residue classes mod $2 \cdot 3^{i-1}$.

Similar questions regarding this filling-in process arise as for the recursive "spiral"s case.

### Strategy of Using a Topology Defined on "Spiral"s

It is natural to wonder if defining an appropriate topology on "spiral"s or their elements or finite paths in "spiral"s might lead us to a proof of the $3x + 1$ Conjecture. We might begin by taking as our set of points the set of all range elements, and then, for each range element $y$, defining a neighborhood of $y$ as the set of all range elements mapping to $y$ in $\leq n$ iterations, where $n$ is a non-negative integer.

Unfortunately, the neighborhood system thus defined fails one of the conditions for a topological space, namely the condition:

If U is a neighborhood of $y$, and $U \subset V$, then V is a neighborhood of $y$.

For, let $y = 1$, and let U be a neighborhood of 1. Let $V = \{U \cup \{z\}\}$, where $z$ is a counterexample. Then clearly $U \subset V$ but, by definition, V is not a neighborhood of U.

The condition is not violated, of course, if we take as our set of points, the set of range elements mapping to a given range element, $y$, e.g., $y = 1$. But in this case, there is a separate topology for the set of range elements mapping to 1, and for each connected set of counterexamples, where a connected set of counterexamples has the property that each element maps to another element of the set, or is mapped to by one or more other elements of the set.

We believe that an investigation of topologies defined on "spiral"s or their elements would be worthwhile.

### Strategy of the Boundary Between Non-Counterexamples and Counterexamples

The motivation for this strategy is the simple fact that whether or not a counterexample exists, a portion of the infinite set of "spiral"s relative to the base element 1, remains the same.

We begin with the observation that, if a counterexample exists, then the intervals in *every* "spiral" in this infinite set contain an *infinity* of non-counterexamples. The reason is that the elements of each "spiral" and the intervals between them constitute all odd, positive integers ≥ the first element of the "spiral". Since, if a counterexample exists, an infinity of counterexamples exists, the observation follows.

Therefore, if a counterexample exists, in each "spiral" in the infinite set of "spiral"s relative to the base element 1, there is a first interval, and a first element of that interval, that is a counterexample.

Suppose we mark with black all "spiral" elements in the above infinite set, and all interval elements in each "spiral", that map to 1. We mark with red all counterexample elements in all "spiral" intervals. Call the set of elements marked in black, $B$. We then ask how it is possible for counterexamples to exist at all, given that beginning with any "spiral" element in $B$, it is not possible to tell, by applying successive inverses of the $3x + 1$ function, whether or not a counterexample exists. Putting it another way, we can in principle represent the infinite set of "spiral"s with a diagram, defining some appropriate scaling factor for the distances between "spiral" elements and base elements. The portion of the diagram representing the set $B$ will be exactly the same whether or not a counterexample exists. We ask how exactly does the remainder of the diagram differ for the two cases? In short, where do the two cases — (1) no counterexamples exist, (2) counterexamples exist — "begin to diverge" from $B$?

# Generalizations of the 3*x* + 1 Function

Numerous generalizations of the 3*x* + 1 function have appeared in the literature. Here we give only two because it appears, on the basis of limited examination, that the tuple-sets structure, including the distance functions, apply to them.

## The 3*x* – 1 Function

Here, division in each iteration is by

$$2^{ord(3x - 1)} .$$

The negative of the range elements of this function are the range elements of the 3*x* + 1 function applied to the odd, negative integers.

It is well-known that at least three cycles exist in this function. They involve 1, 5, and 17.

### Description of Tuple-sets for the 3x –1 Function

<1, 1>
<5, 7>
<9, 13>
<13, 19>
<17, 25>
...

<7, 5>
<15, 11>
<23, 17>
<31, 23>
<39, 29>
...

<1, 1, 1>
<9, 13, 19>
<17, 25, 37>

## The 3*x* + 3*^k^* Function

Here, $k \geq 0$. Each *k* defines a separate function. The 0 case of course gives us our familiar 3*x* + 1 function. Division in each iteration is by

$$2^{ord(3x + 3^k)} .$$

As far as we know, this class of functions was first defined in 1993 by Barry Brent (email 6/27/02). The paper is accessible on Brent's web site, www.home.earthlink.net/~barryb0/.