The Structure of the 3*x* + 1 Function:

An Introduction

by

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Introduction

Statement of Problem

For x a positive odd integer, set

$$C(x) = \frac{3x+1}{2^{ord_2(3x+1)}}$$

where $ord_2(3x + 1)$ is the largest exponent of 2 such that the denominator evenly divides the numerator. Thus, e.g., C(17) = 13, C(13) = 5, C(5) = 1. The $3x + 1$ Problem, also known as the $3n + 1$ Problem, the Syracuse Problem, the Collatz Problem, Ulam's Problem, Kakutani's Problem, and Hasse's Algorithm, asks if all repeated iterations of C eventually terminate at 1. The conjecture that they do is hereafter called the $3x + 1$ Conjecture. We call C the $3x + 1$ function; note that $C(x)$ is by definition odd.

Other equivalent formulations of the 3x + 1 problem are given in the literature; we base our formulation on the C function (following Crandall) because, as we shall see, it brings out certain structures that are not otherwise evident.

Purpose of This Paper

This paper presents two remarkably simple structures underlying the $3x + 1$ function, namely, tuple-sets, which describe the structure of the function "in the forward direction", and recursive "spiral"s, which describe the structure of the inverse function — the structure "in the backward direction". The paper also presents several possible strategies for solving the $3x + 1$ Problem that are based on these structures. These are described under "Possible Strategies for Proving the 3x + 1 Conjecture Using Tuple-sets" on page 19, and "Possible Strategies for Proving the 3x + 1 Conjecture Using "Spiral"s" on page 44.

An Important Fact to Keep in Mind

This paper sets forth numerous results concerning tuple-sets and recursive "spiral"s. However, in all but a few cases, these results apply equally whether a counterexample to the $3x + 1$ Conjecture exists or not. (See details under "Preliminary Discussion of Strategies" on page 19.) That does not necessarily mean that these results are "useless" as far as a proof of the Conjecture is concerned, but it does mean that the reader should be cautious about assuming that each new result necessarily brings us closer to a proof. Of course, the $3x + 1$ function is interesting in its own right, and in this regard each result that adds to our understanding of this function, has value.

On the Style of This Paper

It will be obvious at a glance that this is not a formal paper intended for publication in a journal. Early experience showed that most prospective readers are hard-pressed for time, and typically can only afford to spend a few minutes browsing the text. These persons seldom have the time even to search the text for the definition of a symbol. Furthermore, they have a variety of backgrounds: undergraduate and graduate mathematics, computer science, and electrical engineering majors; and academics and industrial professionals in these subjects. We have therefore done his best to enable the reader to acquire at least a superficial understanding of the underlying concepts and logical arguments as rapidly as possible. As a result, several rules governing the correct style of formal mathematical papers have been broken. For example, whereas in a formal paper one would denote the frequently mentioned sequence {1, 5, 21, 85, 341, ... } (the set of numbers mapping to 1 in one iteration of the $3x + 1$ function) by a symbol, say, S_1, and the sequence of intervals defined by S_1 by another symbol, say, I_1, we have instead frequently written out {1, 5, 21, 85, 341, ... } itself, or referred to it by its formal name (in this paper), *the base sequence relative to 1*.

Numbering of lemmas and figures is the same in this paper as in previous versions of the paper — hence not necessarily consecutive, due to the addition or deletion of lemmas and figures in various revisions.

In the interest of conciseness most proofs of lemmas are omitted in the present paper. For a fuller discussion, including proofs, see "The Structure of the $3x + 1$ Function", available on the web site www.occampress.com.

Some of the results supporting the strategies already exist in the literature. We have tried to indicate these wherever possible. However, as far as we know, the value, as far as suggesting strategies for a solution to the Problem are concerned, of the "graphical" presentations of the two structures described in this paper has not been recognized.

This paper is a work in progress, and thus may contain errors. We will appreciate readers notifying him of any they find.

The reader is encouraged to use the "Table of Symbols and Terms" on page 103 in order to save time in locating definitions.

The reader is also encouraged to contact us at peteschorer@cs.com for explanations of any parts of this paper that the reader finds difficult.

In Memoriam

Many of the lemmas in this paper, and in the paper, "The Structure of the $3x + 1$ Function", which is accessible on the web site, www.occampress.com, were proved by Michael O'Neill. It was with great sadness that we learned that O'Neill died in November, 2003, after a brief illness. He made a major contribution to this research, and he is sorely missed.

Collaborator Sought

We are seeking a qualified collaborator to help develop the ideas in this paper.

Section 1: Tuple-Sets

In the first part of this paper, we describe a structure called "tuple-sets" that underlies iterations of the $3x + 1$ function — in other words, that describes the function in the "forward" direction. The structure called "recursive 'spiral's" is presented in the second part of this paper, and describes the inverse of the $3x + 1$ function — in other words, describes the function in the "backward" direction.

The "spatial", "geometric", "graphical" nature of both structures is important for the strategies it suggests.

We begin with some definitions.

Definitions

Iteration

An *iteration* takes an odd, positive integer, x, to another odd, positive integer, y, via one application of the $3x + 1$ function.

Trajectory

A *trajectory* (sometimes called an *orbit*) is a sequence of one or more successive iterations of C, i.e., if the sequence is finite,

$$(C^k(x))_{k \geq 0} = (x, C(x), C^2(x), \ldots C^k(x))$$

or, if the sequence is infinite,

$$(C^{\infty}(x)) = (x, C(x), C^2(x), \ldots)$$

The last element of the finite sequence need not be 1 and it need not be an infinity of successive 1's in the case of an infinite sequence.

(See definition of *tuple*, below.)

Power of 2

By a power of 2 we mean a positive integer power of 2.

Exponent

If $C(x) = y$, with $y = (3x + 1)/2^a$, we say that *x maps under iteration to y* (or *x maps directly to y*) *via the exponent a*, and that *a is the exponent associated with x*. We will sometimes speak of a as *mapping directly to y*. The sequence $\{a_2, a_3, ..., a_i\}$, where $a_2, a_3, ..., a_i$ are the exponents associated with $x, C(x), ..., C^{(i-2)}(x)$ respectively, is called an ***admissible vector*** in [3]. We define the function $e(x)$ to be the exponent associated with x. We will sometimes refer to y as a *range element*. It is easily shown (Lemma 0.2) that y cannot be a multiple-of-3. Any element x of the domain of the $3x + 1$ function, whether multiple-of-3 or not, we will sometimes refer to as a *domain element*.

Tuple

A tuple is a trajectory, finite or infinite. A finite tuple is denoted <*x*, *y*, *y*′,..., *y*′··′>. An infinite tuple is denoted <*x*, *y*, *y*′, ... >

Tuple-sets

(The reader might find it helpful to refer to Fig. 1 while reading the following.)

Let $A = \{a_2, a_3, ..., a_i\}$ be a finite sequence of positive integers (i.e., exponents), where $i \geq 2$. The *tuple-set* T_A consists of all and only the following *tuple*s:

all tuples <*x*> such that *x* does not map to any number via a_2;

all tuples <*x*, *y*> such that *x* maps to *y* via a_2 (i.e., $e(x) = a_2$) but *y* does not map to any number via a_3;

all tuples <*x*, *y*, *y*′> such that *x* maps to *y* via a_2 (i.e., $e(x) = a_2$) and *y* maps to *y*′ via a_3 (i.e., $e(y) = a_3$) , but *y*′ does not map to any number via a_4;

...

all tuples <*x*, *y*, *y*′,...,*y*′··′, *y*′···′> such that *x* maps to *y* via a_2 (i.e., $e(x) = a_2$) and *y* maps to *y*′ via a_3 (i.e., $e(y) = a_3$) and ... and the (*i*-1)th element *y*′··′maps to *y*′···′ via the exponent a_i (i.e., $e(y'^{\cdot\cdot}{}') = a_i$).

In the case of maximum length tuples *t* only, we say that *t is defined by* the exponent sequence *A*. Similarly, given any tuple *t* of *i* elements, $i \geq 2$, we say that *t produces,* or *defines,* or *generates* the sequence *A* if *t* is defined by the exponent sequence *A*. Finally, we say that the tuple-set T_A *is defined by* the sequence *A*.

Thus, in Fig. 1, where $A = \{a_2, a_3, a_4\} = \{1, 1, 2\}$, the tuple-set T_A includes:

the tuple <1>, because $e(1) \neq a_2$;
the tuple <3, 5> , because $e(3) = a_2 = 1$, but $e(5) = 4 \neq a_3 = 1$;
the tuple <15, 23, 35>, because $e(15) = a_2 = 1$, and $e(23) = a_3 = 1$, but $e(35) = 1 \neq a_4 = 2$.

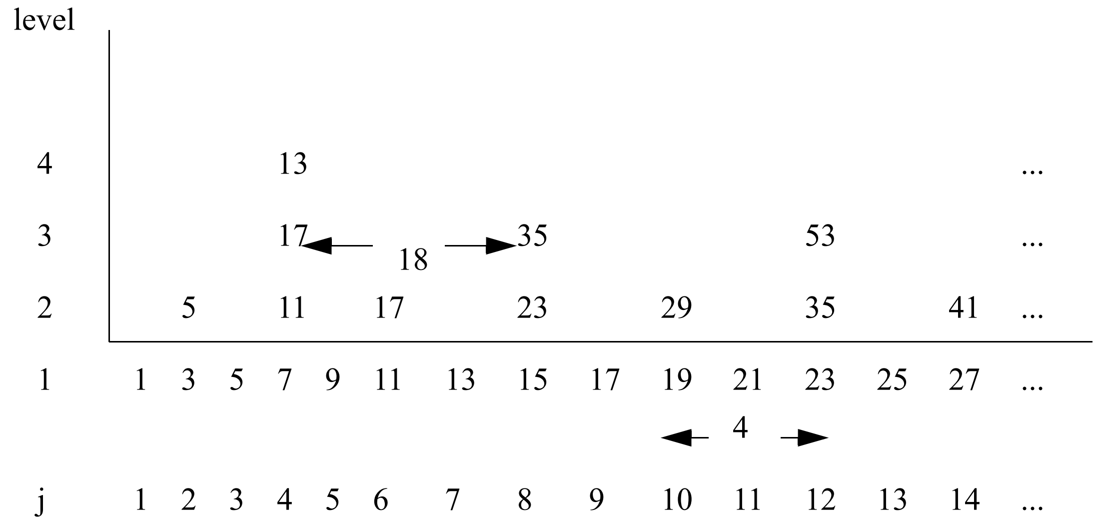

Fig. 1. Part of the tuple-set T_A associated with the sequence A = {1, 1, 2}

Fig. 1 shows part of a tuple-set, namely, the tuple-set T_A associated with the sequence $A = \{1, 1, 2\}$.

The 2nd element of the 8th tuple, $t_{8(2)}$, is 23 because 23 is the range element mapped to by the 1st element, 15, in one iteration ($a_2 = 1$).

The 4th element of the 4th tuple, $t_{4(4)}$, is 13 because 13 is the range element mapped to by the 3rd element, 17, in one iteration ($a_4 = 2$).

There is no 2nd element of the 1st tuple because there is no range element mapped to by 1 such that $a_2 = 1$.

There is no 2nd element of the 5th tuple because there is no range element mapped to by 9 such that $a_2 = 1$.

As stated above, tuples in a tuple-set are ordered according to their first elements. Thus, there is always a unique first (leftmost) tuple in every tuple-set. We adopt the convention of orienting tuples vertically on the page.

Level in a Tuple-set

A *level j* in a tuple-set is defined as follows. If $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, is a finite sequence of exponents, the subscript j in a_j, $2 \leq j \leq i$, denotes the *level j* in T_A. As specified under the definition of tuple-set, we begin numbering our levels with 2 so that level 1 is then the level containing the set of all possible tuple first elements {1, 3, 5, 7, ... } in any T_A, that is, the set of odd, positive integers.

If a tuple has an element at level j, but none at level $j + 1$, we will refer to the tuple as a *j-tuple*, or a *j-level tuple*. If the tuple also has an element at level $j + 1$, we will sometimes refer to the tuple as a ($\geq j$)-tuple. The longest tuple in any tuple-set defined by an exponent sequence of length i - 1 is an i-level tuple.

In the case that $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, we will refer to T_A as an *i-level tuple-set* and we will refer to A as an *i-level exponent sequence*. An i-level exponent sequence consists of (i - 1) exponents. Clearly, every range element mapped to by a given i-level exponent sequence occurs in level i of the corresponding tuple-set.

Tuples Consecutive at Level j

Tuples *consecutive at level j*, $j \geq 2$, are defined as follows. Let t_k, t_m be ($\geq j$)-tuples in some T_A. If there is no ($\geq j$)-tuple between t_k and t_m, we say that t_k and t_m are *tuples consecutive at level j*. Here, "between" means relative to the natural linear ordering of tuples based on their first elements.

Thus, for example, in Fig. 1, tuples 4 and 8 are consecutive at level 3.

Ordering of Tuples in a Tuple-set

See under "Remarks About the Distance Functions" on page 12.

Row

Let $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, be a sequence of exponents, and let T_A be the corresponding tuple-set. Then a *level-j row*, R_j, where $1 \leq j \leq i$, in T_A is the set of all jth tuple-elements in tuples consecutive at level j. We shall see that, as a result of the distance functions defined in lemmas 1.0 and 1.1, that each row is a congruence class — specifically, a reduced residue class mod $2 \bullet 3^{i-1}$, $i \geq 2$. We shall also see that the $3x + 1$ function can — and perhaps should! — be defined as a function *on these congruence classes*, rather than merely on odd, positive integers. This definition holds even if we include negative elements of each congruence class.

Among the questions that it is natural to ask regarding the top (i-level) row R_i in an i-level tuple-set are:

To which (i+1)-level rows in the set of (i+1)-level tuple-sets does R_i map under the set of all exponents 1, 2, 3, 4, ... ?

How is the set of all exponents partitioned in this mapping? (A total of $2 \bullet 3^{i-1}$ of rows R_i over all i-level tuple-sets maps to a total of $2 \bullet 3^{(i+1)-1}$rows R_{i+1} over all i+1 level tuple sets; the infinite number of exponents is partitioned into a finite set of classes.)

If y is an element in a row R_i that maps to an element of a row R_{i+1} under the exponent a_{i+1}, what is the next larger element in row R_i that maps to the row R_{i+1}?

Answers to these and other questions are given in the following lemmas, which are stated in "Appendix A — Statements of Lemmas": Lemmas 7.25, 7.27, 7.3, 7.31, 7.32, 7.35, 7.36, 7,38, 7.4.

Extensions of Tuples and of Tuple-sets

Let T_A be a tuple-set defined by the sequence of exponents $A = \{a_2, a_3, ..., a_i\}$. Then any tuple-set $T_{A'}$ defined by a sequence of exponents $A' = \{a_2, a_3, ..., a_i, a_{i+1}\}$ is called an *extension* of T_A. We define extensions of tuples in a similar manner. Thus, a ($\geq i$)-tuple in $T_{A'}$ is an extension of an i-tuple in T_A.

If $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, is a sequence of exponents, then we define an *initial sub-sequence of the exponent sequence A* as the sequence $\{a_2, a_3, ..., a_j\}$, where $2 \leq j \leq i$. Thus, for example, $\{a_2\}$ is an initial sub-sequence of A, and so is $\{a_2, a_3, a_4\}$, but, for example, $\{a_3, a_4\}$ is not. We define an *inititial sub-sequence of a tuple t_k* similarly.

With the concept of extensions of tuples and tuple-sets established, we can see that every j-tuple, $2 \leq j \leq i$, defined by an initial sub-sequence $\{a_2, a_3, ..., a_j\}$ of A is in the tuple-set T_A.

Non-terminating Tuple (n-t-v-1, n-t-v-c)

As stated under “Trajectory” on page 4, a trajectory (tuple) may be finite or infinite. We will use the term *non-counterexample tuple* to denote a finite tuple whose elements map to 1, and the term *counterexample tuple* to denote a finite tuple whose elements are counterexamples. We will sometimes use the term *n-t-v-1* (non-terminating-tuple-via-1) to denote an infinite tuple whose elements map to 1, and the term *n-t-v-c* (non-terminating-tuple-via-c (*c* for *counterexample*)) to denote an infinite tuple whose elements are counterexamples.

It is possible that a tuple contains a repetition of one of its elements. (The tuple <1, 1, 1, ...> is a trivial example, and the only known example at time of writing.) Clearly, any such tuple is infinite. If the repeated element is not 1, then the tuple contains solely counterexample elements. Results concerning cycles are given in “Appendix A — Statements of Lemmas” on page 49.

Graphical View of a Tuple-set

At this point, it will be helpful if we get an abstract view of the various-length tuples in a tuple-set. Let T_A be any tuple-set, with $A = \{a_2, a_3, ..., a_i\}$. Then, as shown in Fig. 3.05, there is an infinity of tuples consecutive at level *i* and, indeed, at all levels $1 \leq j \leq i$. Between each pair of *i*-level tuples there is a finite set of tuples consecutive at level *i* - 1. Between each pair of these is a finite set of tuples consecutive at level *i* - 2, etc., down to level 1. The distance (numerical difference) between elements of tuples at each level will be specified in Lemmas 1.0 and 1.1.

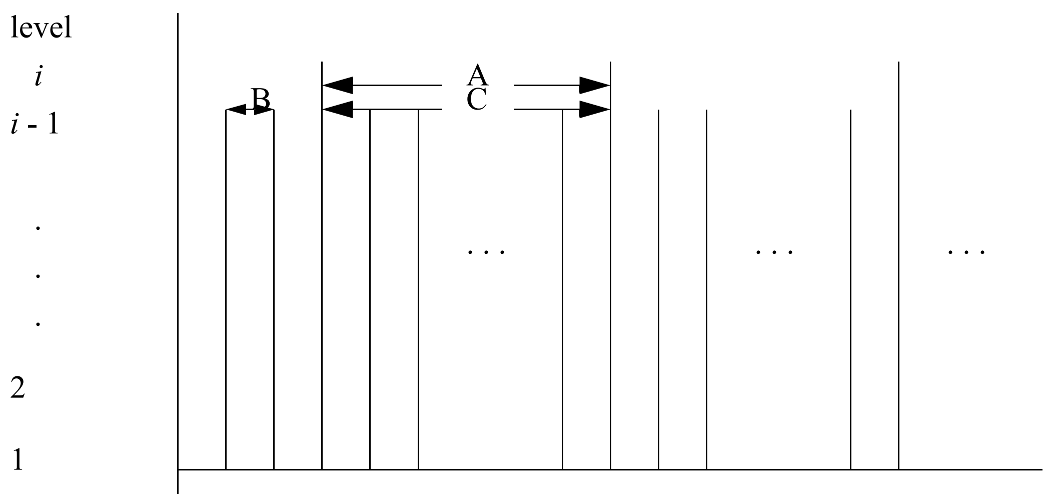

Fig. 3.05. Graphical view of tuples in a tuple-set.

A, the distance (numerical difference) at level *i* between elements of tuples consecutive at level *i*, which is $2 \cdot 3^{i-1}$. (Lemma 1.0 (a))

B, the distance (numerical difference) at level *i* - 1 between elements of tuples consecutive at level *i* - 1, which is $2 \cdot 3^{i-2}$. (Lemma 3.0 (a))

C, the distance (numerical difference) at level *i* - 1 between elements of tuples consecutive at level *i*, which is

$$lcm(2 \cdot 2^{a_i}, 2 \cdot 3^{i-2}) = 2 \cdot 2^{a_i} \cdot 3^{i-2}$$

where *lcm* is the least common multiple. (Lemma 1.1)

The reader may find the following intuitive description of a tuple-set to be helpful.

• Every i-level tuple-set T_A , $A = \{a_2, a_3, a_4, ..., a_i\}$, can be viewed as a “picket fence”, infinitely long to the right. (Pickets correspond to tuples.) There is:

an infinity of tuples of length 1;
an infinity of tuples of length 2;
an infinity of tuples of length 3;
...
an infinity of tuples of length i;

• Furthermore, if counterexamples exist, there is (by Lemma 10.0):

an infinity of counterexample tuples and an infinity of non-counterexample tuples in the infinity of tuples of length 1;
an infinity of counterexample tuples and an infinity of non-counterexample tuples in the infinity of tuples of length 2;
an infinity of counterexample tuples and an infinity of non-counterexample tuples in the infinity of tuples of length 3;
...

an infinity of counterexample tuples and an infinity of non-counterexample tuples in the infinity of tuples of length i;

• Finally, there is, for each exponent a_{i+1} (i.e., for each non-negative integer):

an infinity of counterexample tuples in T_A that are extended by a_{i+1} and
an infinity of non-counterexample tuples in T_A that are extended by a_{i+1}. (Lemma 2.0)

Graphical Views of the Set of All Tuple-sets

The reader may also find it helpful to have a graphical view of the set of *all* tuple-sets, particularly when the reader reviews the lemmas below concerning rows and extensions of tuple-sets.

Probably the best graphical view is that of an infinitary tree, as shown in Fig. 3.07, because the set of exponents by which the i-level row in any i-level tuple-set can be mapped to some $(i + 1)$-level row of some $(i + 1)$-level tuple-set is precisely the set of all possible exponents, namely, $\{1, 2, 3, ...\}$(Lemma 7.25). In the figure, exponents are given next to (some) branches. Each node represents an infinity of tuple-set elements, namely, all j-level elements, $j \geq 1$, this infinite set conceived of as running perpendicularly *into* the page.

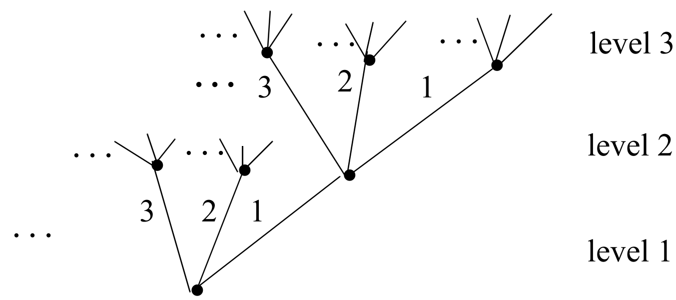

Fig. 3.07. Graphical view of the set of all tuple-sets

We can further simplify the graphical representation of the set of all tuple-sets by recognizing that the top row of each i-level tuple-set is generated by the top rows of (i - 1)-level tuple-sets, $i \geq 3$, and that there are only a finite number of top rows of all i-level tuple-sets, $i \geq 2$ (each row is a reduced residue class mod ($2 \bullet 3^{i-1}$)). For example, Fig. 3.08 shows the generating relationship between the top rows of all 2-level tuple-sets, and the top rows of all 3-level tuple-sets. Each arrow represents the generating function via all exponents. The arrow points to the row generated. Note that, even though each row is identified by its first element, the contents of rows with the same first element at different levels are not identical, because of the distance function $d(i, i)$ (Lemma 1.0 (a)).

By Lemma 7.25, the same generating relationship between successive top levels holds for all higher levels, so that the infinitary tree of all tuple-sets can, without loss of generality, be reduced to a finitary tree, namely, a ($2 \bullet 3^{i-2}$) -ary tree, $i \geq 2$. (($2 \bullet 3^{i-2}$) is the number of reduced residue classes mod ($2 \bullet 3^{i-1}$).)

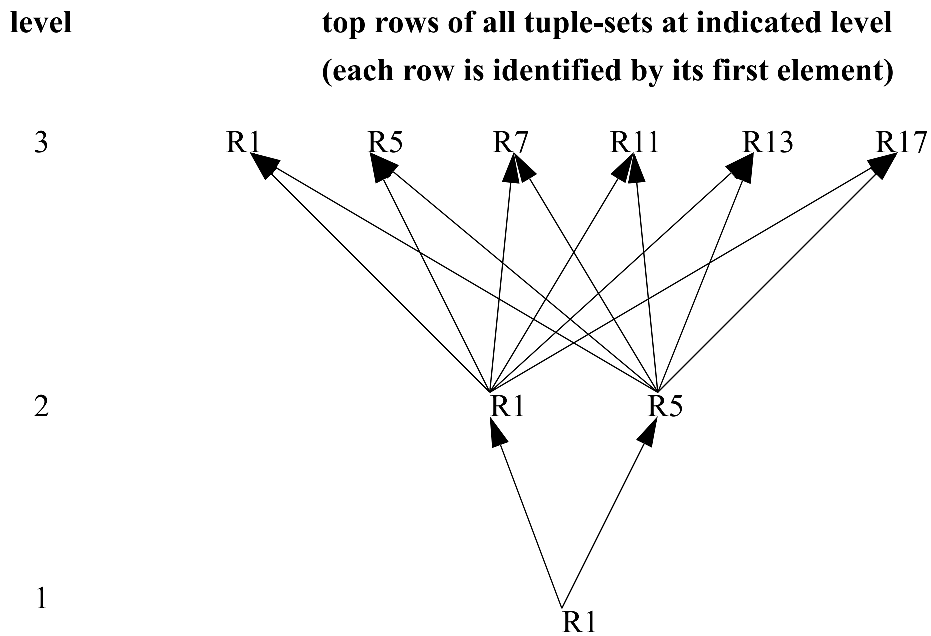

Fig. 3.08. Generating relationship between top levels of all 2-level tuple-sets and top levels of all 3-level tuple-sets

Distance Functions on Tuple-sets

Lemma 1.0
(a) Let $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, *be a sequence of exponents, and let* t_k, t_m *be tuples consecutive at level i in* T_A. *Then d(i, i), the distance between* t_k *and* t_m *at level i, is defined to be the absolute value of the difference between the level i elements of* t_k *and* t_m, *i.e., is defined to be* $|t_{k(i)} - t_{m(i)}|$, *and is given by*:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) Let t_k, t_m *be tuples consecutive at level i in* T_A. *Then d(1, i), the distance between* t_k *and* t_m *at level* 1, *is defined to be the absolute value of the difference between the level 1 elements of* t_k *and* t_m, *i.e., is defined to be* $|t_{k(1)} - t_{m(1)}|$, *and is given by:*

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3})\dots(2^{a_i})$$

Thus, in Fig. 1, the distance $d(3, 3)$ between $t_{8(3)} = 35$ and $t_{4(3)} = 17$ is $2 \cdot 3^{(3-1)} = 18$. The distance $d(1, 2)$ between $t_{12(1)} = 23$ and $t_{10(1)} = 19$ is $2 \cdot 2^1 = 4$.

Remarks About the Distance Functions

(1) Strictly speaking, we should include the sequence A of exponents as arguments of $d(1, i)$, $d(i, i)$, but this notation would be cumbersome and, since typically this sequence is known, unnecessary.

(2) The distance functions make clear that, for each finite sequence of exponents, there exists an infinity of tuples produced by that sequence. (The equivalent of this statement is made in [3] (p. 48).) In particular, there exists an infinity of tuples consecutive at level i for all $i \geq 2$.

(3) In each i-level tuple-set, $i \geq 2$, there exists:

an infinite sequence of 1-level tuples, and
an infinite sequence of 2-level tuples, and
an infinite sequence of 3-level tuples, and
..., and
an infinite sequence of i-level tuples.

The relation between the tuples at any one level relative to those of another level, is outlined in Fig. 3.05, under "Graphical View of a Tuple-set" on page 8, and can be deduced from the table "Distances between elements of tuples consecutive at level i" on page 13.

(4) Lemma 1.0 (a) makes clear that no two i-level tuples in a given i-level tuple-set have the same last element. In fact, the values of the last elements of i-level tuples in an i-level tuple-set always increase as one proceeds along the sequence of i-level tuples.

(5) The formula for $d(1, i)$ implies that it is possible for pairs of tuples consecutive at level i in one tuple-set to be the same distance apart, at level 1, as pairs of tuples consecutive at level 1 in another tuple-set. For example, this would occur between tuples consecutive at level 2 in T_A when $A = \{2\}$ ($d(1, 2) = 2 \cdot 2^2 = 8$) and between tuples consecutive at level 3 in $T_{A'}$ when $A' = \{1, 1\}$($d(1, 3) = 2 \cdot 2^1 2^1 = 8$).

(6) The distance, at level j, $2 \leq j < i$, between elements of tuples consecutive at level i, is given in Lemma 1.1.

(7) It is straightforward to prove that the distance functions carry over into the odd, negative integers as well. (The proof is contained in the proof of Lemma 2.0.)

Lemma 1.1. *Let T_A be a tuple-set defined by a sequence $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$. Then the distance $d(j, i)$ between elements at level j, $1 \leq j \leq i$, of tuples t_k, t_m consecutive at level i is given by the following table:*

Table 1: Distances between elements of tuples consecutive at level i

Level	Distances between elements of t_k, t_m at level
i	$2 \cdot 3^{i-1}$
$i - 1$	$2 \cdot 3^{i-2} \cdot 2^{a_i}$
$i - 2$	$2 \cdot 3^{i-3} \cdot 2^{a_{i-1}} 2^{a_i}$
$i - 3$	$2 \cdot 3^{i-4} \cdot 2^{a_{i-2}} 2^{a_{i-1}} 2^{a_i}$
...	...
2	$2 \cdot 3 \cdot 2^{a_3} \ldots 2^{a_{i-1}} 2^{a_i}$
1	$2 \cdot 2^{a_2} 2^{a_3} \ldots 2^{a_{i-1}} 2^{a_i}$

A Recursive Description of Any Tuple-set

Let $\boldsymbol{x}$ denote the set of odd, positive integers. Let $\boldsymbol{y} = \mathbf{C}\{a_2 \bmod 2 \bullet 3^{(1-1)}\}(\boldsymbol{x})$ denote the set of range elements of the $3x + 1$ function produced by the exponent $a_2 \bmod 2 \bullet 3^{(1-1)}$ operating on all the elements of $\boldsymbol{x}$. As we know from Lemma 1.0, $\boldsymbol{y}$ is one of two sets, namely, the set of all $y \equiv 1 \bmod 2 \bullet 3^{(1-1)}$ (if a_2 is even) or the set of all $y \equiv 5 \bmod 2 \bullet 3^{(1-1)}$ (if a_2 is odd).

We can repeat the process recursively, so that, if $A = \{a_2, a_3, ..., a_i\}$, then

(1)
$$T_A = \mathbf{C}\{a_i \bmod 2 \bullet 3^{((i-1)-1)}(... \mathbf{C}\{a_3 \bmod 2 \bullet 3^{(2-1)}\}(\mathbf{C}\{a_2 \bmod 2 \bullet 3^{(1-1)}\}(\boldsymbol{x}))...).$$

The reason that this is a recursive description of the tuple-set T_A is that it is precisely the sequence of tuple-set extensions,

$$T_{\{a_2\}}, T_{\{a_2, a_3\}}, T_{\{a_2, a_3, a_4\}}, \ldots, T_{\{a_2, a_3, a_4, \ldots a_i\}}$$

The reason we only need to consider the indicated finite set of exponents at each level is established by Lemmas 7.0 and 7.1 in the first part of the second file of the paper, “The Structure of the $3x + 1$ Function: An Introduction” on the web site occampress.com.

We remind the reader that if $\boldsymbol{y}''^{\cdots}''$ is a set mapped to by $\mathbf{C}\{a_i...\}(\boldsymbol{y}''^{\cdots}')$, then we know by “Lemma 1.0” on page 11 that $\boldsymbol{y}''^{\cdots}''$ is a reduced residue class mod $2 \bullet 3^{((i+1)-1)}$.

Equation (1) describes the behavior of the $3x + 1$ function over its entire domain, namely, the set of all odd, positive integers, regardless if counterexamples exist or not.

Summary of Properties of Tuple-sets

We now provide a table that summarizes our results on tuple-sets and rows in a tuple-set. (Recall that a row is simply the set of elements at a given level in a given tuple-set.) We break the properties of rows into three parts: those concerning top rows, those concerning middle rows, and those concerning the bottom (i.e., first) rows. The phrase "extension of a top row R_i" means the same thing as "the top row R_{i+1} mapped to by a top row R_i".

The table entry for each property whose value is known includes a reference to definitions or lemma(s) that establish the value.

Note: some table-rows may have the same content as other rows, though under different properties. This redundancy is deliberate, the purpose being to aid understanding and to make the looking up of properties easier.

Statements of all referenced lemmas are given in the Appendix.

Table 2: Some important properties of tuple-sets

Property	Value of property	Reference
Sequence of exponents, A, that define a tuple-set T_A	$A = \{a_2, a_3, ..., a_i\}$, $a_i \geq 1$.	Definition of tuple-set
Structure of tuple-sets (not of tuples within tuple-sets)	Infinitary tree, equivalent to a $2 \bullet 3^{i-2}$-ary tree. Thus, in the latter, finitary, tree: **level 2** has $2 \bullet 3^{2-2} = 2$ nodes (the 2 top rows of all 2-level tuple-sets), mapped to by 2 equivalence classes of exponents; **level 3** has $2 \bullet 3^{3-2} = 6$ nodes (the 6 top rows of all 3-level tuple-sets), mapped to by 6 equivalence classes of exponents; **level 4** has $2 \bullet 3^{4-2} = 18$ nodes (the 18 top rows of all 4-level tuple-sets), mapped to by 18 equivalence classes of exponents; etc.	Lemma 7.3
$2 \bullet 3^{i-1}$	Distance between elements of tuples successive at level i in an i-level tuple-set	Lemma 1.0

Table 2: Some important properties of tuple-sets

Property	Value of property	Reference
$2 \cdot 3^{i-2}$	Number of top rows of all *i*-level tuple-sets; also Number of exponent equivalence classes (and the maximum exponent), from which exponents mapping to the top row of any *i*-level tuple-set, from the top rows of all *i*- 1 level tuple-sets, must be selected.	Lemmas 3.055, 3.057 Lemma 7.3

Table 3: Some important properties of the top (i.e., level *i*) row of an *i*-level tuple-set

Property	Value of property	Reference
Distance $d(i, i)$ between successive elements of a top row, i.e., between *i*-level elements of tuples consecutive at level *i*	$d(i, i) = 2 \cdot 3^{i-1}$	Lemma 1.0 (a)
Total number of different top rows over the set of all *i*-level tuple-sets	$\phi(2 \cdot 3^{i-1}) = 2 \cdot 3^{i-2}$ = the number of reduced residue classes mod $2 \cdot 3^{i-1}$	Lemmas 3.055, 3.057
Distance between successive exponents in an exponent equivalence class mapping from an (*i*-1)-level top row to an *i*-level top row. All members of a class map to the same level-*i* top row from the same (*i* - 1) - level top row.	$2 \cdot 3^{i-2}$	Lemma 7.3
Total number of exponent equivalence classes mapping a level- (*i* - 1) top row to all level-*i* top rows	$2 \cdot 3^{i-2}$	Lemma 7.3

Table 3: Some important properties of the top (i.e., level *i*) row of an *i*-level tuple-set

Property	Value of property	Reference
Smallest exponent mapping to any given top row of an *i* -level tuple-set from any top row of an *(i - 1)*-level tuple-set	≤ 4	Lemma 7.35
Upper bound on exponents mapping from any given top row of an *(i* - 1)-level tuple-set to the top row of any *i*-level tuple-set	$2 \cdot 3^{i-2}$ (All larger exponents are elements of equivalence classes having smaller minimum elements)	Lemma 7.3
Beginning of sequence of exponents mapping to any given *i*-level top row from *any* (*i*- 1)-level top rows	*For an i-level top row mapped to by odd exponents*: 1,3, *, or 1, *, 5, or *, 3, 5. *For an i-level top row mapped to by even exponents*: 2, 4, *, or 2, *, 4, or *, 4, 6, where * denotes a "missing" exponent due to absence of a multiple-of-3 in the *i*-level top row. The * recurs after every two non-* exponents.	Lemma 15.0
Sequence of exponents mapping from any given *(i* - 1)-level top row to *all i* -level top rows	1, 2, 3, ..., $2 \cdot 3^{i-2}$, with each exponent mapping to a unique *i*-level top row. A larger exponent a'_i then maps to the same row as one of the above exponents a_i does if $a'_i \equiv a_i \mod 2 \cdot 3^{i-2}$.	Lemma 7.3
Minimum element in a top row	Minimum residue in a reduced residue class mod $2 \cdot 3^{i-1}$	Lemmas 3.055, 3.057
Formula for the minimum element of the top row of an *i*-level tuple-set, given only the sequence of exponents defining the tuple-set	See Lemma 7.38	Lemma 7.38

Table 3: Some important properties of the top (i.e., level *i*) row of an *i*-level tuple-set

Property	Value of property	Reference
Formula for the minimum element of the top row of an (*i*+ 1)-level tuple-set mapped to by the top row of an *i*-level tuple-set via an exponent a_{i+1}	See Lemma 7.36	Lemma 7.36
Distance between successive elements of (sub-row of) top row of an *i*-level tuple-set that generates a top row of an (*i* + 1)-level tuple-set via the exponent a_{i+1}	$lcm(2 \cdot 3^{i-1}, 2 \cdot 2^{a_{i+1}})$, where *lcm* denotes least common multiple	Lemma 1.1
Successive elements of (sub-row of) top row of i-level tuple-set map to successive elements elements of top row of (i + 1)-level tuple-set?	Yes.	Lemma 7.40
Set of elements in all top rows of all *i*-level tuple-sets	Set of range elements, i.e., set of odd, positive integers not multiples of 3	Lemma 3.28
Relationship between top rows of all *i*-level tuple-sets and top rows of all (*i* + 1)-level tuple-sets	(1) *Each* top row in an *i*-level tuple-set generates, via all exponents a_{i+1}, the top rows of *all* (*i* + 1)-level tuple-sets. (2) For *each* (*i* + 1)-level top row, if it is desired to generate the row via all possible exponents, then *all* *i*-level top rows are required .	(1) Lemma 7.25 (2) Lemma 7.27.

Table 4: Some important properties of the middle (i.e., levels 1 < *j* < *i*) row of an i-level tuple-set

Distance, $d(j, i)$ between elements at level j of successive tuples consecutive at level i	$d(j, i) =$ $$lcm(2 \cdot 3^{j-1}, 2 \cdot 2^{a_{j+1}} \cdot 2^{a_{j+2}} \ldots \cdot 2^{a_i})$$ where *lcm* is the least common multiple.	Lemma 1.1
For each i and each j, minimum elements of level j rows over all i-level tuple-sets	General formula not yet known; must be determined empirically for each given tuple-set	
For each j, set of elements in all j-level rows of all i-level tuple-sets	Set of range elements, i.e., set of odd, positive integers not multiples of 3	Lemma 3.28

Table 5: Some important properties of the bottom (i.e., level 1) row of an *i*-level tuple-set

Property	Value of property	Reference
Distance, $d(1, i)$, between successive tuple elements at level 1 of tuples consecutive at level i	$d(1, i) =$ $2 \cdot 2^{a_2} \cdot 2^{a_3} \cdot \ldots \cdot 2^{a_i}$	Lemma 1.0 (b)
Set of elements in bottom row of all i-level tuple-sets	Set of odd, positive integers	Lemma 3.28

Tuple-sets and Finite Stopping Times

In the literature on the $3x + 1$ Problem, the term *stopping time* is defined as the smallest k such that $C^{(k)}(n) < n$, in other words, the smallest number of iterations of the $3x + 1$ function on n such that a value smaller than n is produced. (Note: in the literature this definition of stopping time usually is made relative to a different definition of the $3x + 1$ function than ours, namely, one in which a division by 2 is considered an iteration of the function.

Tuple-sets make it easy to show the equivalent of a well-known result, namely (and here stated informally) that "most" exponent sequences have finite stopping times. For, consider, first, that $C(x) = y$ and $y > x$ iff $ord_2(3x + 1) = 1$. Thus, e.g., $C(7) = 11$ and $ord_2(3 \cdot 7 + 1) = 1$. Now consider any exponent sequence of length $(i - 1)$, $i \geq 2$. Then there are $2^{(i-1)} - 1$ ways that an exponent equal to 1 can appear in such a sequence. For each such way except for that in which there are $(i - 1)$ exponents each equal to 1, there is an infinity of possible exponent sequences of length $(i - 1)$, since an exponent can be any positive integer. Take any such way w containing j, $0 < j < (i$

- 1), exponents each equal to 1. Then all but a finite number of sequences corresponding to w will have finite stopping times, for (informally) it is always possible to find a single exponent sufficiently large to "overcome" the above-mentioned increasing effect of the exponents equal to 1, and all larger exponents will likewise overcome this effect.

Complete List of All Our Results

Before beginning our discussion of possible strategies for proving the 3x + 1 Conjecture, we should mention that a complete list of all results (lemmas) we have obtained so far in our $3x + 1$ research is available in the first part of the second file of this paper on the web site occampress.com.

Possible Strategies for Proving the 3*x* + 1 Conjecture Using Tuple-sets

Preliminary Discussion of Strategies

One of the characteristics of the $3x + 1$ function that makes proving the $3x + 1$ Conjecture so difficult, is that virtually every fact we prove about the function applies equally to counterexamples and to non-counterexamples. This is true of virtually every lemma in this paper, including, e.g., such well-known elementary facts as the following. (The "exponent of 2" is, of course, a in $(3x + 1)/2^a = y$.)

If x maps to y in one iteration of the 3x + 1 function, then:
If $x \equiv 1 \bmod 4$ *then the exponent of* 2 *is* ≥ 2*;*
If $x \equiv 3 \bmod 4$ *then the exponent of* $2 = 1$*;*
If $y \equiv 1 \bmod 3$ *then the exponent of* 2 *is even*;
if $y \equiv 2 \bmod 3$ *then the exponent of 2 is odd*. (Lemmas 5.5, 5.7)

Proving such facts can lead us to believe that we are making progress toward proving the $3x + 1$ Conjecture, when all that we can say for certain is that we are increasing our knowledge of the properties of the $3x + 1$ function.

In attempting to prove the Conjecture using tuple-sets, we must realize that, whether or not there is a counterexample, each odd, positive integer, whether counterexample or non-counterexample, will occupy exactly the same place in each tuple of which the integer is a member. This is in contrast to the recursive "spiral"s structure described in the second part of this paper. This structure describes the inverse of the $3x + 1$ function. There, the existence of counterexamples to the Conjecture would make a definite difference in the set of all "spiral"s describing the inverse of the function. If the Conjecture is true, then every odd, positive integer has a position in the infinite set of "spiral"s whose base element is 1. If the Conjecture is false, then *some* odd, positive integers, namely, those that map to 1, have positions in the infinite set of "spiral"s whose base element is 1, but in addition, there are other integers, namely, counterexamples, that occupy positions in at least one other infinite set of "spiral"s. Clearly, the integers in the infinite set that map to 1, constitute a set disjoint from the integers (counterexamples) in the other infinite set or sets of "spiral"s. (For a further elaboration on the material of this paragraph, see "Appendix D — A Curious Fact About the Inverse of the 3x + 1 Function" on page 89 and "Appendix E — A Curious Fact About Tuple-sets" on page 95.)

Perhaps the argument that whether or not there is a counterexample, the set of all tuple-sets will remain unchanged, will be more convincing if the reader considers a version of the $3x + 1$ function that initially acts *simultaneously* on the entire set of odd, positive integers. Then, if the exponent is 1, the result is the *set* of range elements congruent to 5 mod $2 \cdot 3^{(2-1)} = 5$ mod 6. If the exponent is 2, the result is the *set* of range elements congruent to 1 mod $2 \cdot 3^{(2-1)} = 1$ mod 6..

We can designate this initial behavior of the $3x + 1$ function as $\mathbf{C}_{\{1\}}(\mathbf{x}) = \mathbf{y}$ in the first case, and as $\mathbf{C}_{\{2\}}(\mathbf{x}) = \mathbf{y}'$ in the second case.

We then apply **C**, the set-argument version of the function, to the set **y** or the set **y′**, for any exponent a_3, and again arrive at a set of range elements, in this case, a set whose elements are congruent mod $2 \cdot 3^{(3-1)} =$ mod 18. And so on.

It should be clear that this process always yields the same results (the same sets of range elements) regardless if counterexamples exist or not.

In any case, it is important that we ask the following question:

Question 1. How Does the Existence of Counterexamples "Make a Difference" in the Set of All Tuple-Sets?

We can give at least two answers to this question. First, some definitions.

Definition. Let M_i denote the set of all minimum residues of reduced residue classes mod $2 \cdot 3^{i-1}$. There are $2 \cdot 3^{i-2}$ such residues as stated above under "Summary of Properties of Tuple-sets" on page 13, and below in Lemma 3.0574 . Thus:

for level $i = 2$, M_i has $2 \cdot 3^{2-2} = 2$ elements, namely $M_i = \{1, 5\}$;
for level $i = 3$, M_i has $2 \cdot 3^{3-2} = 6$ elements, namely $M_i = \{1, 5, 7, 11, 13, 17\}$;
for level $i = 4$, M_i has $2 \cdot 3^{4-2} = 18$ elements, namely $M_i = \{1, 5, 7, 11, 13, 17, 19, 23$, and all other odd, positive integers up to and including 53 that are not multiples of $3\}$.
M_i is the set of last elements of all first i-level tuples in all i-level tuple-sets (by Lemma 1.0) .

Lemma 3.057. *The set of minimum elements of all top rows in all i-level tuple-sets is the set of minimum residues of the set of reduced residue classes mod* $2 \cdot 3^{i-1}$.

Proof: follows directly from the fact that a row is a reduced residue class mod $2 \cdot 3^{i-1}$ . .□

Lemma 3.0574. *For each* $i \geq 2$, *the number of elements of* M_i, *which we will denote* $|M_i|$, *is* $\varphi(2 \cdot 3^{(i-1)}) = 2 \cdot 3^{(i-2)}$, *where* φ *is Euler's totient function, i.e., the function that returns the number of numbers less than its argument and relatively prime to its argument.*

Proof: The number of numbers less than $2 \cdot 3^{(i-1)}$ and relatively prime to it is given by Euler's totient function φ, which for powers of two primes p^n, q^m is $\varphi(p^n q^m) = (p-1)p^{n-1}(q-1)q^{m-1}$. Applying this formula to $2 \cdot 3^{(i-1)}$, we get $2 \cdot 3^{(i-2)}$ . .□

Definition. We call the elements of M_i, *anchors at i*, and we call the the i-level tuples they are the last elements of, *anchor tuples at i*. (We do not give a special name to the *first* element of the first i-level tuple in an i-level tuple-set. i.e., to the first element of an anchor tuple)

Clearly, because the tuples in each tuple-set are linearly ordered in the natural way by first elements of tuples, there is exactly one *i*-level anchor tuple in each *i*-level tuple-set. Furthermore, by definition of M_i, this anchor tuple is the *first* *i*-level tuple in each *i*-level tuple-set.

Thus, for example:

at level 2, the total number of anchors is $2 \bullet 3^{(2-2)} = 2$. These anchors are 1 and 5. The tuple <1, 1> is the 2-level anchor tuple of the 2-level tuple-set T_A, where $A = \{2\}$. The tuple <13, 5> is the 2-level anchor tuple of the 2-level tuple-set T_A, where $A = \{3\}$.

at level 3, the total number of anchors is $2 \bullet 3^{(3-2)} = 6$. These anchors are 1, 5, 7, 11, 13, 17. The tuple <13, 5, 1> is the 3-level anchor tuple of the 3-level tuple-set T_A, where $A = \{3, 4\}$. The tuple <7, 11, 17> is the 3-level anchor tuple of the 3-level tuple-set T_A, where $T_A = \{1, 1\}$.

A helpful tabular representation of anchors and anchor tuples, namely, the "anchor rectangle at *i*" and the "Infinite Anchor Rectangle" is given in "Appendix A1 — Lemmas and Definitions Used in Implementations of the "Pushing Away" and "Missing Sequences" Strategies" on page 66.

First Answer to Question 1

We express the answer in the form of a lemma:

Lemma 10.96.

(a) *If a counterexample exists, then for all* $i \geq i_0$, *where* i_0 *is the smallest i such that a counterexample is an anchor at i, the set of anchor tuples at i is partitioned into two disjoint sets: the set* $\{t_c\}$ of *counterexample anchor tuples and the set* $\{t_{nc}\}$ *of non-counterexample anchor tuples. Otherwise, if there are no counterexamples, the set of anchor tuples at i,* $i \geq 2$, *consists exclusively of non-counterexample anchor tuples.*

(b) *For each* $i \geq i_0$, *let* $\{A_{nc}\}$ *denote the set of all exponent sequences defined by* $\{t_{nc}\}$ *in part (a), and let* $\{A_c\}$ *denote the set of all exponent sequences defined by* $\{t_c\}$ *in part (a). Then* $\{A_{nc}\} \cap \{A_c\} = \emptyset$.

Proof: (a) follows directly from the fact that no tuple can be simultaneously a non-counterexample and a counterexample tuple, and **(b)** from the fact that the set of all anchor tuples at any *i* defines the set of all *i*-level exponent sequences. □

Thinking about how Lemma 10.96 might lead us to a proof of the 3*x* + 1 Conjecture brings us, sooner or later, to the following question:

Why Are There An Infinite Number of Tuples in Each Tuple-set?

The answer is: because every sequence of positive integers defines a tuple-set, and because the last element of each tuple maps directly to one and only one odd, positive integer. For, consider the tuple-set T_A defined by the exponent sequence $A = \{a_0, a_1, a_2, ..., a_i\}$. T_A has an extension for each positive integer a_{i+1}, otherwise there would exist a sequence of positive integers that did not define a tuple-set. But since the last element of each tuple in T_A maps directly to one and only one odd, positive integer, and since each tuple-set $T_{A'}$, $A' = \{a_0, a_1, a_2, ..., a_i, a_{i+1}\}$, likewise has an extension for each positive integer a_{i+2}, it follows that, for *each* a_i, there exists an *infinity* of tuples in T_A whose last elements directly map to their respective odd, positive integers *via* a_i. (This is what Lemma 2.0 states:

Lemma 2.0: *Every i-level tuple-set can be extended by any exponent* a_{i+1}. *Or, in other words, for each i-level tuple-set and for each* a_{i+1}, *every i-level row — though not every element in every i-level row — maps to a non-empty row in some (*i+1*)-level tuple-set.*)

Thus, e.g., in the tuple-set T_A, $A = \{2\}$, the last element of the tuple <9, 7>, namely, 7, maps to 11 via the exponent 1. And similarly, in the same tuple-set, the last element of the tuple <25, 19>, namely 19, maps to 29 via the same exponent, 1. (However, 11 then maps to 17 via the exponent 1, whereas 29 maps to 11 via the exponent 3.)

Each *i*-level tuple in an *i*-level tuple-set has an extension via some exponent a_i. An infinity of such tuples have an extension via the exponent 1, another infinity have an extension via the exponent 2, another infinity via exponent 3, etc. The details are given in Lemmas 7.25 through 7.4 (see Appendix A).

Now Lemma 10.0 implies that, whether or not a counterexample exists, there is an infinity of non-counterexample tuples in each tuple-set. Or, in other words, the set of all (finite) non-counterexample tuples defines the set of all (finite) exponent sequences, hence the set of all tuple-sets, regardless whether a counterexample exists or not.

The question we must ask ourselves is this: if we remove an infinity of elements (namely, counterexample elements) from the top row of each tuple-set — and not merely an infinity of elements, but an infinity of elements that guarantee that the set of tuples so removed (i.e., the counterexample tuples) do likewise define the set of all (finite) exponent sequences, as Lemma 10.0 requires, hence the set of all tuple-sets — if we remove all these elements, is it possible that the set of non-counterexample tuples (i.e., the tuples that remain) can still define the set of all finite exponent sequences, especially given that there is no redundancy in the set of anchor tuples for each *i* (i.e., each anchor tuple defines one and only one *i*-level exponent sequence, and all *i*-level exponent sequences are defined by the set of all anchor tuples)?

The answer is not clear. All that we can say at this point is that for each $i \geq i_0$, where i_0 is as defined in Lemma 10.96 above, the presence of counterexamples removes an infinite set of exponent sequences from those defined by non-counterexample anchor tuples at *i* if there are no counterexamples.

Second Answer to Question 1

The second answer to Question 1 follows directly from the definition of a counterexample, namely, a number that does not eventually map to 1.

If a counterexample exists, then for all $i < i_0$, *where* i_0 *is as defined in Lemma 10.96, all elements of* M_i *map to 1, and hence are "connected" to elements of the infinite set of recursive "spiral"s (see second part of this paper) with base element 1. At* $i = i_0$, *however, there is at least one element of* M_i *that is not "connected" to the set of elements that map to 1. In particular, this element (a counterexample) is not connected to any element of* M_i *for any* i, $2 \leq i < i_0$. *Otherwise, if there are no counterexamples, the set of elements of* M_i, *for all* $i \geq 2$, are *"connected" to elements of the infinite set of recursive "spiral"s with base element 1.*

Strategies for Proving the 3x + 1 Conjecture That Are Suggested by the Answers to Question 1

The above answers suggest the following strategies.

We would have a proof of the Conjecture if we could show:

(1) that the assumption of a counterexample implies that one or more non-counterexamples were not mapped to by exponent sequences that an existing result required.

A major problem connected with this strategy is that, at each $i \geq i_0$, where i_0 is the smallest level at which a counterexample is an anchor at i, there *does* exist a set of non-counterexample range elements that is, in fact, mapped to by every i-level exponent sequence, and similarly for counterexample anchors. This is not a contradiction to Lemma 10.96, above, because to obtain each such set of range elements requires that we go outside the set of anchors at i. Further details on these sets are given in "Appendix A1 — Lemmas and Definitions Used in Implementations of the "Pushing Away" and "Missing Sequences" Strategies" on page 66.

Or we would have a proof of the Conjecture if we could show:

(2) that a counterexample never becomes an element of an anchor tuple at any level i (for this implies that no counterexample exists).

A brief description of this strategy is given in the next sub-section. Several possible proofs of the $3x + 1$ Conjecture derived from this strategy — which we call the "Pushing Away" Strategy — are given in "Appendix B — Possible Proofs of the 3x + 1 Conjecture Using the "Pushing Away" Strategy" on page 80.

Or we would have a proof of the Conjecture if we could show:

(3) that there is no minimum counterexample.

A discussion of this strategy is given under "Strategy of Proving There Is No Minimum Counterexample" on page 26.

The "Pushing Away" Strategy in Brief

In the "Pushing Away" Strategy we attempt to show that every tuple containing an assumed counterexample is "pushed away" from tuples whose elements map to 1, i.e., every tuple containing a counterexample must always be the second, or third, or fourth, or ... tuple in any tuple-set, but never the first. Thus counterexample tuples never become anchor tuples, hence counterexample tuples do not exist (by the Corollaries to Lemmas 10.90 and 10.91 (see Appendix A1)).

How the Pushing Away Strategy Resolves a Seeming Paradox Concerning Tuple-sets

The basic idea underlying the Pushing Away strategies can be used to resolve a seeming paradox concerning the cardinality of tuples and of tuple-sets. Stated informally, the seeming paradox arises as follows. It is easily shown that the cardinality of tuple-sets is countably infinite (Lemma 1.2). But tuple-sets are defined by finite sequences of positive integers. Since we know that every tuple $<x>$, x an odd, positive integer, has an infinite sequence of extensions, $\{<x>, <x, y>,$

<*x*, *y*, *y'*>, ... }, and that such a sequence defines an infinite sequence of exponent sequences that we shall denote {*A*({}), *A*(<*x*>), A(<*x*, *y*>), A(<*x*, *y*, *y'*>), ... } we can speak of tuple-sets that are, "in the limit", defined by infinite sequences of positive integers. The cardinality of all infinite sequences of positive integers is easily shown to be uncountably infinite. But then what are the contents of this uncountable infinity of tuple-sets, given that only a countable subset contain infinite tuples generated by odd, positive integers? Are "most" of the tuple-sets in this uncountable infinity empty? But if so, how does the tuple-set generated by an infinite sequence of positive integers that is *not* a sequence generated by an odd, positive integer, "know" when to stop containing tuples, given that *every* finite sequence of positive integers generates a tuple-set containing an infinity of tuples, even if the finite sequence is the initial part of an infinite sequence not defined by an odd, positive integer *x*? In other words, the seeming paradox is that a sequence of tuple-sets, each set containing an infinity of tuples, can, "in the limit", be empty (if that is in fact the case).

The resolution of this seeming paradox is as follows. Let *x* be any odd, positive integer. (It is irrelevant here whether *x* is a counterexample or not.) Then {<*x*>, <*x*, *y*>, <*x*, *y*, *y'*>, ... } is an infinite sequence of tuples that gives rise to an infinite sequence of tuple-set extensions. (Each tuple defines an exponent sequence that defines a tuple-set.)

Now it is easily shown (Lemmas 3.0, 4.0) that there exists an *i* such that some tuple *t* in the above sequence, *t* having length (i - 1), is the first *i*-level tuple in its *i*-level tuple-set[1], and that all tuples that are extensions of *t* remain first ($i + k$)-level tuples in their respective ($i + k$)-level tuple-sets, $k \geq 1$.

But now consider an infinite sequence *s* of positive integers that is *not* one of the infinite sequences generated by extensions of tuples <*x*>, where *x* is any odd, positive integer. In this case, no first *i*-level tuple in an *i*-level tuple-set remains a first ($i + k$)-level tuple in all ($i + k$)-level tuple-sets generated by $s(i + k)$, where $k \geq 1$ and $s(i + k)$ is the first ($i + k$ - 1) elements of *s*. In other words, if we could observe the sequence of tuple-sets generated by the sequence of exponent sequences *s*(2), *s*(3), *s*(4), ... we would observe that the first *i*-level tuple in each corresponding *i*-level tuple-set does not permanently remain an extension of the same tuple <*x*> ! Informally, the first *i*-level tuples "keep moving to the right", meaning that they keep having higher and higher numbers *x* as their first elements. (This phenomenon is explained in more detail in the next sub-section. Examples of infinite sequences *s* of positive integers that are *not* one of the infinite sequences generated by extensions of tuples <*x*>, where *x* is any odd, positive integer, are given.) Thus, indeed, "in the limit", the tuple-sets generated by infinite exponent sequences *s* different from those generated by odd, positive integers *x* are "empty". For, if you specify any *x* you claim is the first element of a tuple in one of these tuple-sets , I can show you a tuple-set defined by some sequence *s*(*i*) in which *x* is not the first element of any tuple.

However, in the case of tuple-sets generated by sequences corresponding to those generated by odd, positive *x* (regardless whether *x* ultimately maps to 1 or *x* is a counterexample), "in the limit" the infinite tuple <*x*, *y*, *y'*, ...> is the first *and only* tuple in the corresponding tuple-set, for the distance functions defined in Lemmas 1.0 and 1.1 imply that all other tuples are pushed infinitely far away. Thus the tuple-set defined by the infinite sequence of positive integers defined by <*x*, *y*, *y'*, ...> is not empty.

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1. Such a tuple is called an *anchor tuple*. See definition in "Preliminary Discussion of Strategies" on page 19.

The fact that the same "pushing away" phenomenon that underlies our above-described Pushing Away strategies for proving that no counterexamples exist, also resolves the seeming paradox concerning (some) tuple-sets defined by infinite sequences of positive integers — a paradox having nothing to do with counterexamples — lends support, at least in our opinion, to the importance of the pushing away phenomenon.

Some Infinite Exponent Sequences That are Not Generated by Any Odd, Positive Integer, *x*

It is easily shown that the cardinality of all infinite sequences of positive integers is uncountable, whereas the cardinality of the odd, positive integers is countable, and each such integer generates (defines) exactly one infinite sequence of exponents. This is the simplest proof that there are infinite exponent sequences that are not generated by any domain element, *x*. (The reader is encouraged to also read "Some Infinite Inverse Exponent Sequences That are Not Generated by Any Range Element, y" on page 43.)

An obvious next question is, can we give specific examples of infinite exponent sequences that are not generated by any odd, positive integer? The answer is yes.

Lemma 1.5. *Each cycle in the odd,* ***negative*** *integers defines an infinite exponent sequence A such that no odd, positive integer x generates A. Examples of such sequences A are: A * {1, 1,... }, A * {1, 2, 1, 2, ..., } and A * {1, 1, 1, 2, 1, 1, 4,1, 1, 1, 2, 1, 1, 4,...}, where A is a finite (possibly empty) exponent sequence, and "*" denotes concatenation of sequences.*

Proof:

1. The reader can easily verify for himself that the sequences following *A* in the statement of the Lemma, do, in fact, define cycles in the odd, *negative* integers: {1, 1, 1, ... } is generated by –1 (the cycle is <–1, –1, ... >). {1, 2, 1, 2, ... } is generated by –5 (the cycle is <–5, –7, –5, ...>). {1, 1, 1, 2, 1, 1, 4} is generated by –17 (the cycle is <–17, –25, –37, –55, –41, –61, –91, –17, ...>).

2. As stated under "Remarks About the Distance Functions" on page 12, it is easily shown that the distance functions defined by Lemma 1.0 (a) and (b) extend to the odd, *negative* integers. We will call a tuple-set that includes the odd, negative integers, an *extended tuple-set*.

3. Assume an odd, positive integer *x* exists such that *x* generates the sequence {1, 1, 1, ..., }. Then *x* and -1 are first elements of tuples in the infinite sequence of extended tuple-sets defined by {1}, {1, 1}, {1, 1, 1,}, ...

4. But this means that eventually the distance functions defined by Lemma 1.0 (a) and (b) will be contradicted, e.g., -1 and *x* will, for all *i* greater than some minimum *i*, be the first elements of consecutive *i*-level tuples, which contradicts Lemma 1.0 (b).

4. A similar argument applies to the other two cycles.

5. Lemma 2.0 assures us that the argument holds following any arbitrary finite exponent sequence A. □

Strategy of Proving There Is No Minimum Counterexample

Assume a counterexample exists. Then there must be a minimum counterexample. If we can show that there is no minimum counterexample, then we will have shown that counterexamples do not exist.

Consider the tuple $<x, y>$ where $x \neq y$. Then if $x < y$, y cannot be the minimum counterexample, because it is mapped to by a smaller odd, positive integer, namely, x. On the other hand, if $x > y$ then x cannot be the minimum counterexample, because it maps to a smaller odd, positive integer, namely, y.

If in any i-level tuple-set, regardless how large i is, there exists an i-level tuple t such that the i-level element y of t is less than the first element x of t, then x cannot be the minimum counterexample.

We can say more.

Infinities of Odd, Positive Integers None of Which Can Be the Minimum Counterexample

Lemma 25.0:

For each i-level tuple-set such that 2 • 3(i – 1) < $(2)2^{a_2}2^{a_3}\ldots2^{a_j}$*, and such that the i-level element y of the first i-level tuple is less than the first element x of the first i-level tuple, the i-level element of the nth i-level tuple is less than the first element of the nth i-level tuple, where n ≥ 1.*

Proof: The distance from y to the i-level element of the *second* i-level element of the tuple-set is $2 \cdot 3(i - 1)$ (part (a) of "Lemma 1.0" on page 11). The distance from x to the first element of the *second* i-level tuple is $(2)2^{a_2}2^{a_3}\ldots2^{a_j}$ (part (b) of "Lemma 1.0" on page 11). Therefore, the i-level element of the second i-level tuple is less than the first element of the second i-level tuple. And so on for the third, fourth, fifth ... i-level tuples. □

Example

In the 3-level tuple-set $T_{\{1,4\}}$ the first few 3-level tuples are

<3, 5, 1>,

<67, 101, 19>,

<131, 197, 37>,

<195, 293, 55>

The last element of each tuple is less than the first.

It is easy to show that no element of the level 1 congruence class $x + n((2)2^1 2^{a_3})$, where x is the minimum residue of the class, $n \geq 0$, in any tuple-set T_A, where $A = \{1, a_3\}$, and $a_3 \geq 3$, is the minimum counterexample.

Thus, since for each i, there is an infinity of $(2)2^{a_2}2^{a_3}\ldots2^{a_j}$ that are greater than $2 \cdot 3^{(i-1)}$, and since there is an infinity of first level elements of i-level tuples in each such tuple-set, there is an *infinity of infinities* of odd, positive integers, none of which can be the minimum counterexample.

Lemma 26.0:

If a counterexample to the 3x + 1 Conjecture exists, then the minimum counterexample must be an element of the congruence class $3 + n((2)2^1)$*, where* $n \geq 1$ *— in other words, the class {3, 7, 11, 15, ... }.*

Proof: For each exponent $a_2 \geq 2$, each first element of each tuple in the tuple-set T_A, where $A = \{a_2\}$, is greater than the second element of the tuple, because

$$x > \frac{3x+1}{2^{a_2}}$$

or, in other words, the first element, x, is always greater than the second element, and therefore x cannot be the minimum counterexample.

However, if $a_2 = 1$, then

$$x < \frac{3x+1}{2^1}$$

and so it is possible that x is the minimum counterexample. □

All odd, positive integers up to at least $10^{18} + 1$ are known, by computer test, to be non-counterexamples. Therefore, by Lemma 25.0, each such odd, positive integer in the congruence class {3, 7, 11, 15, ... }gives rise to a congruence class mod 4, no element of which can be the minimum counterexample.

Lemma 27.0

Let x be a non-counterexample > 1, and let $A = \{a_2, a_3, ..., a_i\}$ *be the sequence of exponents associated with the tuple <x, ..., 1>.*

Then each i-level tuple <y, ..., z> in T_A *has the property that y > z, and hence that y cannot be the minimum counterexample.*

Proof:

The result follows from Lemma 25.0. □

The Minimum Counterexample

If there is a minimum counterexample, then it may be either a multiple-of-3 (not mapped to by any odd, positive integer), or a range element.

Consider the minimum counterexample y_c that is a range element.

Then y_c has the following properties:

(1) for all z resulting from iterations of y, $z \geq y_c$;

(2) for all x mapping directly or indirectly to y_c, $x \geq y_c$. In particular, this means that y_c must be mapped to, in one iteration of the $3x + 1$ function, by exponents of even parity. (Lemma 5.0 states that each range element is mapped to by all exponents of one parity only; if y were mapped to, in one iteration of the $3x + 1$ function, by exponents of odd parity, then y would be mapped to by the exponent 1, and that would mean that y_c was mapped to by an $x < y_c$, contradicting the assumption that y_c is the minimum counterexample that is a range element.)

(3) It must be the case that y_c is the first element of an infinite "spiral" (see "Section 2. Recursive "Spiral"s" on page 39). That is, there cannot exist an x such that $y_c = 4x + 1$.

If y_c is not a range element, that is, if it is a multiple of 3, then only (1) and (3) apply.

The existence of a minimum counterexample y_c puts a further constraint on counterexamples, for it must be the case that no counterexample y, in any counterexample tuple, in any tuple-set, is less than y_c. If we can show that this constraint implies a contradiction to Lemma 10.0 — which states that, if counterexamples exist, then every tuple-set contains an infinity of non-counterexample tuples and an infinity of counterexample tuples — ***then we will have a proof of the 3x + 1 Conjecture.***

A further reason why we might be skeptical about the existence of a minimum counterexample is the following:

If a minimum counterexample y_c exists, then all elements of the tree in which y_c is an element must be $\geq y_c$. If we can prove that no tree containing odd, positive integers has this $\geq y_c$ property, ***then we have a proof of the 3x + 1 Conjecture***.

It is important to keep in mind that each finite prefix $<y_c, ..., z>$ of the (infinite) minimum counterexample tuple $<y_c, ... >$ is an element of a tuple-set, namely, the tuple-set whose exponent sequence is associated with the prefix. Each such tuple-set contains an infinity of non-counterexample tuples, as well as an infinity of counterexample tuples. So there is no chance of the minimum counterexample tuple diverging from non-counterexample tuples as far as exponent sequences are concerned.

Above all, let us not forget that if the minimum counterexample y_c exists, then an infinity of counterexamples exists, each of which is the first element of an infinite tuple, no element of which is less than y_c. Furthermore, for each such infinite tuple, there exists a minimum i such that there exists an i-level tuple-set such that the i-level element z of the first i-level tuple t has the property that $1 \leq z < 2 \bullet 3^{(i-1)}$. The $(i + k)$-level element , where $k \geq 0$, of the first $(i + k)$-level tuple in each $(i + k)$-level extension of t has the corresponding property if each extension exists in the tuple-set defined by that extension.

Extensions of Tuple-Sets into the Odd, Negative Integers

We should not fail to take advantage of the fact that tuple-sets can be extended into the odd, negative integers. The result is the negative of the $3x - 1$ function over the negative integers. Each element of the assumed infinite counterexample tuple with first element y_c must obey the Distance Functions set forth in Table 1., "Distances between elements of tuples consecutive at level i" on page 13. Informally, it might be that the successive i-level distances between i-level elements of our counterexample infinite tuple, and corresponding elements of $3x - 1$ tuples, force our counterexample infinite tuple to give up its exponent sequence, and pass on its non-decreasing property to a tuple with a larger first element, etc.

The "Beaking" of Tuples

Another approach that definitely seems worth investigating is based on the phenomenon of the *breaking* of tuples in a tuple-set. The term simply means the non-continuation of a tuple in the tuple-set beyond a certain level. If we could show that some extension of each counterexample tuple in each tuple-set must eventually break, ***then we would have a proof of the 3x + 1 Conjec-***

ture, because that would imply that counterexample tuples, which by definition must lie in infinite extensions of one tuple-set, do not exist. Here is an example of the phenomenon.

Consider the tuple <15, 23, 35, 53>. This tuple lies in the tuple-set $T_{\{1, 1, 1\}}$. However, 53 maps to 5 via the exponent 5. Thus we say that the tuple breaks at 53 in the tuple-set $T_{\{1, 1, 1, 1\}}$.

By part (a) of Lemma 1.0, the distance interval at level 6 in a 6-level tuple-set is 486, and clearly, –1 + 486 does not equal 5(!). Nor does it equal 80, which is (3(53) + 1)/2. So we can say, informally, that it is necessary, in $T_{\{1, 1, 1,1\}}$ to go to a tuple farther to the right, which is accomplished by the breaking of our original tuple. That tuple farther to the right is <63, 95, 143, 215, 323, 485>, and sure enough –1 + 486 = 485.

The astute reader will, of course, ask if there will always be a known infinite tuple in the odd, negative integers, to break potential infinite tuples in the odd, positive integers, and if there will not be, how can we guarantee the breaking of potential infinite tuples?

In attempting to answer that question, we need to consider the only infinite tuples we know about, namely non-counterexample tuples, each of which terminates in < ..., 1, 1, 1, ...>. For each i-level tuple whose i-level element is 1 (this tuple is necessarily the first i-level tuple in its tuple-set), there must be an i-level element of a negative tuple at the distance $2 \cdot 3^{(i-1)}$ to the left of 1, and an i-level element of a positive tuple at the same distance to the right of 1. But no counterexample tuple can contain a 1. So is it possible that eventually (for some i), there is no longer an element on the left, or on the right, or both, that fulfills this distance requirement? The answer is no, because that would be a violation of the distance function defined in part (a) of Lemma 1.0.

Of course, the astute reader will point out that at each level i, 1 is only one of the odd, positive integers less than $2 \cdot 3^{(i-1)}$ (excluding multiples of 3). How is it that all those other non-counterexample integers less than $2 \cdot 3^{(i-1)}$ (excluding multiples of 3), don't run into trouble? The answer may rest upon the fact that each tuple is associated with the sequence of tuple-set extensions defined by its own sequence of extensions, So different non-counterexamples live inside of different sequences of tuple extensions. each associated with different exponent sequences, hence different tuple-sets. This may give each non-counterexample tuple a different length of time before the tuple element 1 occurs, which assures immortality, that is, satisfaction of the left and right difference requirements, for all larger i.

How to Prevent the Breaking of a Tuple

We can prevent the breaking of a given tuple by simply allowing its extensions to define the tuple-sets the extensions reside in.

Counterexample First Tuples vs. Non-Counterexample First Tuples

We need to investigate how the behavior of arbitrarily long extensions of the tuple $<y_c>$, all of whose elements must always be $\geq y_c$ affects the set of first i-level tuples in i-level tuple-sets. We know that, up to at least $i = 35$, the first i-level tuples in *all* i-level tuple-sets are non-counterexample tuples, and therefore these are associated with the set of *all* i-level exponent sequences. Each exponent sequence defines exactly one tuple-set, hence no two i-level first tuples of i-level tuple-sets, can be associated with the same exponent sequence. If we could show that, for all i, the set of i-level first tuples of all i-level tuple-sets were always non-counterexample tuples, ***then we would have a proof of the 3x + 1 Conjecture***, because there would be no "room" for a counterexample first i-level tuple.

How the Minimum Counterexample Becomes a First *i*-Level Tuple...

Let us review how the minimum counterexample y_c becomes a first *i*-level tuple *t* in an *i*-level tuple-set, and necessarily remains a first $(i + k)$ - level tuple in a sequence of $(i + k)$ -level tuple-sets that are associated with extensions of *t* , where $k \geq 0$.

For some smallest *i*, the *i*-level element of the *i*-level extension of $<y_c>$, is less than $2 \cdot 3^{(i-1)}$. It is tempting to argue that if counterexamples did not exist, this tuple would be a non-counterexample tuple, and then claim that this constitutes a contradiction to Lemma 8.8 — which states that each non-counterexample, hence each non-counterexample tuple, remains a non-counterexample, hence a non-counterexample tuple, whether or not counterexamples exist, but we are unsure about this reasoning.

Comparison of Tuples $<x, ..., 1, 1, 1, ...>$ and $<y_c, ..., w_1, w_2, w_3, ... >$

The crucial question is: Is it possible that, on the one hand, every non-counterexample tuple is eventually $<x, ..., 1, 1, 1, ...>$, and on the other hand, our minimum counterexample tuple is eventually $<y_c, ..., w_1, w_2, w_3, ... >$, where all elements w_i are greater than 1 and less than $2 \cdot 3^{(i-1)}$ for an infinite succession of increasing *i*'s? If we can prove that it is impossible, ***then we will have a proof of the 3x + 1 Conjecture***.

One reason it might be impossible is the following. By Lemma 8.8 non-counterexample tuples are unchanged by the existence or non-existence of counterexample tuples. But if counterexample tuples exist, then each tuple-set defined by extensions of counterexample tuples, must have an infinity of non-counterexample tuples that are the same as those in the tuple-sets defined by extensions of non-counterexample tuples. This does not seem possible, given that the first element of the first *i*-level (counterexample) tuple is not a non-counterexample tuple, and given that the tuple-set has an infinity of non-counterexample tuples. The only way we could avoid a contradiction is if each of the infinity of non-counterexample tuples in each such tuple-set, was the same as a non-counterexample in a tuple-set defined by extensions of non-counterexample tuples.

But perhaps there is another approach. First, we must remember that whether or not a counterexample exists, for all $i \geq 2$, there is a first *i*-level tuple in each *i*-level tuple-set for all *i*-level exponent sequences. This tuple is always a non-counterexample tuple if counterexamples do not exist, or, if counterexamples, e.g., y_c, exist, then from some *i* on, and for the exponent sequence associated with the counterexample tuple, this is a counterexample tuple. Now here is **the crucial point**:

If counterexamples do not exist, then the smallest non-counterexample associated with any *i*-level exponent sequence (including any that a counterexample tuple "could" be associated with) is the first element of the first *i*-level tuple in the tuple-set defined by that exponent sequence.

But if counterexamples exist, hence if y_c exists, then the smallest non-counterexample associated with the *i*-level exponent sequence associated with the y_c tuple, is *not* the first element, as previously explained.

If this violates Lemma 8.8 — if the existence of a counterexample changes which odd, positive integers are non-counterexamples — ***then we have a proof of the 3x + 1 Conjecture***. (See "A Second Possible Strategy for a Proof of the 3x + 1 Conjecture" on page 33.)

Let us consider the matter more closely. For each finite sequence of 1's, there is associated with the non-counterexample tuple $<x, ..., 1, 1, ..., 1>$ a tuple-set. That tuple set, by Lemma 5.0, contains an infinity of non-counterexample tuples and, if counterexamples exist, an infinity of counterexample tuples. In the case of the minimum counterexample y_c, there is associated with the counterexample tuple <y_c, ..., w_1, w_2, w_3,...w_k>, a tuple-set. That tuple-set, by Lemma 5.0, contains an infinity of non-counterexample tuples and an infinity of counterexample tuples.

How Tuple-sets "Work"

To answer this question, we need to consider how tuple-sets "work". Each *i*-level tuple-set, where $i \geq 2$, can be extended by any positive integer. If the positive integer *m* is that by which the first *i*-level tuple is extended, then the extended tuple remains the first (*i* + 1) - level tuple in the resulting (i + 1) - level tuple-set. If not, then the first tuple in the (*i* + 1) level tuple-set is the first one, in the linear ordering of *i*-level tuples in the *i*-level tuple-set, that is extended by *m*. An infinite tuple results from infinite extensions of a tuple, each of which establishes the tuple-set that the tuple is in.

In any case, the set of elements of first tuples in any *i*-level tuple-set is the set $\{x + (n((2)2^{a_2}2^{a_3}\ldots2^{a_j})\}$, where *x* is the the first element of the first *i*-level tuple, and $n \geq 0$. If counterexamples exist, then an infinity of these *n* must give non-counterexamples, and an infinity must give counterexamples. If we can show that the existence of counterexample first i-level tuples gives different non-counterexamples than if counterexamples do not exist, ***then we will have a proof of the 3x + 1 Conjecture***. (See "A Second Possible Strategy for a Proof of the 3x + 1 Conjecture" on page 33.)

In thinking about this strategy, we must keep in mind that:

(1) for each *i*-level exponent sequence, there is exactly one tuple-set defined by that sequence. It contains exactly one *i*-level first tuple.

(2) counterexamples are not odd, positive integers that exist if counterexamples exist, but do not exist if they don't. The set of odd, positive integers is present in the set of all tuple-sets whether or not counterexamples exist, because, at the least, the set of first elements of all tuples in each tuple-set, is the set of odd, positive integers (follows from definition of *tuple-set*).

(3) If the minimum counterexample y_c exists, then starting at the first *i* such that an element of the tuple <y_c ... > lies between 1 and $2 \bullet 3^{(i-1)}$, first elements of (*i* + *k*) - level tuples in (*i* + *k*) - level tuple-sets, where $k \geq 0$, and where each tuple set is associated with the extension of the first (i + *k* – 1) - level counterexample tuple in the previous (*i* + *k* – 1) - level tuple-set, are given by:

$y_c + (n((2)2^{a_2}2^{a_3}\ldots2^{a_{i+k}}))$, where $n \geq 0$.

Recall that, for each $i \geq 2$, the set of *i*-level exponent sequences is the set of *i*-level exponent sequences associated with the set of all first *i*-level tuples in the set of all *i*-level tuple-sets. Since there is one and only one such tuple for each such exponent sequence, this means that, for each *i* + *k*, there is *no* non-counterexample first (*i* + *k*) - level tuple associated with the exponent sequence that a counterexample first (*i* + l) - level tuple is associated with.

Yet in each (*i* + *k*) - level tuple-set having a counterexample first (*i* + *k*) - level tuple, there is an *infinity* of non-counterexample tuples associated with that tuple-set's exponent sequence! (By definition of *tuple-set*.) Does that give us the basis for a proof of the 3*x* + 1 Conjecture? We are not sure, because on the one hand, this is precisely the case in the 3*x* – 1 function, but on the other, the first counterexample first tuple in an *i*-level tuple-set for that function occurs already at *i* = 2, the tuple being <7, 5>.

A Possible Strategy for a Proof of the 3*x* + 1 Conjecture

Each non-counterexample x yields an infinity of successive tuple extensions <x, ..., 1, 1, ..., 1> each extension terminating in a finite sequence of 1's, with the number of 1s increasing with each extension. Each such extension is associated with the tuple-set in which it is a tuple. That tuple-set is associated with an exponent sequence A = {..., 2, 2, ..., 2}. If counterexamples exist, that tuple-set must contain an infinity of counterexample tuples.

If counterexamples exist, then each of an infinity of counterexamples y yields an infinity of successive tuple extensions <y, ..., w_1, w_2, w_3,...w_k>, each extension terminating in a finite sequence w_1, w_2, w_3,...w_k,, no w_i being equal to 1, and k increasing with each extension. Each such extension is associated with the an exponent sequence B ={..., b_1, b_2, ..., b_k}.

But for all sufficiently large i, no counterexample tuple, which is necessarily associated with an exponent sequence B, can be an element of a tuple-set having an exponent sequence A = {..., 2, 2, ..., 2} unless (1) The "..." part of B equals the "..." part of A, and (2) w_1, w_2, w_3,...w_k are each equal to 2.

Regarding (1): if the first element of a tuple is the minimum counterexample y_c, then "..." in the associated exponent sequence can never equal "..." in the exponent sequence "..." of A, because the latter, like all exponent sequences associated with a tuple <x, ..., 1> , is associated only with tuples in which the first element is greater than the last element. Clearly, the minimum counterexample y_c cannot be the first element of such a tuple. *But then y_c can never be the first element of a tuple-set associated with a tuple* <x, ..., 1>!

Regarding (2): except in the case of the tuple-element 1, the exponent 2 always produces an element *less than* the element on which the iteration was performed (for example, C(9) = 7 via the exponent 2). Thus successive w_i each equal to 2 produce decreasing tuple elements.

But this cannot continue indefinitely, for no counterexample tuple can contain an element less than the minimum counterexample, y_c.

On the other hand, the sequence of 1s terminating a non-counterexample tuple *can and does* continue indefinitely, and each such sequence is associated with a sequence of exponents = 2.

If we can show that these facts lead to a contradiction to Lemma 5.0 — which states that if counterexamples exist, then each tuple-set contains an infinity of counterexample tuples, and an infinity of non-counterexample tuples — then we will have a proof that counterexamples do not exist.

Remark 1: **An argument against this strategy** is the fact that there *is* a minimum counterexample to the 3x – 1 Conjecture. It is 5. So we feel that every strategy employing the possible minimum counterexample for a proof of the 3x + 1 Conjecture, must be checked against the 3x – 1 function.

Remark 1: **Another argument against this strategy** is the following. The exponent sequences associated with counterexample tuples of increasing length must terminate in increasingly long sequences of 2's, each of which decreases the value of the last element of the tuple relative to the first, and thus sends these values in the direction of a value less than that of the minimum counterexample, a contradiction that would give us a proof of the Conjecture. However, the first element of each of these tuples grows larger and larger, as the counterexample tuples are "pushed" farther and farther to the right, in accordance with the distance function for first elements of *)*-level tuples (see "Lemma 1.0" on page 11. So this fact may nullify any hope we might have of the tuple-sets eventually "running out of" counterexample tuples.

Remark 3: A question whose answer might give us a proof of the Conjecture is the following: how do the *i*-level tuples in any *i*-level tuple-set associated with the non-counterexample tuple <*x*, ..., 1, 1, 1, ..., 1> differ if (1) counterexamples do not exist, (b) counterexamples exist? If the answer is that they do not differ, then that implies that counterexample tuples and non-counterexamples are the same, which, of course is impossible. Therefore counterexamples do not exist.

A Second Possible Strategy for a Proof of the 3*x* + 1 Conjecture

1. Assume a counterexample exists, and let y_c be the minimum counterexample. The infinite tuple $<y_c, ... >$ has the property that no element of $<y_c, ... >$ — and, indeed, no element of any counterexample infinite tuple — is less than y_c, otherwise, y_c would not be the minimum counterexample.

2. For all levels $i + k$, where $k \geq 0$ and i is the smallest i such that an element of an extension of $<y_c>$ is less than $2 \bullet 3^{i-1}$, y_c is the first element of the first $(i + k)$-level tuple in the $(i + k)$-level tuple-set associated with the $(i + k)$-level extension of $<y_c>$. The set of all first elements of all $(i + k)$-level tuples in that tuple-set are given by

$$y_c + (n((2)2^{a_2}2^{a_3}\dots2^{a_{i+k}})) \text{ , where } n \geq 0.$$

y_c is a minimum residue of the modulus $(2)2^{a_2}2^{a_3}\dots2^{a_{i+k}}$.

A countable infinity of the *n* must yield non-counterexamples, by Lemma 5.0 in our paper, "Are We Near a Solution to the 3*x* + 1 Problem" on occampress.com..

3. The exponent sequence of each infinite tuple, counterexample or non-counterexample, is approximated by the exponent sequences associated with an infinite sequence of finite non-counterexample tuples.

Thus if counterexamples did not exist, then for the same exponent sequences associated with extensions of $<y_c>$, we would have

$$\{x + (n((2)2^{a_2}2^{a_3}\dots2^{a_{i+k}})\}\text{, where } n \geq 0.$$

x is a minimum residue of the modulus $(2)2^{a_2}2^{a_3}\dots2^{a_{i+k}}$. but the *x's* change as the modulus increases, or at least after a finite number of increases in the moduli.. If they didn't change, then we would simply have the y_c case again[1].

---

1. Each *i*-level tuple-set, where $i \geq 2$, has an extension for each positive integer, i.e., for each possible exponent . If the exponent is the one by which the first *i*-level tuple is extended, then the extended tuple will be the first $(i + 1)$-level tuple in the $(i+1)$- level tuple-set that the extended tuple occupies. Otherwise, the first (i+1)-level tuple in the (i + 1)-level tuple-set that the tuple occupies, will be the extension of the first tuple in the i-level tuple-set that is extended by the exponent.
Thus, for example, <3, 5> is the first 2-level tuple in the 2-level tuple-set $T_{\{1\}}$. It is extended by the exponent 4, yielding the first 3-level tuple <3, 5, 1> in the 3-level tuple-set $T_{\{1, 4\}}$. If we extend the tuple-set $T_{\{1\}}$ by the exponent 1, then the first 3-level tuple in the 3-level tuple-set T{1, 1} is <7, 11, 17>, because <7, 11> is the first 2-level tuple in T{1} that is extended by 1.

A countable infinity of the n must yield non-counterexamples, by Lemma 5.0 in our paper, “Are We Near a Solution to the $3x + 1$ Problem? on occampress.com, but they must be the *same* counterexamples as in the y_c case, by Lemma 8.8 in the first part of our paper, “The Structure of the 3x + 1 Function: An Introduction” on occampress.com.. This Lemma simply states in effect that the existence of a non-counterexample is *not* subject to the existence or non-existence of counterexamples. Thus, for example, 13 maps to 1 today, and if the $3x + 1$ Conjecture is proved true tomorrow, it will continue to map to 1, and if the $3x + 1$ Conjecture is proved false tomorrow, it will *still* map to 1.)

4. But it is not possible for the same set of non-counterexamples to be yielded via the same modulus *if* the minimum residues are different (as they are in the case of y_c and x), just as in elementary congruence theory, the integers congruent to r mod m are all different from the integers congruent to s mod m, when r, s are different minimum resides.

If our reasoning is correct, this contradiction gives us a proof of the $3x + 1$ Conjecture.

Remark

The possible proof does not apply to the 3x – 1 function, since in that case, we know that counterexamples exist (5 and 7 are the smallest ones), whereas in our case we do not know that, and hence we can apply our reasoning.

More on Minimum Counterexamples

The reader might be interested in several results regarding the minimum counterexample given in “Appendix G — Results on the Minimum Counterexample” in the second file of this paper, on the web site occampress.com. Further results are contained in the section, “Strategy of Proving There Is No Minimum Counterexample” in the second file of our paper, “The Structure of the 3x + 1 Function” on occampress.com.

Testing for Counterexamples

We cannot *necessarily* determine by computer testing of each successive odd, positive integer, if counterexamples exist. Of course, if such a test reveals a cycle, then we have determined that a counterexample exists. However, if the test program simply keeps running beyond the time our computer resources allow, we cannot know whether the reason is that the original number x being tested is a counterexample, or whether the reason is that it simply takes an inordinate length of time for n to yield 1.

A Way to Reduce Computation Time in Computer Testing of the 3*x* + 1 Conjecture

The existence of exponent sequences with the less-to-greater property suggests a method for reducing the computation time for testing the $3x + 1$ Conjecture.

If a counterexample exists, then there is a minimum counterexample. Consider any sequence A of exponents having the less-to-greater property. Now since, according to reliable reports, the Conjecture has been tested and found valid for all odd, positive integers through 56×10^{15}, and since 56×10^{15} is greater than 2^{55}, we know, by Lemma 1.0(b), that all exponent sequences A having the less-to-greater property, and whose sum is ≤ 54, have been tested and have failed to reveal a minimum counterexample. Therefore the only candidates x for minimum counterexamples must lie at distances of $2 \cdot 2^{54} = 2^{55}$. Furthermore, there exists an algorithm for generating all sequences of a given length ($P(i - 1)$), or a given sum, having the less-to-greater property, and so testing can continue, up to the limits of modern computing power, for a minimum counterexample, making use of the distance function established by Lemma 1.0 (b).

Strategy of Using a Topology Defined on Tuples or Tuple-sets

It is natural to wonder if defining an appropriate topology on tuples or on tuple-sets might lead us to a proof of the $3x + 1$ Conjecture. For example, if we could define a Hausdorff topology on tuples, then show that the assumption of a counterexample to the Conjecture implies that an infinite sequence of tuple-extensions converges to two or more points, we would have a proof of the Conjecture, because that would be a contradiction, since in a Hausdorff space, if an infinite sequence converges, it converges to only one point. Another possibility might be the following: assume that the set of odd, positive integers mapping to 1 is not connected — which would be the case if counterexamples exist — and from that assumption derive a contradiction.

A Possible Topology, TT

We define a *separate* topology TT on the tuples (prefixes of infinite tuples) in the tuple-sets of *each* infinite sequence of tuple-set extensions. Thus, if T_A is a tuple-set, $A = \{a_2, a_3, ..., a_j, ..., a_i\}$, $i \geq 2$, then the topology TT relative to T_A is defined on the tuples in the sequence of tuple-set extensions defined by the exponent sequences $A*\{a_{i+1}\}$, $A*\{a_{i+1}\}*\{a_{i+2}\}$, $A*\{a_{i+1}\}*\{a_{i+2}\}*\{a_{i+3}\}$, ... , where a_{i+j}, $j \geq 1$, is any exponent. Each tuple-set is a neighborhood of the tuples it contains.

The reader can verify for him- or herself that the topologies so defined fulfill the requirements of a topology, namely, that it is a collection of the subsets of the set $\{T\}$ of all tuple-sets in the sequence of tuple-set extensions such that:

$\emptyset$ and $\{T\}$ are in *TT*;
The union of any subcollection of $\{T\}$ is in TT;
The intersection of the elements of any finite subcollection of $\{T\}$ is in TT.[1]

Lemma 10.8. *The topology TT is Hausdorff.*

Proof:

A topology is Hausdorff if, for every pair of points p, p' in the space X, there exist disjoint neighborhoods U_1, U_2 such that U_1 is a neighborhood of p, and U_2 is a neighborhood of p'.

By the distance functions established in Lemmas 1.0 and 1.1, the distance between first elements of tuples consecutive at level i in any i-level tuple-set, increases with i. Therefore, no two infinite tuples can remain indefinitely in the same sequence of tuple-set extensions. Or, in other words, any two infinite tuples <x, ...> and <x', ...>, $x \neq x'$, are eventually in different tuple-sets.
□

Lemma 10.83. *A metric exists on the topological space TT.*

1. From Munkres, James R., *Topology: A First Course*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975, p. 76.

Proof: We define a metric μ on tuples in a tuple-set as follows.

Let t, t' be tuples. Then:

If $t = t'$, $\mu(t, t') = 0$.

If t, t' are tuples in the unique 1-level tuple-set {<1>, <3>, <5>, ...}, $\mu(t, t') = 1/(|(t(1) - t'(1))|)$, where |...| denotes absolute value, and $t(1)$, $t'(1)$ are the first (and only) elements of the tuples t, t' respectively. By definition of the $3x + 1$ function, we know that the denominator in the value of μ in this case is the absolute value of the difference of two odd numbers.

If t, t' are i-level tuples in an i-level tuple-set, $i \geq 2$, $\mu(t, t') = 1/(|t(i) - t'(i)|)$, where |...| denotes absolute value, and $t(i)$, $t'(i)$ are the i-level elements of t, t', respectively. By Lemma 1.0(a) we know that the denominator is the absolute value of a multiple of $2 \bullet 3^{(i-1)}$.

The proof that μ is in fact a metric follows directly from the definition of a metric, namely, a function d from pairs of elements (x, y) of a nonempty set X to the nonnegative real numbers such that:

$d(x, y) = 0$ if and only if $x = y$.

$d(x, y) = d(y, x)$.

$d(x, y) \leq d(x, z) + d(z, y)$ for all z in X. □

Similarity of μ to a Frequently-Used *p*-adic Metric

Readers who are acquainted with p-adic number theory will recognize a similarity between our metric μ and the frequently-used p-adic metric $|x - y|_p$ defined as:

$$|x-y|_p = \frac{1}{p^{ord_p(x-y)}}$$

where $ord_p(x - y)$ is the exponent of the largest power of p that evenly divides $x - y$. Observe that, if x, y are both divisible by different powers of p, then their difference, $x - y$, is divisible by the *lower* of the two powers [5]. This fact corresponds to the fact, derived directly from our definition of the topology TT, that two tuples t, t' are in all tuple-sets defined by exponent sequences that are the same as initial sub-sequences of the exponent sequences for t and t'. At the least, t and t' are in the 1-level tuple-set $T_{\{\}}$. Thus, in the above p-adic metric, two numbers are p-adically "close" (i.e., the p-adic distance between them is small) if they are both divisible by a large power of p. Similarly, two tuples are "close" in terms of our metric μ if they are both elements of a tuple-set defined by a "long" exponent sequence. (They are even closer if they are separated from each other by a "large" number of other tuples in the tuple-set. Further experience with the metric μ is necessary in order to determine if this additional factor — the actual distance between the tuples in a given tuple-set — is necessary for our purposes.)

Observe that our metric μ differs from the above p-adic metric in that, in general, $\mu = 1/(m \bullet 2 \bullet 3^{(i-1)})$, $m \geq 1$.

Using the Topology TT and the Metric μ to Prove the 3*x* + 1 Conjecture

We now show how the topology TT and the metric μ might be used to prove the $3x + 1$ Conjecture. In particular, we describe a possible implementation of the strategy described at the start of this sub-section, namely, that of showing that the assumption of a counterexample implies that the same infinite sequence converges to two points, which is not possible in a metric space.

Let x be an odd, positive integer that ultimately maps to 1. By definition of tuple and of tuple-set, each sequence of tuples, {<*x*>, <*x*, *y*>, <*x*, *y*, *y*′>, ... }, establishes a corresponding sequence of exponent sequences, $\{\{\}, A, A', A'', ...\}$, which in turn defines a sequence of tuple-set extensions, $\{T_{\{\}}, T_A, T_{A'}, T_{A''}, ...\}$.

Assume a counterexample exists. Then by Lemma 10.0, each tuple-set extension contains an infinity of tuples (n-t-v-1s) whose elements map to 1, and an infinity of tuples (n-t-v-cs) each containing counterexamples, and this is true regardless how long the exponent sequence $A''^{\cdots}{}'$ defining the tuple-set is.

Then, in a sense that we believe can be made precise, in the limit, each infinite sequence of exponents converges to two points, namely, a point defined by numbers that map to 1, and a point defined by counterexamples.

Remark About the Above Strategies

Occasionally, a reader will argue that none of the above strategies can be considered valid until we show that it will not also prove that there are no “counterexamples” in the domain of the odd, negative integers. For, if the strategy should in fact prove there are no such “counterexamples”, then the strategy could not possibly be correct, since at least one “counterexample” is known in that domain, namely, –17, which gives rise to an infinite loop.

Our reply to the above argument is the following:

The “Statement of Problem” on page 2 makes clear that the domain of the $3x + 1$ function in this paper is the odd, positive integers. Furthermore, all the proofs of lemmas (and the illustrative examples), and the above strategies of the $3x + 1$ Conjecture, are carried out in this domain. In particular, the number 1, which is explicitly mentioned in several lemmas, and the other minimum residues of the reduced residue classes mod $2 \bullet 3^{(i - 1)}$, $i \geq 2$, which play such an important role in some of the proofs and, in particular, in the above strategies, are odd, positive integers (as these minimum residues always are in number theory). For example, a first i-level tuple in an i-level tuple-set is identified by the fact that its last element is such a minimum residue. (What residues will take the place of these minimum residues in the negative-integer domain?)

Having said all that, we will be the first to admit that the behavior, in terms of tuple-sets, of the $3x + 1$ function on the odd, negative integers, is definitely of interest. In fact, it is easily shown that the distance functions (Lemmas 3.0, 4.0) carry over directly to the odd, negative integers. Thus, e.g., $(3 \bullet (13) + 1)/2^3 = 5$. The distance functions say that the next 2-level tuple in the negative direction should have $13 - 2 \bullet 2^3 = -3$ as first element. And indeed, we find that $(3 \bullet (-3) + 1)/2^3 = -1$, and $-1 + 2 \bullet 3^{(2-1)} = 5$, as the distance functions require.

Nevertheless, either the lemma proofs, and the above strategies, are correct as they stand, or they are not. The question why the strategies show there is no counterexample among the odd, positive integers, and why it is a fact that there is at least one “counterexample” among the odd, negative integers, is a separate issue.

Of course, if the strategies, when taken over the odd, negative and the odd, positive integers, enable us to show that there both is, and is not, a counterexample among the odd, positive integers, then we have discovered something whose importance far exceeds that of the $3x + 1$ Problem, namely, the inconsistency of number theory!

Turning Tuple-sets "Inside-out"

Natural curiosity compels us to ask if it might be worthwhile to investigate the relationship between the sequences of numbers in tuples, and the sequences of numbers that define tuple-sets. Of course, the number of exponents that define any i-level tuple-set T_A, $i \geq 2$, is (i - 1), whereas the number of elements in any tuple in T_A is i. But, as a start, we might consider the question, Is there anything of interest to be learned in taking any tuple in T_A, allowing the sequence of its elements to define another tuple-set $T_{A'}$, picking any tuple in $T_{A'}$ and allowing its elements to define a tuple-set $T_{A''}$, etc.?

Section 2. Recursive "Spiral"s

In the first part of this paper, we described a structure called "tuple-sets" that underlies iterations of the $3x + 1$ function — in other words, that describes iterations of the function in the "forward" direction. In this part, we describe a structure called "recursive 'spiral's" that describes the inverse of the $3x + 1$ function — in other words, describes iterations of the function in the "backward" direction.

The "spatial", "geometric", "graphical" nature of both structures is important for the strategies it suggests.

We begin with some definitions.

Definitions

Recursive "Spiral"

A *recursive "spiral"* is the infinity of odd, positive integers that map to a given range element in one iteration of the $3x + 1$ function, as established in the proof of Lemma 5.0. (See Fig. 4.) Each range element in the infinity of elements in turn sets up a recursive "spiral", etc. Thus the infinite set of all "spiral"s relative to a given range element are a *self-similar* structure ([4], p. 34).

The recursive "spiral" structure has been independently discovered by at least two researchers besides us, although we are not aware of anything in the literature that deals explicitly with this structure.

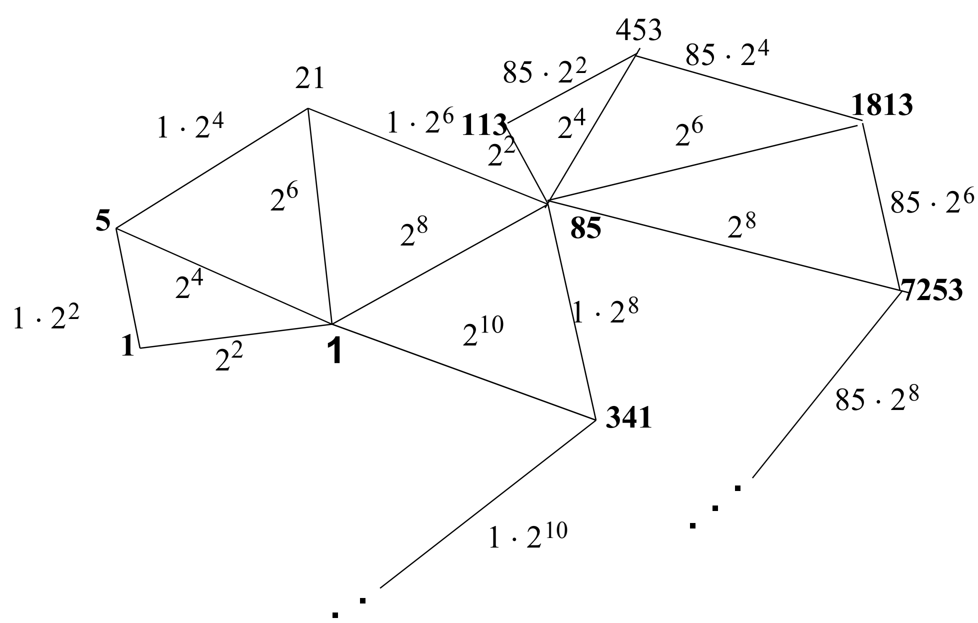

Fig. 4. Recursive "spirals" structure of computations produced by the $3x + 1$ function.

Bold-faced numbers are range elements (21 and 453 are multiples of 3, hence not range elements). Partial "spirals" surrounding the base elements 1 and 85 are shown. The line connecting

1813 to 85 is marked with a 2^6 because $(3 \cdot 1813 + 1)/ 2^6 = 85$. The line connecting 453 to 1813 is marked $85 \cdot 2^4$ because $453 + 85 \cdot 2^4 = 1813$. The quantity $85 \cdot 2^4 = 3 \cdot 453 + 1$, and similarly for the difference between successive elements of a "spiral" in all "spiral"s. These facts follow from the fact that if x, y are consecutive elements of a "spiral", with $x < y$, then $y = 4x + 1$.

The exponents of 2 are not even in all "spiral"s, of course. For example, the "spiral" of numbers (not shown) mapping to 341 has odd exponents.

Level *i* and Base Sequence of Elements

Let y be a range element, e.g., 1. We define y to be at *level 0 relative to y*. We now define all x that map to y in a single iteration to be at *level 1 relative to y*. (*Warning*: no suggestion is intended that the term *level* as defined here is the same as the term *level* as defined above for tuple-sets, although, as we shall see, there is a strong relationship between the two terms.) We define all odd, positive integers of level 1 iterations that map to y to be level 1 *elements*. These elements constitute a sequence (a "spiral" in Figure 4). Specifically, they constitute a unique *base sequence* (relative to y). Thus, for example, the base sequence relative to 1 is the sequence {1, 5, 21, 85, 341, ... }. We define all x that map to a level 1 element in a single iteration to be at level 2 relative to y, and similarly for level 2 elements. And so on for all levels i. When y is understood, we will sometimes eliminate the phrase, "relative to y". The expressions *level i sequence* and *level i "spiral"* thus mean the same thing.

We define the range element mapped to, in a single iteration, by each element of a "spiral", to be a *center element*, because it is the center of a "spiral", as shown in Figure 4. Often, we will call a center element a *base element*. The infinite set of elements that map to a given base element corresponds, in [3] (p. 21), to a *predecessor set*, although unlike a predecessor set, the infinite set of elements that map to a given base element contains no even numbers.

We say that elements of the base 1 sequence map *directly* to the base element, and that elements of level i, $i > 1$, map *indirectly* to the base element.

We define a *path*, relative to any base element y, to be a finite sequence of elements of "spirals" at levels $i, i - 1, i - 2, ... 0, i \geq 0$, such that the "spiral" element at level j, $1 \leq j \leq i$ maps directly to the element at level $j - 1$ in a single iteration. Thus, e.g., <13, 5, 1> is a path. A path is thus the equivalent of a tuple. Each path defines an exponent sequence, e.g., in the case of our example, the sequence {3, 4}.

Some examples of elements at different levels: If $y = 1$, then 1, 5, 21, 85, 341, ... are level 1 elements relative to 1. They also constitute the base sequence relative to 1, which is the center, or base element. For the center element (or base element) 5 in this base sequence, the level 1 elements are 3, 13, 53, 213, 853, ... These are level 2 elements relative to 1.

We define the set of odd, positive integers lying *between* any two successive elements of a level i sequence to be *interval*s of that sequence. Thus, for example, 7, 9, 11, 13, 15, 17, 19, are the elements of the second interval in the level 1 sequence, 1, 5, 21, 85, 341, ... When necessary, we number the intervals in a given sequence starting with 1.

Distance Functions on "Spiral"s

The proof of Lemma 5.0 implicitly defines two distance functions on "spirals": one, between any "spiral" element and the base element of the "spiral", and the other between successive elements of a "spiral". We will refer to these simply as *"spiral" distance functions*, specifying which one we mean as required. We give these functions in the next lemma.

Lemma 11.0. *(a) The distance between the jth element, $j \geq 1$, of a "spiral", and the base element y of the "spiral", is given by $|(2^k y - 1)/3 - y|$, where k is the jth element in the sequence <1, 3, 5, ..., > or the sequence <2, 4, 6, ..., > as established by y.*

(b) The distance between successive elements x, x′ of a "spiral" is given by $3x + 1$, i.e., $x' = 4x + 1$;

(c) If x, x′ are elements of a "spiral" then $x, x' \equiv 5 \bmod 8$.

Proof:

(a) Follows directly from Lemma 5.0.

(b) By Lemma 5.0 we have

$$\frac{3x+1}{2^j} = y$$

and

$$\frac{3x'+1}{2^{j+2}} = y$$

so that

$$\frac{3x+1}{2^j} = \frac{3x'+1}{2^{j+2}}$$

and hence

$$2^2x + 1 = x'$$

and thus

$$x' - x = (2^2x + 1) - x = 3x + 1$$

(c) Follows directly from (b). □

Summary of Properties of Recursive "Spiral"s

We now provide a table that summarizes our results on recursive "spiral"s.

Note: some table-rows may have the same content as other rows, though under different properties. This redundancy is deliberate, the purpose being to aid understanding and to make the looking up of properties easier.

Statements of all referenced lemmas are given in "Appendix A — Statements of Lemmas" on page 49.

Table 6: Some important properties of recursive "spiral"s

Property	Value of property	Reference
Self-similarity	Each (non-multiple-of-3) element of a "spiral" is the base element of a "spiral" each (non-multiple-of-3) element of which is the base element of... Also, for all base points y, the infinite set of "spiral"s relative to y is path-similar to every other such infinite set, i.e., all paths, as defined by finite exponent sequences, exist in each such infinite set.	Lemmas 0.2, 5.0, 15.85.
Set of elements in a "spiral"	$\{x \mid x = (2^k y - 1)/3\}$, where y is the base element of the "spiral" and all k are either even or odd, depending on y. Thus, the number of elements in a "spiral" is infinite.	Lemma 5.0
Distance between jth element of a "spiral" and its base element y	$\lvert(2^k y - 1)/3 - y\rvert$, where k is the jth element of $\{1, 3, 5, ...\}$ or $\{2, 4, 6, ...\}$, depending on y.	Lemma 11.0
Distance between successive elements x, x', of a "spiral"	$3x + 1$	Lemma 11.0
Number of levels in the infinite set of "spiral"s relative to any given base element	Infinite	Lemma 5.0

Table 6: Some important properties of recursive "spiral"s

Property	Value of property	Reference
In the infinite set of "spiral"s relative to any given base element, number of paths defined by any given exponent sequence $A = \{a_2, a_3, ..., a_i\}$.	Infinite, i.e., there are an infinite number of paths for *each* exponent sequence, as in tuple-sets.	Lemma 7.0
Congruence classes to which base element and "spiral" elements belong	For all $i \geq 2$, and for each base element y: (1) y is an element of a reduced residue class mod $2 \cdot 3^{i-1}$; (2) the elements of the base sequence (i.e., of the "spiral" having y as base element) are elements of a sequence s of all reduced residue classes mod $2 \cdot 3^{(i-2)}$, with s being repeated endlessly over all elements of the"spiral".	Lemma 15.85.

Some Infinite Inverse Exponent Sequences That are Not Generated by Any Range Element, *y*

It is natural to ask, regarding recursive "spiral"s, the equivalent of the question we asked, regarding tuple-sets, under "Some Infinite Exponent Sequences That are Not Generated by Any Odd, Positive Integer, x" on page 25, namely, are there infinite exponent sequences that are not generated by any infinite path in the infinite set of "spiral"s defined by any range element, y? The answer is yes.

Lemma 12.5. *Let $\underline{A}$ = ...A*A*A*A′ be an infinite sequence of positive integers, where A, A′ are finite sequences, A is not empty, A′ is possibly empty, A is repeated infinitely and successively, and is such that in every path <x, ..., y> defined by A, x is less than y. Then no range element y′ defines $\underline{A}$.* An example of such an $\underline{A}$ is {...1, 1, 1}.

Proof:

Each successive *A* (moving from right to left) is produced by successively smaller *x*'s. The existence of $\underline{A}$ therefore implies the existence of an infinitely decreasing sequence of odd, positive integers, which is impossible. □

Possible Strategies for Proving the 3*x* + 1 Conjecture Using "Spiral"s

Fractal-Based Strategy

Since an infinite set of recursive "spiral"s relative to a base element constitute a self-similar structure, it is natural to ask if such a structure has a fractal dimension, and if so, what the dimension is. Then, if we know the dimension, we can perhaps use it to prove that only one such set is required to "cover" the odd, positive integers. But we must keep in mind that the 3x – 1 function has a structure very similar to that of the 3x + 1 function, and yet it has known counterexamples, e.g., a cycle that involves 17.

To compute the fractal dimension *d* of an infinite set of recursive "spiral"s relative to a base element, we must be able to know the scaling ratio of successive approximations to the fractal object that constitutes the limit of the approximations. That is, we need to be able to compute:

$$d = \frac{\log n}{\log s}$$

where (informally),

n is the number of "sides" in the next approximation;

s is the size of each "side" in the next approximation relative to the length of a "side" in the previous approximation.

(See any of the well-known Mandelbrot works for a formal definition.)

If we take the level 1 "spiral" to be the first approximation to the final fractal object that constitutes the infinite set of "spiral"s, then it is natural to take the "line" connecting two successive elements of this "spiral" to be a "side" (see Fig. 4), and the length of this "side" to be that defined by the distance functions, namely, $4x + 1$, where x is the smaller of the two elements.

However, the total length of the first approximation is then clearly infinite, as is the total length of the second approximation, which we take to be the total length of all level 2 "spiral"s. This does not enable us to compute *d*. Another approach is to take only the first *k* "side"s of the level 1 "spiral", the length of which is finite; and then compute the length of the first *k* "side"s of the level 2 "spiral"s yielded by the first *k* elements of the level 1 "spiral".

We must temporarily leave it to the reader to work out the details from this point. We will welcome reader comments.

Strategy of Proving Existence of a Certain Map Between Tuples and Paths in "Spirals"

Probably the most direct approach to a proof of the 3*x* + 1 Conjecture using "spiral"s would be by proving there exists a one-one onto map between tuples in tuple-sets and finite paths in the infinite set of "spiral"s whose base element is 1. Such a proof would prove the 3*x* + 1 Conjecture because the set of all tuple-sets represents the set of all finite computations by the 3*x* + 1 function.

We must confess that we spent an inordinate amount of time trying to discover such a mapping by trying to figure out where, in the infinite set of "spirals" having base element 1, each tuple "belonged" — in other words, by trying to map tuples onto paths in this set of "spirals". A much better idea *initially* (which, after the fact, is obvious) is to proceed in exactly the opposite direction: to try to discover where, in the set of all tuple-sets, each infinite set of "spiral"s (regardless of its base element) "belongs" or "fits in". If we view the matter in this way, we see immediately that *each element (except a multiple of 3) of each tuple in each tuple-set is the base element of an infinite set of recursive "spirals"!* (Recall that multiples of 3 only occur at level 1 in any tuple-set.) An awesome structure, indeed! Of course, we still retain the converse goal, namely, that of discovering where, in the set of infinite "spirals" whose base element is 1, each tuple in each tuple-set "belongs" or "fits in", always keeping in mind that, if a counterexample exists, this goal will not be achievable, in which case our goal then becomes that of discovering where, in the set of all possible infinite sets of "spiral"s, each tuple in each tuple-set belongs, or fits in.

In any case, we can say, now that we know how recursive "spirals" fit into tuple-sets, that recursive "spiral"s show how tuple-sets are related to each other, and that answers a question that has confronted us since we first discovered tuple-sets.

As an aid in conceiving the structure resulting from the insertion of an infinite set of "spirals" at each element of each tuple in each tuple set defined by a sequence of length ≥ 2, we may imagine each infinite set of "spiral"s as lying in a plane perpendicular to the page, the page containing (some of) the tuples in a tuple-set. The base element (which is a tuple element) of each infinite set of spirals is, in turn, an element of another infinite set of "spiral"s, namely, that established by the tuple element mapped to by the base element tuple element in accordance with the sequence of exponents that define the tuple-set.

But to show where tuples fit into "spirals", we need to "split the nodes", i.e., "split the base elements" in each infinite set of "spirals" having a given base element. That is, we must remember that each element of a tuple in a tuple-set (except the first element) is mapped to by only one element and, in turn, may or may or may not have one or more extensions in that tuple-set. In an infinite set of "spirals", on the other hand, an infinite set of elements maps to each base element (node) (unless the node is a multiple of 3). So in order to bring the form of such an infinite set into closer conformity with the form of tuple-sets, we must "split" each node (base element) *y* into an infinite set of nodes each of which is equal to *y*. An example is shown in the following figures.

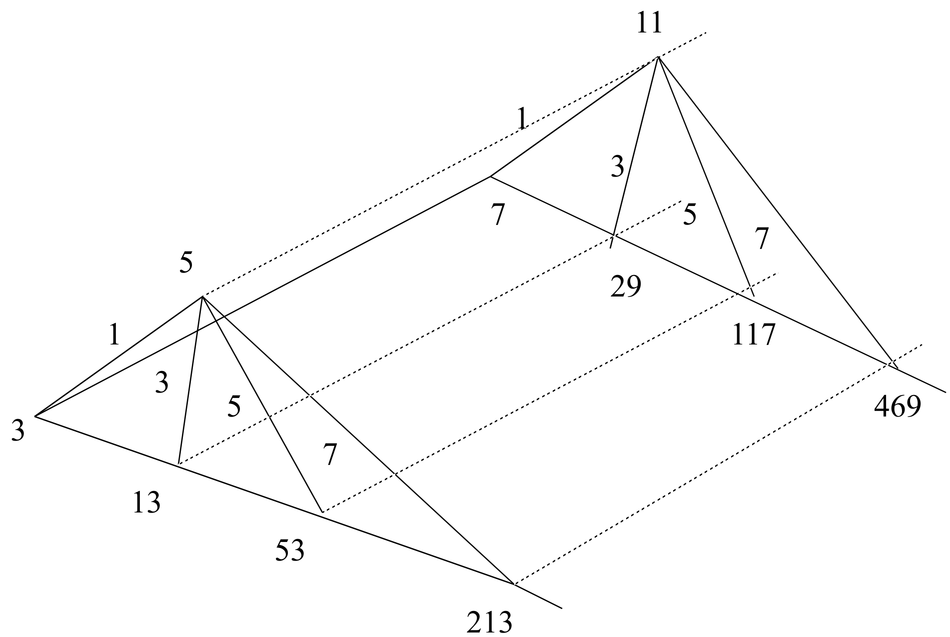

Fig 7. Example of the merging of tuple-sets and recursive "spiral"s: first stage.
1, 3, 5, 7, ... are exponents of 2.

The line 5, 11, ... represents the top row of every 2-level tuple-set defined by an odd exponent.

The other lines running diagonally into the page to the right represent bottom rows of 2-level tuple-sets.

Thus, e.g., we see the first two tuples in each of the tuple-sets defined by *A* = {1}, {3}, {5}, {7}. These tuples are, respectively, <3, 5> and <7, 11>; <13, 5> and <29, 11>; <53, 5> and <117, 11>, and <213, 5> and <469, 11>.

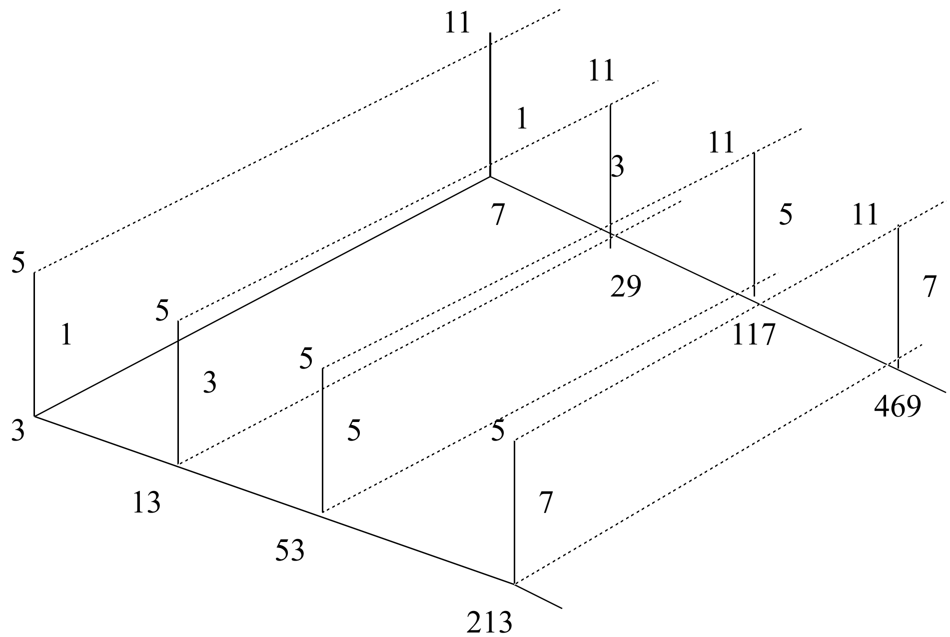

Fig 7. 5. Example of the merging of tuple-sets and recursive "spiral"s: second stage, **showing the "splitting of nodes" in recursive "spiral"s (see Fig. 7.).**

Finding "Locations" of Range Elements in Tuple-sets

We can now attempt to correlate the "locations" (defined below) of a given odd, positive integer u in the set of all tuple-sets, with its "locations" in the set of all recursive "spirals" relative to a given base element, in particular, the base element 1. If this correlation allows us to show that every assumed counterexample has a "location" in the infinite set of recursive "spirals" having base element 1 (which would be a contradiction) then we will have proved the $3x + 1$ Conjecture.

To begin our search for this correlation, let us ask a seemingly meaningless question, namely, "*Where* is the integer n mod m?" To show that, from the right point of view, the question is not meaningless, we recall the fundamental fact of elementary congruence theory, namely, that for each non-negative integer n, and for each modulus m (also a non-negative integer), there exists an r such that $n \equiv r \bmod m$, where r is a minimum residue mod m. This congruence in turn means that there exists a non-negative integer k such that $n = r + km$.

We can therefore say that, for each modulus m, each n has a "location" that is defined by the ordered triple (r, k, m). (This definition is a case of "what" = "where": *what* the value of a variable n is, is a function of *where* it is, i.e., of its location (r, k, m). The benefits of assigning geometric locations to numbers is an old one in mathematics, going back to the beginnings of analytic geometry in the 1600s, and further extended through the use of the complex plane, beginning in the early 1800s, and given new impetus by Minkowsi's *Geometry of Numbers* (1896), which set forth a way of assigning coordinates to the elements of a module.)

Now, the distance functions established in Lemmas 1.0 and 1.1 in effect tell us that each sequence A of exponents, $A = \{a_2, a_3, ..., a_i\}$, establishes a sequence of moduli, namely, the moduli

$$m_i = 2 \cdot 3^{i-1}$$

$$m_{i-1} = lcm(2 \cdot 3^{i-2}, 2^{a_i})$$

$$m_{i-2} = lcm(2 \cdot 3^{i-3}, 2^{a_{i-1}} 2^{a_i})$$

.

.

.

$$m_1 = 2 \cdot 2^{a_2} 2^{a_3} \ldots 2^{a_i}$$

where *lcm* denotes the least common multiple.

Let y be a range element in any tuple in the tuple-set T_A. Then for each of these moduli, y has an address, (r_j, k_j, m_j), where $1 \leq j \leq i$. This is the same thing as saying that y is an element of many tuples in T_A, which is the same thing as saying that y is an element of many "spiral"s defined by elements of tuples in T_A. (Note: as of yet, we do not know a general formula for computing the r_j except in the case of $j = i$.)

Thus we would like to define a function F which, for any tuple-set T_A, and for any range element y, will return all the locations of y in T_A, i.e., all the tuples containing y, and the index in each tuple of y, each such location being simultaneously the location of an element in a recursive "spiral". Formally, $F(A, j, y) = (r_j, k_j, m_j)$, where $A = \{a_2, a_3, ..., a_i\}$, $1 \leq j \leq i$, y is any range element, and (r_j, k_j, m_j) is as defined above. Clearly, any given y has an infinite number of locations, even if i is fixed, because y is mapped to by an infinity of exponents, hence y is an element of a different tuple in each of an infinity of tuple-sets.

Let us consider a few examples of the function F. $F(\{a_2\},1,1)$, where a_2 is any even exponent, $= (1, 0, 2 \cdot 2^{a_2})$; $F(\{a_2\},2,1) = (1, 0, 2 \cdot 3^{2-1})$. (Note that a value of $(r_j, 0, m_j)$ means that r is a minimum residue mod m_j.)

For $A = \{2, 1, 1\}$, $j = 3$, $y = 29$, we have $F(A, 3, 29) = (11, 1, 2 \bullet 3^{(3 - 1)})$

Now let us ask: What is the unique characteristic of any counterexample? Answer: that it never appears in the infinite set of "spiral"s whose base element is 1. Thus if we can use the function F to show that every assumed counterexample is an element of the infinite set of 'spiral"s having base element 1, this contradiction will give us a proof of the $3x + 1$ Conjecture.

We conclude with the observation that each element y in the infinite set of recursive "spiral"s relative to a given base element, also has a "location" if it is in that infinite set — a location that can be specified by the sequence of exponents that lead from the base element to y. Note that this sequence is the *reverse* of the sequence that would lead to the base element from y in a tuple-set.

Thus, we can label each of the elements in an infinite set of recursive "spiral"s relative to a base element, by one or more of the "location"s of that element in one or more tuple-sets, and, conversely, we can label any element of a tuple by one or more of the "location"s of that element in the infinite set of recursive "spiral"s relative to one or more base elements.

Strategy of "Filling-in" of Intervals in the Base Sequence Relative to 1

At present, we believe there is at least one mathematician (and probably several) in the world who could, from the material in this section (and/or the material in the related section referenced in Note 1 below), either construct a proof of the $3x + 1$ Conjecture, or make a major, publishable, advance toward such a proof. The main reason for this belief is contained in the sub-section, "The Most Promising Implementations of the Filling-in Strategy" in the section referenced in Note 1.

Note 1: an introductory version of this section, with our latest results, is given in the section "Strategy of "Filling-in" of Intervals in the Base Sequence Relative to 1" in our paper, "Are We Near a Solution to the $3x + 1$ Problem?" on occampress.com. We recommend that the reader begin with that section.

Note 2: in this section, we will sometimes refer to the infinite set of "spiral"s whose base element is 1, as *the 1-tree*.

Definition of "Filling-in" Strategy

We begin with the following conjecture, which defines our strategy, and is clearly equivalent to the $3x + 1$ Conjecture:

Conjecture 4.0 Every interval in the base sequence relative to 1, i.e., in the sequence $S_1 = \{1, 5, 21, 85, 341, ..., \}$, is eventually filled by elements that map to 1.

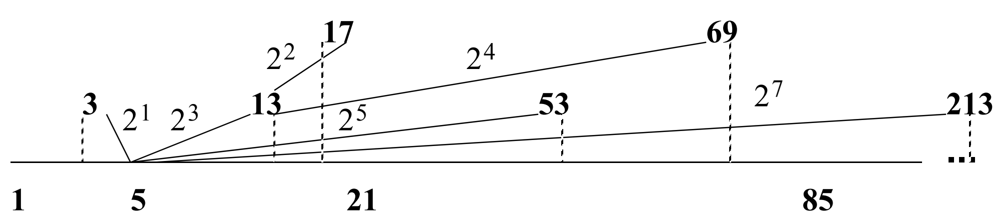

Fig. 5. Illustration of part of the "filling-in" process.

Before we discuss possible proofs of this conjecture, we will state several conjectures each of which, if true, implies the truth of Conjecture 4. These conjectures are as follows.

Several Conjectures That Imply the Truth of the 3x + 1 Conjecture

Conjecture 5.0

Let y be any base element, and let x, x' be successive elements of the base sequence relative to y. We know that $x' = 4x + 1$ (Lemma 11.0).. Then all elements of the interval between x and x' are filled by "spiral" elements at no higher level than $4x + 1$ beyond the level of y.

Thus, e.g., let $y = 1$ be our base element. The base sequence relative to 1 is $S_1 = \{1, 5, 21, 85, 341, ...\}$ Let $x = 1$ and $x' = 5$. The only element in the interval is 3, and 3 maps to 1 in two iterations of the $3x + 1$ Conjecture, hence is at level 2, which is at a level less than 4(1) + 1 = 5, in accordance with the Conjecture.

Or let $x = 5$ and $x' = 21$. We find that 9 maps to 1 via the tuple <9, 7, 11, 17, 13, 5, 1>, which means that 9 is at level 6, and 6 is less than 4(5) + 1 = 21, in accordance with our Conjecture.

It is known that 27 requires one of the largest number of iterations, for small odd, positive integers such as we are dealing with, to reach 1, namely, 41. So 27 is at level 41, which is still considerably less than 4(21) + 1 = 85. So once again, the Conjecture holds.

Obviously, the truth of Conjecture 5 implies the truth of the $3x + 1$ Conjecture.

Conjecture 6.0

An infinity of intervals of the base sequence {1, 5, 21, 85, 341, ... } are filled in solely by odd, positive integers that map to 1.

Note that Conjecture 6 differs from Conjecture 4 in that the latter states that *all* intervals are filled in by odd, positive integers that map to 1, whereas the former merely specifies an infinity of intervals. The reason the truth of Conjecture 6 implies the truth of the $3x + 1$ Conjecture is that the presence of an infinity of intervals I each of which is filled in solely by odd, positive integers that map to 1 implies that counterexample sequences are forced to "leap over" these I, and this is prohibited by Lemma 14.0 (see under "Three Important Lemmas" on page 52).

Observe that a proof of Conjecture 6 need not specify what the first such filled-in interval is (we know, by computer test, that at least the first 26 intervals are filled in solely by odd, positive integers that map to 1[1]), nor does the proof need to specify the distance between any two of the infinity of intervals.

Conjecture 7.0

If a counterexample exists, then at least one interval in the base sequence relative to 1 is filled in solely by counterexamples.

The reason the truth of this conjecture implies the truth of the $3x + 1$ Conjecture is that such a filled-in interval would force it to be "leaped over" by successive elements of a higher-level sequence that maps to 1, thus contradicting Lemma 14.0 (see under "Three Important Lemmas" on page 52).

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1. See "Plausibility Argument for the Truth of Conjecture 7.0" on page 51.

Plausibility Argument for the Truth of Conjecture 7.0

For each base element, whether counterexample or not, the structure of the infinite set of recursive "spiral"s relative to that base element is "similar" to the structure of the "spiral"s for any other base element — "similar" based on the properties described in Table 6, "Some important properties of recursive "spiral"s," on page 42.

We have heard, from a source we consider reliable, that as of Nov., 1998, the $3x + 1$ Conjecture had been verified for all integers (even and odd) up to about 56 quadrillion[1], i.e., $56 \cdot 10^{15}$.

Now the number of numbers (even and odd) in successive intervals of the base sequence relative to 1 are $2^2, 2^4, 2^6, \ldots$ This means that, **at least the first 26 intervals in the base sequence are known to be filled with odd, positive integers that map to 1**. The proof is:

(1)

$$((2^2)^1) + ((2^2)^2) + ((2^2)^3) + \ldots + ((2^2)^{26}) = \frac{(2^2)^{27} - 1}{2^2 - 1} - 1 < 2^{54} < 56 \cdot 10^{15}$$

For odd, positive integers in which the "spiral" element preceding each interval is included in the interval, we have:

$$(2^1) + (2^3) + (2^5) + \ldots + (2^{2 \bullet 26 - 1}) = 2(1 + 2^2 + 2^4 + \ldots + 2^{2 \bullet 26}) =$$

(2)

$$= 2\left(\frac{(2^2)^{27} - 1}{2^2 - 1}\right) < 2^{54} < 56 \cdot 10^{15}$$

But since, for *any* base element y, the structure of the infinite set of recursive "spiral"s relative to y is "similar" (in the sense stated above) to the structure for any other base element y', it seems plausible that a similarly long sequence of intervals must be completely filled in by integers (even and odd) that map to the smallest counterexample that is a base element. But this is prohibited by Lemma 14.0 (see under "Three Important Lemmas" on page 52), in particular, by this lemma as applied to the base sequences of range elements in the first 26 intervals.

Conjecture 8.0

Let s be a "spiral". As we know from the proof of Lemma 18.0 in the paper, "A Solution to the $3x + 1$ Problem on occampress.com, the elements of s are in the sequence ... 3, e, o, 3, ... where "3" denotes a multiple-of-3, "*e*" denotes a range element that is mapped to by all even exponents only,

1. The web site www.ericr.nl/, which we consider reliable, reported in June, 2006 that the number was then more than $48.4 \cdot 10^{16}$.

and "*o*" denotes a range element that is mapped to by all odd exponents only. Then there exist moduli m_1, m_2 and m_3 such that all multiple-of-3 elements are congruent mod m_1, all "*e*" elements are congruent mod m_2, and all "*o*" elements are congruent mod m_3.

The proof seems straightforward.

If this Conjecture is true, then we have a means of identifying infinite sets of non-counterexample 2-tuples in the tuple-sets $T_{\{1\}}$ and $T_{\{2\}}$. That is, if we know that a 2-tuple t is a non-counterexample tuple, then we know an infinity of 2-tuples t', t'', ... that are also non-counterexample tuples. (These have first elements that are congruent mod m_1, m_2, or m_3 to the first element of t.)

Three Important Lemmas

We now state three lemmas that are important, perhaps essential, for proving any of the Conjectures 4.0, 5.0, 6.0, or 7.0.

The first states, informally, that no element of a higher level sequence is "wasted" by being mapped "on top of" one of the base sequence elements. In other words, we never need to worry about "overfilling" an interval. This is Lemma 13.5, whose formal statement is as follows:

Lemma 13.5.
For all base sequences, and for all levels i ≥ 2 relative to a base sequence, no element of a level i sequence is an element of the base sequence. Thus, in particular, no level i element, i ≥ 2, is an element of the base sequence relative to 1, i.e., of the base sequence {1, 5, 21, 85, 341, ... }.

Proof: See second file of the paper, "The Structure of the 3x + 1 Function" on occampress.com

The second lemma states, informally, that no interval of the base sequence is "leaped over" by successive elements of a higher level sequence once that sequence gets started. This is Lemma 14.0, whose formal statement is as follows:

Lemma 14.0.
For any "spiral" at any level i ≥ 1, the sequence of elements of the "spiral" map to successive intervals of the base sequence.

Proof: See second file of the paper, "The Structure of the 3x + 1 Function" on occampress.com

The third lemma states, informally, that an infinity of successive intervals of the base sequence can always be filled with an arbitrary number of elements that map to 1 (up to the number of elements in each interval). This is Lemma 17.0, whose formal statement and proof[1] are as follows. We begin with a lemma.

1. The proof of this lemma is given here because the lemma is not at present stated and proved in any of our other papers on the $3x + 1$ problem.

Lemma 17.0.
For all $m \geq 2$, and for all $k \geq 2$, there exists an infinity of consecutive intervals in the base sequence relative to 1, i.e., in the sequence {1, 5, 21, 85, 341, ... }, such that each such interval contains $((m^k - 1)/(m - 1) - 1)$ numbers that map to 1.

Proof: The infinite set of recursive "spiral"s relative to base element 1 (i.e., the 1-tree) can be regarded as an infinitary tree, since every non-multiple-of-3 in the set is mapped to by an infinity of non-multiples-of-3 (Lemmas 5.0, 15.85), and since there can be no cycles in the set because all numbers map to 1. (It is easily shown[1] that only every third element of a recursive "spiral" is a multiple of 3. Thus, e.g., in the recursive "spiral" {7, 29, 117, 469, 1877, 7509, ...} (whose base element is 11), 117 and 7509 are the only multiples of 3 in the first six elements.)

Therefore, for all $m \geq 2$, we can select the first m non-multiples-of-3 in each "spiral" that map to the base element y, where "first" here means in the order of increasing values of the x that map to y. We thus get an m-ary tree. (The reader may find it helpful to refer to Table 7 while reading the following.)

Level $k = 0$ contains solely the number 1, hence no "spiral";
Level $k = 1$ contains solely the "spiral" {1, 5, 21, 85, 341, ... }; higher levels of "spiral"s will be filling in intervals in this "spiral", so we do not count it;
Level $k = 2$ contains m "spiral"s. (In Table 7, $m = 3$ "spiral"s.);
Level $k = 3$ contains $m \bullet m$ "spiral"s. (In Table 7, $m \bullet m = 3 \bullet 3 = 9$ spirals.);
etc.

Table 7: "Spiral"s for $m = 3$, $k = 3$

level k	No. of infinite "spiral"s	Infinite "spiral"s	Base element of "spiral"s
0	0		
1	1	{1, 5, 21, 85, 341,...}	1
2	3	{3, 13, 53, 213, 853, ...} {113, 453, 1813, 7253, ...} {227, 909, 3637, 14549, ...}	5 85 341

1. See proof of Lemma 18.0 in our paper, "A Solution to the 3x + 1 Problem" on occampress.com.

Table 7: “Spiral”s for *m* = 3, *k* = 3

level k	No. of infinite “spiral”s	Infinite “spiral”s	Base element of “spiral”s
3	9	{17, 69, 277, 1,109,...}	13
		{35, 141, 565, 2261,...}	53
		{1137, 4549, 18197, 72789,...}	853
		{75, 301, 1205, 4821, ...}	113
		{2417, 9669, 38677, 154709, ...}	1813
		{4835, 19341, 77365, 309461,...}	7253
		{151, 605, 2421, 9685, ...}	227
		{4849, 19397, 77589, 310357,...}	3637
		{9699, 38797, 155189, 620757, ...}	14549

Thus the number of different “spiral”s is given by $m + m \bullet m + m \bullet m \bullet m + ... + m \bullet m \bullet m \bullet ... \bullet m$ [k-1 m’s in the last term], which, by a basic fact of elementary algebra = $((m^k - 1)/(m - 1) - 1)$ “spiral”s. Lemmas 13.5 and 14.0 assure us that there will be no duplicate numbers in any two different “spiral”s. □

Thus, as the reader can see from Table 7, for $m = 3$, $k = 3$, we have $(3^3 - 1)/(3 - 1) - 1 = 26/2 - 1 = 12$ “spiral”s, ignoring the “spiral”s for $k = 0$ and 1.

Is this lemma sufficient for a proof of Conjecture 6, and hence of the $3x + 1$ Conjecture? The following facts are relevant to an answer to this question.

First, by Lemma 5.0 we know there is an infinite set of recursive “spiral”s for each counterexample, if any exists. Can there be an infinite number of *disjoint* such sets, each consisting (solely) of counterexamples? Lemma 17.0 implies the answer is no, because otherwise there would be an infinite number of numbers in each of an infinite number of intervals in the base sequence relative to 1, which is impossible.

Second, it is not possible that in any “spiral” in the infinite set of all such “spiral”s relative to any base element, all but a finite number of elements are cycle elements. (If it were, then we might not have an m-ary tree for all $m \geq 2$ in the case of an infinite set composed of counterexamples in which there were an infinite number of cycle elements.) However, Conway or Thompson proved already by 1973 that only a finite number of cycles is possible,[1] and therefore at most a finite number of numbers in any “spiral” can be cycle elements.

Given these two facts, it might be possible to prove Conjecture 6, and hence the $3x + 1$ Conjecture, by arguing that since each non-multiple-of-3 in the infinite set of recursive “spiral”s relative to the base element 1 (i.e., the 1-tree) adds a number mapping to 1 to each of an infinite sequence of successive intervals in the base sequence relative to 1, the assumption of a counterexample implies that in at least one interval, a counterexample must be “mapped on top of” by a number mapping to 1, contradicting Lemma 13.5.

1. We do not have the reference. The existence of the proof was mentioned in a lecture by H. Hasse at the University of New Zealand, New Zealand, on 10/26/73.

Further thoughts on the possibility of proving the 3x + 1 Conjecture using Lemma 17.0 are given in "Appendix F — Further Thoughts on the "Filling-in" Strategy" on page 97.

A Difficulty With the "Filling-in" Strategy

It is tempting to try to implement the "Filling-in" Strategy by arguing that since there is an infinite set of range elements *y* in the infinite set of recursive "spiral"s relative to 1 (the proof is simple), and since each range element *y* is itself mapped to by an infinite set of recursive "spiral"s relative to *y*, then we only need the show that it is always possible to add at least one more element to an infinite set of intervals in the base sequence relative to 1. (Recall that the base sequence relative to 1 is {1, 5, 21, 85, 341, ...}; interval 1 in this sequence consists of the number 3; interval 2 consists of the numbers 7, 9, 11, 13, ..., 17, 19; interval 3 consists of the numbers 23, 25, 27, ..., 81, 83; etc.). Thus, in particular, there can be no fixed number of counterexample elements in each of an infinite set of intervals.

The problem with this implementation is the following.

Suppose there exists a non-counterexample range element y_1 that is mapped to by a base sequence (i.e., a "spiral") that places a number in interval 2, a number in interval 3, a number in interval 4, etc. (We are not concerned here with determining which non-counterexample range elements, if any, fulfill this condition, or any of the following similar conditions, since the exact elements are not relevant to the point we are trying to make.)

Suppose, further, that there exists a non-counterexample range element y_2 that is mapped to by a base sequence (i.e., a "spiral") that places a number in interval 3, a number in interval 4, a number in interval 5, etc.

Suppose, further, that there exists a non-counterexample range element y_3 that is mapped to by a base sequence (i.e., a "spiral") that places a number in interval 4, a number in interval 5, a number in interval 6, etc.

Etc.

And suppose, finally, that there are no other non-counterexample range elements.

We see immediately that for any number $n \geq 1$ of non-counterexamples we name, there exists an infinity of successive intervals each of which contains *n* non-counterexamples. And yet we also see that no interval is filled with non-counterexamples. Informally, we say that the problem is that the first elements of the base sequences "move forward" too rapidly. The next few sub-sections contain a discussion of possible ways to overcome this problem.

It is essential that the "forward" movement of new "spiral"s not be too fast. If it is too fast, then all new "spiral"s will begin at intervals beyond an interval that has not been filled in. The worst case for our purposes is the exponent 2, which is the exponent by which the first element of a "spiral" maps to the base element in the case where all even exponents map to the base element.

The reason that 2 is the worst case is that if *x* maps to *y* via the exponent 2, then $x \approx (4/3)y$. This does not seem like too rapid a forward movement. Only after five successive iterations with exponent 2 is it the case that $x > 4y + 1$, thus forcing *x* to be in the next interval. The above table of numbers of "spiral"s suggests that this number of levels down will give us many "spiral"s in the next few triples.

But we must hasten to point out that a valid proof based on the Filling-in Strategy must consider the worst-case forward movement resulting from the tree of all possible triples. There are three such triples from each range element, namely, the 3, *e*, *o* triple, the *o*, 3, *e* triple, and the *e*, *o*, 3 triple, where "3" means the range element is mapped to by a multiple-of-3, "e" means that the range element is mapped to by all even exponents, and "o" means that the range element is mapped to by all odd exponents. We must determine the maximum forward movement that is possible via a descent of arbitrary length in this tree. This is unquestionably the most difficult task that faces us in the Strategy. Certainly the literature on trees should be searched to see if any useful results have already been obtained.

The Filling-in Process in More Detail

The base sequence ("spiral") with respect to 1 is {1, 5, 21, 85, 341, ... }. We will call this "spiral" S_1. We number the intervals between successive elements of the "spiral", $I_1, I_2, I_3, ...$ Thus I_1 = {3}, I_2 = {7, 9, 11, 13, 15, 17, 19}, etc.

We assert without proof that there are 2^{2k} integers in the interval I_k. Thus, there are $2^{2•1} = 4$ integers in I_1 (5 – 1 integers); $2^{2•2} = 16$ integers in I_2 (21 – 5 integers), etc.

We also assert without proof that there are $2^{2k-1} - 1$ odd integers in the interval I_k. Thus, there is $2^{2•1-1} - 1 = 1$ odd integer, namely, 3, in I_1. There are $2^{2•2-1} - 1 = 7$ odd integers, namely, 7, 9, 11, 13, 15, 17, 19 in I_2, etc.

Each "spiral" contains an infinite number of elements. These can be grouped in successive fours, yielding successive threes, or triples, of intervals, starting with the first interval of the "spiral". Thus, in the case of S_1, the first two triples are those defined by {1, 5, 21, 85} and {85, 341, 1365, 5461}. We know that each triple contains one multiple of 3 (which no odd, positive integer maps to); one integer that is mapped to by all even exponents, and one integer that is mapped to by all odd exponents (Lemma 15.0 in file 2 of "The Structure of the 3x + 1 Function" on occampress.com). Thus each triple *r* is mapped to by two "spiral"s , s_1 and s_2, one that maps to an element of *r* by even exponents and one that maps to an element of *r* by odd exponents.

Let us consider only the first triple in each of the two "spiral"s s_1 and s_2 that map to *r*. Call these first triples r' and r''. Each of r' and r'' is mapped to by two "spiral"s. So our original triple *r* has yielded 2 + 4 "spiral"s in a descent of only two levels below the original "spiral" *r*. (Remember that the more "spiral"s, the more we fill up the infinity of successive intervals beyond those in which *r* resides.

Let *w* ("width") denote a number of consecutive *triples* in a "spiral", starting from the first triple that we wish to consider.

Let us say that the "spiral" S_1 is at level 1. Fix *w*.

Then the number of "spiral"s yielded by the width *w* at level 2 is $w•2$. And the number of "spiral"s yielded by the width *w* at level 3 is $(w•2)2$. And the number of spirals yielded by the width *w* at level 4 is $((w•2)2)2$, etc.

Let *j* denote the level number we wish to descend to. Then the total number of spirals yielded by *w* and *j* is clearly $w•2 + (w•2)2 + ((w•2)2)2 + ... + ...\ w•2^{j-1}$ or $w(2^1 + 2^2 + 2^3 + ... + ...2^{j-1}) = w(2^j - 2)$.

By Lemma 14.0 we know that successive elements of a "spiral" occupy successive intervals of S_1. So we can say:

(1)

For each w and each j, there exists a countable infinity of successive intervals of S_1 *that contain* $w(2^j - 2)$ *non-counterexamples.*

The process we have described gives rise, for *each* "spiral" r, an *infinite binary tree*. The base element of the "spiral" can be considered the root of the tree. The "spiral" r consists of a countable infinity of successive triples. (So the root has an infinity of branches.) Each triple has two branches. The node at the end of each branch is a "spiral". Each "spiral" in turn consists of a countable infinity of successive triples. Each triple has two descending branches. Etc.

Unfortunately, this does not give us a proof of the $3x + 1$ Conjecture by showing that all intervals I_k, where $k \geq 1$, are filled in, because it is always possible that the first interval (and all subsequent intervals) containing the $w(2^j - 2)$ non-counterexamples have room for counterexample elements. Furthermore, the number of odd, positive integers in each interval is odd (see above in this sub-section) whereas $w(2^j - 2)$ is even. This problem might be remedied by always considering, in addition to w triples (each of which gives rise to two "spiral"s), just one range element in the next triple, or by including the "spiral" element that immediately precedes an interval, as part of the interval. (See material regarding I_{i+} below.)

Nevertheless, we can define a function $F(w, j)$ that returns the first interval containing $w(2^j - 2)$ non-counterexamples. Clearly, all subsequent intervals will contain at least the same number of non-counterexamples.. The values of $F(w, j)$ can then be sorted into non-decreasing order. If we can show that the rate at which interval numbers increase in this order, is sufficiently smaller than the rate at which $w(2^j - 2)$ increases, then we might be able to construct a proof that eventually every interval is filled with non-counterexamples, which, of course, would be a proof of the $3x + 1$ Conjecture.

It is important to observe that the use of triples may help us to overcome some of the annoying complexity involved in considering tuples that map to a given range element y. The complexity arises from the the presence of multiples-of-3 in the "spiral" mapping to y, and the fact that if x maps to y via odd exponents, then the first element of the "spiral" is less than y ($x \approx (2/3)y$), whereas if x maps to y via even exponents, then the first element is greater than y ($x \approx (4/3)y$). The reader can quickly verify the complexity of the tree of possiblities that results. However, triples introduce a certain stability into the tree. And we may be able to make statements about the range of values at any level j below y by considering the tree of triples instead of individual elements mapping to y.

But we must utter a word of warning here: strategies that rely solely on structural arguments are almost certain to fail, because these same arguments probably apply to $3x + 1$-like functions in which counterexamples are known to exist (e.g., the $3x - 1$ function). At some point in a proposed proof, facts that are unique to the $3x + 1$ function must be introduced. Among these facts are the $4x + 1$ distance between successive "spiral" elements, and the minimum increase in the value of any x that maps to a "spiral" element. This increase is calculated in the next sub-section. We must show that it is sufficiently small to guarantee eventual filling of each interval in $S_1 = \{1, 5, 21, 85, 341, ... \}$.

On the other hand, we must keep in mind that whatever structural arguments apply to the infinite set of "spiral"s with base element 1 (i.e., the 1-tree), also apply to the infinite set(s) of "spiral"s with base element a counterexample.

We must not forget the following reasoning. Let $I_1, I_2, ...$ denote intervals in the "spiral" having base element 1, as described at the start of this sub-section. Then the number of odd numbers in interval j is given by $2^{2j-1} - 1$. Thus, the number of odd numbers in interval 1 is $2^{2 \cdot 1 - 1} - 1 =$ 1, namely, the number 3, and the number of odd numbers in interval 2 is $2^{2 \cdot 2 - 1} - 1 = 7$, and these numbers are 7, 9, 11, 13, 15, 17, and 19, etc. Now we know by computer tests that at least the first

26 intervals are filled with non-counterexamples. The same test results imply that all tuple-set *anchors* for levels 2 through 35 are non-counterexample elements (*range* elements). So the set of all 35-level tuple-sets gives us a view of the tuples that map to the *range* elements in the first 26 intervals — it shows these tuples 35 levels down from their last elements.

The 26th interval contains $2^{2 \cdot 26 - 1} - 1$ odd numbers all of which are non-counterexamples. This means that each of a total of $2^{2 \cdot 26 - 1} - 1$ "spiral"s has an element in the 26th interval, and that therefore (by Lemma 14.0, above) there are this number of "spiral" elements in each interval beyond the 26th.

We ask, now, how many additional odd numbers are needed to fill the 27th interval. Our answer is $2^{2 \cdot 27 - 1} - 1 - (2^{2 \cdot 26 - 1} - 1) = 2^{2 \cdot 27 - 1} - 2^{2 \cdot 26 - 1} = 2^{2 \cdot 26 - 1}(2^2 - 1) = 2^{51}(3)$. Using our formula in (1) above, we see that we can *add* to at least the *30th* interval and beyond $3(2^{26} - 2)$ more non-counterexample "spiral"s. (The 30th interval because it is the fourth interval after the triple of intervals I_{27}, I_{28}, and I_{29}.) The reason we cannot at present say the 27th interval is given by Lemma 18.0, below.

Do we have the basis of an inductive proof here?

We must also not fail to point out another connection between non-counterexamples in the intervals $I_1, I_2, \ldots$ and tuples in tuple-sets. In particular, since there are a total of at least $2^{2 \cdot 26 - 1} - 1$ "spiral" elements in each interval beyond the 26th, we have a confirmation of Lemma 10.0 (second file of this paper, on occampress.com), which asserts that there exists a countable infinity of non-counterexample tuples in each tuple-set, whether or not counterexamples exist. But we know more, namely, that for each non-counterexample y in the 26th interval, there exists a non-counterexample $4y + 1$, $4(4y + 1) + 1$, ... etc. So there exists $2^{2 \cdot 26 - 1} - 1$ *infinities* of non-counterexamples, and we can describe each infinity. In particular, if v is the smallest non-counterexample in the 26th interval, then $v + 2$ is also a non-counterexample in the interval, and $v + 4$, ... (because non-counterexamples are odd, positive integers) and thus the first few countable infinities are $\{v, 4v + 1, 4(4v + 1) + 1, \ldots\}$, $\{v + 2, 4(v + 2) + 1, 4(4(v + 2) + 1) + 1, \ldots\}$, etc.

We conclude this section with a reminder to the reader that we will have a proof of the $3x + 1$ Conjecture if we can prove that *the assumption that there exists an infinity of successive intervals $I_1, I_2, \ldots$ that are only partially filled by non-counterexamples, leads to a contradiction*. The contradiction might, for example, arise from the fact that it is impossible for all the range elements in an interval, and in previous intervals, not to yield additional elements in the interval beyond those that are assumed to be the only ones that exist in the interval. Part (2) of Lemma 18.0, below, implies that if an infinity of successive intervals $I_1, I_2, \ldots$ are only partially filled by non-counterexamples, then in each of these intervals, each range element y is mapped to by even exponents, and furthermore the first element of the base sequence relative to y is a multiple-of-3. This seems unlikely.

More on Levels

Let us define a level in the 1-tree, or simply, a level, as the number of iterations of the $3x + 1$ function required to take odd, positive integers at the level to 1. Thus, for example, 5 is at level 1, 3 is at level 2, 17 is at level 3.

The 1-tree clearly has odd, positive integers at all levels.

We can associate with each odd, positive integer x, its level number. We can also associate with x the element immediately preceding the interval in the "spiral" $S_1 = \{1, 5, 21, 85, 341, \ldots\}$ in which x occurs. In the ordered pair $[m, n]$ associated with each such integer x, m will be the level number, n the "spiral" element. Thus, for example, associated with 13 is the ordered pair

[2, 5].

We can now define the level and interval in S_1 at which the start of a "spiral" occurs as being the [*m*, *n*] associated with its first element. The first element of each "spiral" is a descendant of exactly one element in the "spiral" $S_1 = \{1, 5, 21, 85, 341, ... \}$.

(Is there a possibility of a proof of the $3x + 1$ Conjecture by contradiction in this fact? If a counterexample exists, then either it maps to an element of a very long cycle, or else there is no maximum counterexample to which it maps. In each case, we ask if it is meaningful to speak of the ordered tuple [*m*, *n*] for the first element of a "spiral". If not, then does that imply that counterexample "spiral"s do not exist?)

In passing, we must not fail to point out that if any element of a "spiral" maps to 1, then all elements do. (*Proof*: Each element *x* of a "spiral" maps to the base element *y* of the "spiral". If *x* maps to 1, this is only possible if *y* does. Hence, since each "spiral" element maps to *y*, the result follows. □

We should not fail to investigate the possibility of a proof of the Conjecture by proceeding down through successive levels in the 1-tree, and observing how many levels are required to fill successive intervals in $S_1 = \{1, 5, 21, 85, 341, ... \}$. For example, we observe that it takes two levels to fill interval I_1, since 3 is the only element of this interval, and 3 is at level 2 by virtue of the tuple <3, 5, 1>. It takes six levels to fill interval I_2, since, e.g., 9 is at the highest level among odd, positive integers in I_2 that map to 1, and this level is level 6 by virtue of the tuple <9, 7, 11, 17, 13, 5, 1>. If Conjecture 5.0 is true, then it will take less than 85 levels to fill interval I_3, etc. Of course, we must always remember that each odd, positive integer that maps to 1, and thus is an element of an interval, is also an element of a "spiral", which has an element in an infinity of successive intervals.

We observe that it takes two levels and one "spiral" to fill interval I_1. It takes six levels and seven "spiral"s (not eight, because 13 maps to 5 in one iteration of the $3x + 1$ function, and 13 and 5 are both in the same interval) to fill interval I_2, as the reader can confirm.

Example of Filling-in Process

We begin with the interval I_{26}, which we know, by computer tests, is filled with non-counterexamples. There are $2^{2\cdot 26-1} - 1 = 2^{52-1} - 1$ odd, positive integers in I_{26}, which, with the "spiral" element preceding the interval, yields 2^{52-1} in I_{26+} (see definition below in this sub-section). Therefore there are that number of "spiral"s,. each with an element in each successive interval from I_{27+} on. We must see if what we have established above can result in the filling in of all these intervals. We know, from the previous sub-section, that in interval I_{27+} we must add $(2^{2\cdot 26-1})(3)$ non-counterexamples to fill up interval I_{27} with non-counterexamples.

We consider successive triples of intervals beginning with I_{27+}. Since two out of every three elements of the "spiral" $S_1 = \{1, 5, 21, 85, 341, ... \}$ are range elements, we must include them in the following argument. We include the "spiral" element preceding each interval as part of the larger interval, which we denote I_{i+}. There are 2^{2i-1} odd, positive integers in I_{i+}, hence $2^{2\cdot 26-1}$ in I_{26+}. We know that in each triple, each "spiral" has three elements, one in each interval in the triple, and that two of these elements are range elements, one mapped to by even exponents, one mapped to by odd exponents.

So in each interval I_{27+}, I_{28+}, I_{29+}, and in each interval in all successive triples beyond, there are $(2^{52-1}) + 2\,(2^{52-1})$ non-counterexamples.

In each interval in the next triple, I_{30+}, I_{31+}, I_{32+} and in each interval in all triples beyond there are $(2^{52-1}) + 2\,(2^{52-1}) + 2\,((2^{52-1}) + 2\,(2^{52-1}))$ non-counterexamples.

In each interval in the next triple, I_{33+}, I_{34+}, I_{35+} and in each interval in all triples beyond, there are $(2^{52-1}) + 2\,(2^{52-1}) + 2\,((2^{52-1}) + 2\,(2^{52-1})) + 2((2^{52-1}) + 2\,(2^{52-1}) + 2\,((2^{52-1}) + 2\,(2^{52-1})))$ non-counterexamples.

Etc.

Considering that we have so far been concerned with only one level down from each existing "spiral" in each interval, is there reason to hope that we can prove that all "spiral"s are eventually filled in? The reader should keep in mind that we do not need to prove explicitly that each interval beyond I_{26} is eventually filled in. We only need to prove, for example, that there is no room in these intervals for a corresponding filling-in process by counterexamples. Or that just one later interval is completely filled in (see above under "Conjecture 6.0").

The Problem of the Rate of Increase of the Smallest Element at Each Level

In thinking about a proof of any of the Conjectures 4.0, 5.0, 6.0, 7.0, we confront the problem of the rate of increase of the the smallest element at each level.

If x yields y in one iteration of the $3x + 1$ function via the exponent $a_j = 1$, then $x < y$ — in particular, $x \approx (2/3)y$, and $a_j = 1$ is the only exponent when this occurs. This is in our favor, of course. Otherwise, x will be greater than the base element, except in the case when $y = 1$. We need to be sure that x will never be "too large". Let us consider the "spiral" whose base element is 13, namely, the sequence {17, 69, 277, 1109, 4437, 17749, ...}, which produces 13 via the exponents 2, 4, 6, 8, 10, 12, ... respectively. We see that 17 is larger than 13, but not a great deal larger.

We can make some progress by the following reasoning:

Each "spiral" contains a countable infinity of elements.

By Lemma 10.0 in the second file of this paper on occampress.com, there exists a countable infinity of non-counterexamples whether or not a counterexample exists. Therefore there exists a countable infinity of non-counterexample range elements.

By definition of a "spiral", the first element of a "spiral" maps to the base element via either the exponent 1 or the exponent 2. Thus we need only concern ourselves with these exponents, since the remaining elements of each spiral map to the base element either by the exponents 3, 5, 7, ... (in the case of 1) or by the exponents 4, 6, 8, ... (in the case of 2). Each of these exponents yields x that is greater than y.

Fortunately, we have a list of all x that map to their base element via the exponent 1 in the first elements of the 2-tuples in the tuple-set T_A, where $A = \{1\}$. And similarly, we have a list of all x that map to their base element via the exponent 2 in the first elements of the 2-level tuples in the tuple-set T_A, where $A = \{2\}$.

Furthermore, the elements in each list are in sorted order because tuples are ordered by the natural order of their first elements.

We must now ask: are we guaranteed that the rate of increase of these x in each case is sufficiently slow to allow for the filling in of the intervals established by the elements of the level 1 "spiral" $S_1 = \{1, 5, 21, 85, 341, ... \}$? If, for example, and contrary to fact, the rate of increase of these x were such that each successive x occupied a successive interval in the sequence of intervals established by S_1, then the filling-in strategy would fail.

On the other hand, if x increased at the rate of increase of interval lengths, so that the first x remained in interval 1 (there is only odd, positive integer, namely, 3, in interval 1 of {1, 5, 21, 85, ... }), then the next 16 x remained within interval 2, and the next 64 x remained in the interval 3, etc., our filling-in strategy would work. (Actually, to fill each interval j, we would only need a number of x equal to the total number of odd, positive integers in the interval ($2^{2j-1} - 1$) minus the number of odd, positive integers in all preceding intervals, since their “spiral”s would have already placed numbers in interval *j*.)

We know by computer test that all odd, positive integers less than $2 \cdot 3^{35 - 1}$ are non-counterexamples. That immediately tells us — see “Plausibility Argument for the Truth of Conjecture 7.0” on page 51 — that the first 26 intervals in S_1 are filled by non-counterexamples. We must now ask about the remaining infinity of intervals.

Let y be a non-counterexample that is the first element of a “spiral” S. There are three possibilities:

(1) y is a range element mapped to by all odd exponents;
(2) y is a range element mapped to by all even exponents.
(3) y is a multiple of 3 (and hence is not mapped to by any odd, positive integer, that is, not a range element);

Case (1). By definition of the $3x + 1$ function, there exists an x such that $(3x + 1)/2^1 = y$. Thus $x = (2y - 1)/3$. Here, $x < y$, so x is in the same or a previous interval as y, which is in our favor.

Case (2) By definition of the $3x + 1$ function, there exists an x such that $(3x + 1)/2^2 = y$. Thus $x = (4y - 1)/3$. Here $x > y$, but only about $4/3\ y$, so x is in the same interval as y depending on near the end of the interval x is, which is again in our favor.

Case (3) This is our worst case. For in this case, in order to find the smallest x that maps to an element of the “spiral” S, we must go to the first “spiral” element after our initial y. This element has the value $4y + 1$. Then we must make the same calculations we did for Cases 1 and 2. The result is either $x = (2(4y + 1) – 1)/3$ or($4(4y + 1) – 1)/3$. Obviously, x in each case is considerably larger than y. But in the first case, $x \approx (8/3)y$, which is less than $3y$, hence x is not in the next interval (recall that if x is in an interval, then $4x + 1$ is in the next interval). In the second case, $x \approx (16/3)y$, which is just a little more than $5y$, so only in this case is x in the next interval. Keeping in mind that multiples of 3 constitute only about a third of “spiral” elements, can we show that the second case does not prevent all intervals from eventually being filled in by non-counterexamples?

Can we show that if we select any interval established by the “spiral” S_1, there is a sufficiently large sequence of non-counterexamples, beginning with 1, that are the first elements of “spiral”s, such that the sequence gives rise to a sequence of “spiral”s whose elements must fill the interval?

We conclude with a plausibility argument that all intervals in {1, 5, 21, 85, 341, ... } are eventually filled in with non-counterexamples. Assume the contrary. Then although there are elements of “spiral”s in all intervals, an infinity of successive intervals have odd, positive integers that are not non-counterexamples. We ask if that is possible, given that two-thirds of “spiral” elements are range elements, each of which is mapped to by a “spiral”, two-thirds of whose elements are likewise mapped to by range elements, etc. Realizing that *each* range element in each of these “spiral”s is itself the base element of an infinite set of recursive “spiral”s, we ask where all these“spiral” elements *are*. How can there be an infinity of intervals in {1, 5, 21, 85, 341, ... } that are only partially filled with non-counterexamples?

An Upper Bound on the Rate of Increase of the Smallest Element at Each Level

Let s_n, $n \geq 1$, be the smallest element of level n in the infinite set of recursive “spiral”s relative to base element 1. Thus, as the reader can verify, $s_1 = 5, s_2 = 3, s_3 = 17, s_4 = 11, s_5 = 7, s_6 =$

9, $s_7 = 49$, $s_8 = 65$, $s_9 = 43$. Our goal here is to find an upper bound on the rate of increase of the s_n.

Let $y = s_n$ for some n. The worst case occurs if for an arbitrary number of levels $n + 1, n + 2, ...,$ it is the case that $s_{n+1}, s_{n+2}, ...,$ is a multiple of 3. Then, by the following lemma, the smallest element at each of these levels is two intervals beyond the previous interval, which is definitely not in our favor.

Lemma 18.0. *Let y be a range element of interval k, where $k \geq 1$, in any "spiral". Then y is mapped to, in one iteration of the 3x + 1 function, by an odd, positive integer that lies either in interval $k - 1$, k, or $k + 2$.*

Proof: It is easily shown (see Lemma 15.0 in file 2 of "The Structure of the 3x + 1 Function" on occampress.com) that, in any "spiral", successive "spiral" elements follow the pattern, ...3, e, o, 3, ..., where "3" means the element is a multiple-of-3, hence not a range element, "e" means that the element is mapped to by all even exponents, "o" means that the element is mapped to by all odd exponents.

For our purposes, the two worst cases we must consider are

(1) the range element y is mapped to by odd exponents;
the first element of the base sequence relative to y is not a multiple-of-3;
y is the first element of interval k.

In this case, x is in interval $k - 1$, because $(3x + 1)/(2^1) = y$ implies that x is about 2/3 y.

(2) the range element y is mapped to by even exponents;
the first element of the base sequence relative to y is a multiple-of-3, hence the second element maps to the base element via the exponent 4;
y is near to the last element of interval k.

In this case, x is in interval $k + 2$, because $(3x + 1)/(2^4) = y$ implies that x is about 16/3 y, which is greater than $4y + 1$, which thus places x in interval $k + 2$, by Lemma 11.0. □

Remark: This result applies both to non-counterexample and counterexample range elements y.

Can Tuple-sets Provide a Means of Solving the Rate-of-Increase Problem?

We conclude our discussion of the rate-of-increase problem by considering whether tuple-sets might provide us with a means of solving the problem.

In our discussion, "Strategy of Proving There Is No Minimum Counterexample" on page 26, we defined the *less-to-greater* property of an exponent sequence. Recall that an exponent sequence A has this property if, for any tuple t in a tuple-set T_A defined by such a sequence, the last element of t is greater than the first. In an infinite set of recursive "spiral"s relative to some base element y — an infinite set that contains the last element of t — t occurs as a sequence of "spiral" elements. Here, the last element of t is at some level k, and the first element is at a level $k + i - 1$, where i is the number of elements in t. So, given a tuple t with the less-to-greater property, we can find a downward sequence of elements in the infinite set of "spiral"s containing t and we can say that this sequence has a *greater-to-less* property.

But the existence of such sequences of elements solves the problem of $\mathbf{y}'^{\cdots}{}'$ growing too rapidly! For it means that we can always find a greater-to-less sequence of arbitrary length in the infinite set of recursive "spiral"s relative to the base element 1. (Proof: by Lemma 10.0, every tuple-set contains an infinity of tuples whose elements map to 1, regardless of whether a counter-example exists or not. Therefore, for any i-level exponent sequence having the less-to-greater property, we can always find an infinity of tuples that define inverse paths having the greater-to-less property in the infinite set of recursive "spiral"s relative to the base element 1. And we know that for each $i \geq 2$, there exists at least one exponent sequence having the less-to-greater property, namely, the sequence $A = \{1, 1, ..., 1\}$ ((i - 1) 1s).)

However, there is a catch. In order to use greater-to-less sequences, we must prove that the last element y of the tuple t (i.e., the first element of the corresponding greater-to-less sequence of elements in the set of infinite recursive "spiral"s relative to 1) is never so large that the number of elements in t (in the inverse path) is insufficient to fill in at least one interval as desired. In other words, suppose that y occurs at level n, and suppose the inverse path contains i elements. If we fill an interval I in the base sequence relative to 1 with m elements via elements at level $n + 1$, and then with m more via elements at level $n + 2$, etc., then we are assured of being able to fill the interval with $m(i - 1)$ elements that map to 1. Is that sufficient to fill the interval completely?

Summary of Major Results Concerning the Filling-in Strategy

Let $S_1 = \{1, 5, 21, 85, 341, ... \}$. This is the "spiral" — the set of odd, positive integers — that maps to 1 in one iteration of the $3x + 1$ function. We say that an odd, positive integer that maps to 1 in k iterations of the function is *at level k*.

Let I_i, where $i \geq 1$, denote the ith interval in S_1. Thus $I_1 = \{3\}$, $I_2 = \{7, 9, 11, 13, 15, 17, 19\}$.

Let I_{i+}, the "expanded interval", where $i \geq 1$, denote I_i preceded by the ith element of S_1. Thus $I_{2+} = \{5, 7, 9, 11, 13, 15, 17, 19\}$.

Let $|I_i|$ denote the number of elements in I_i. Then $|I_i| = 2^{2i-1} - 1$.

Let $|I_{i+}|$ denote the number of elements in I_{i+}. Then $|I_{i+}| = 2^{2i-1}$ and $|I_{(i+1)+}| = 4|I_{i+}|$.

A total of $|I_{i+}|$ "spiral"s are represented in I_{i+}. Each "spiral" has exactly one element in I_{i+}.

Recall from the above section, "Plausibility Argument for the Truth of Conjecture 7.0" on page 51, that computer tests have shown that all positive integers to at least $56 \cdot 10^{15}$ are non-counterexamples. The total number of odd, positive integers in the first 26 consecutive expanded intervals is given by

$$(2^1) + (2^3) + (2^5) + \ldots + (2^{2 \bullet 26 - 1}) = 2(1 + 2^2 + 2^4 + \ldots + 2^{2 \bullet 26}) =$$

(2)

$$= 2\left(\frac{(2^2)^{27} - 1}{2^2 - 1}\right) < 2^{54} < 56 \cdot 10^{15}$$

The number of additional "spiral" elements that are needed to fill the 27th expanded interval is $|I_{27+}| - |I_{26+}| = 4|I_{26+}| - |I_{26+}| = 3|I_{26+}|$. These additional elements must be obtained by descending a certain number of levels below each "spiral" that was present in I_{26+}.

Each “spiral” has an infinite number of elements.

The level of a “spiral” is the level of its first element (which is the same as the level of all elements in the “spiral”). Since the 1-tree is oriented vertically, we will speak of a level that is “lower” than a given level, or a certain number of levels “down” from a given level, even though the level number is higher (larger).

The descendant “spiral”s of a given “spiral” whose first element is in interval I_i, place “spiral” elements in countable infinities of successive intervals beyond I_i.

Since the first element of a “spiral” maps to the base element either via the exponent 1 or the exponent 2, the set of first elements of all “spiral”s is a subset of the set of first elements of all 2-tuples in the tuple-sets $T_{\{1\}}$ and $T_{\{2\}}$.

Each “spiral” — that is, the first element of each “spiral” — is a descendant of exactly one element of $S_1 = \{1, 5, 21, 85, ... \}$. Each of these elements is, of course, a descendant of the element 1.

Each successive element of a “spiral” is in exactly one successive interval I_i.

The elements of a “spiral” follow the pattern ...3, *e*, *o*, 3, ..., where “3” means: “multiple-of-3, hence not mapped to by any odd, positive integer”; “*e*” means “mapped to by all and only exponents of even parity”; “*o*” means “mapped to by all and only exponents of odd parity”. (See proof of Lemma 18.0 in “A Solution to the $3x + 1$ Problem” on occampress.com.)

For each “spiral”, and for each element of the “spiral” (recall that each element is in exactly one interval I_i), if the “spiral” element is an “e” or an “o”, then it has at least one descendant. When we are considering successive triples of intervals, then for each “spiral”, the triple of elements in the triple of intervals gives rise to an unbounded descending tree. Each descendant in turn gives rise to a new “spiral” having an element in each of the countable infinity of successive intervals.

In reckoning the “spiral” elements in a given expanded interval I_{i+}, we must include the “spiral” elements generated by the first element of I_{i+}, that is, by the *i*th element of S_1, and also the “spiral” elements generated by the first element of $I_{(i+1)+}$.

By computer test, we know that at least the first 26 intervals are filled with non-counterexmples. There are thus elements of $2^{2•26–1}$ “spiral”s in I_{26+}. Not all of these “spiral”s are at the same level, however! Thus, e.g., the “spiral” {3, 13, 53, ... }, which has an element in I_{26+}, is at level 2, because 3 maps to 1 in two iterations of the $3x + 1$ function. But the “spiral” {7, 29, 117, ... }, which has an element in I_{26+}, is at level 5, because 7 maps to 1 in five iterations of the $3x + 1$ function. There exists a maximum level “spiral” in I_{26+}. Since it is known, from computer tests, that the anchors of all 35-level tuple-sets are non-counterexamples, this suggests that the maximum level “spiral” in I_{26+} is about 35.

We now list the number of “spiral”s that are created starting with the triple of intervals following I_{26}. The reason we introduce triples of intervals is that in each triple we are guaranteed that each “spiral” has an element that is a multiple-of-3, a range element that is mapped to solely by even exponents, and a range element that is mapped to solely by odd exponents. The multiplier 2 in the following comes from the fact that in each triple, each “spiral” has two range elements.

In each interval, there are two kinds of “spiral” elements: those having descendants within the interval, and those not. Thus, for example, in the interval I_2 (= {5, 7, 9, 11, 13, 15, 17, 19}), 13 is an example of the first kind of “spiral” element, because it is an element of the “spiral” {3, 13, 53,

... } and 13 is mapped to by 17, which is also an element of I_2. On the other hand, 9 is an example of the second kind of "spiral" element, because it is an element of the "spiral" {9, 37, 149, ... } and 9, being a multiple-of-3, is not mapped to by any odd, positive integer.

Let us call the second kind of element, a *bottom-level* element (in a given interval).

Let a = the number of "spiral"s having bottom-level elements in I_{26}, which, per our remarks above, we take to be the largest interval that is completely filled with non-counterexamples. Then we know that there are at least a non-counterexample elements in each interval beyond the 26th. The number of "spiral"s produced by the a "spiral"s is as follows:

1st triple after I_{26+}, <u>one</u> level down: $a + 2a = 3a$ "spiral"s.
2nd triple after I_{26+}, <u>one</u> level down: $a + 2a + 2(a + 2a) = 9a$ "spiral"s.
3rd triple after I_{26+}, <u>one</u> level down: $a + 2a + 2(a + 2a) + 2(a + 2a + 2(a + 2a)) = 27a$ "spiral"s.

1st triple after I_{26+}, <u>two</u> levels down: $a + 2a + 4a = 7a$ "spiral"s.
2nd triple after I_{26+}, <u>two</u> levels down: $a + 2a + 4a + 2(a + 2a + 4a) = 21a$ "spiral"s.
3rd triple after I_{26+}, <u>two</u> levels down: ... = $63a$ "spiral"s.

1st triple after I_{26+}, <u>three</u> levels down: conjecture = $15a$ "spiral"s.
2nd triple after I_{26+}, <u>three</u> levels down: conjecture = $45a$ "spiral"s.
3rd triple after I_{26+}, <u>three</u> levels down: conjecture = $135a$ "spiral"s.

Etc.

These facts offer some hope that we can prove that the filling-in process actually occurs, even though $|I_{(i+1)+}| = 4|I_{i+}|$. For example they suggest that with <u>five</u> levels down, the first triple after I_{26+} will have elements of $63a$ "spiral"s.

It is essential that the "forward" movement of new "spiral"s not be too fast. If it is too fast, then all new "spiral"s will begin at intervals beyond an interval that has not been filled in. The worst case for our purposes is the exponent 2, which is the exponent by which the first element of a "spiral" maps to the base element in the case where all even exponents map to the base element.

The reason that 2 is the worst case is that if x maps to y via the exponent 2, then $x \approx (4/3)y$. This does not seem like too rapid a forward movement. Only after five successive iterations with exponent 2 is it the case that $x > 4y + 1$, thus forcing x to be in the next interval. The above table of numbers of "spiral"s suggests that this number of levels down will give us many "spiral"s in the next few triples.

But we must hasten to point out that a valid proof based on the Filling-in Strategy must consider the worst-case forward movement resulting from the tree of all possible triples. There are three such triples from each range element, namely, the 3, e, o triple, the o, 3, e triple, and the e, o, 3 triple, where "3" means the range element is mapped to by a multiple-of-3, "e" means that the range element is mapped to by all even exponents, and "o" means that the range element is mapped to by all odd exponents. We must determine the maximum forward movement that is possible via a descent of arbitrary length in this tree. This is unquestionably the most difficult task that faces us in the implementation of the Strategy. Certainly the literature on infinite trees should be searched to see if any useful results have already been obtained.

The next lemma states, informally, that no element of a higher level sequence is "wasted" by being mapped "on top of" one of the base sequence elements. In other words, we never need to worry about "overfilling" an interval.

Lemma 13.5. *For all base sequences, and for all levels $i \geq 2$ relative to a base sequence, no element of a level i sequence is an element of the base sequence. Thus, in particular, no level i element, $i \geq 2$, is an element of the base sequence relative to 1, i.e., of the base sequence {1, 5, 21, 85, 341, ... }.*

The next lemma states, informally, that no interval of the base sequence is "leaped over" by successive elements of a higher level sequence once that sequence gets started.

Lemma 14.0. *For any "spiral" at any level $i \geq 1$, the sequence of elements of the "spiral" map to successive intervals of the base sequence.*

The next lemma may help show that the first elements of "spiral"s do not move "forward" too rapidly. If they do, then that would argue against the success of the filling-in strategy.

Lemma 18.0. *Let y be a range element of interval k, where $k \geq 1$, in any "spiral". Then y is mapped to, in one iteration of the 3x + 1 function, by an odd, positive integer that lies either in interval $k - 1$, k, or $k + 2$.* [But see our remarks above regarding "forward" movement of new "spiral"s relative to triples.]

Possible Implementations of the Filling-in Strategy

• Show that all intervals in S_1 are filled in. One way of doing this might be by exploiting the recursive structure of the 1-tree. For example, one might assume that no interval in any "spiral" not known, by computer test, to be filled in with non-counterexamples, is ever filled in by non-counterexamples, and from that derive a contradiction. But then one might argue that the filling in of at least one such interval implies that all intervals in S_1 are filled in.

• Show that if all intervals up to some interval I_k are filled in, this provides sufficiently many elements in all subsequent intervals (namely, via "spiral" elements) so that, in particular, interval I_{k+1} will be filled in. This proof will probably require the hard, laborious work of studying the tree of triples that descend from any given triple.

• Show that the number of non-counterexamples in each non-empty interval is always increasing. The fact that if x maps to a range element y in a single iteration of the $3x + 1$ function via an even power of 2, then x must be greater than y, implies that it is possible for the "forward movement" of "spiral"s to be sufficiently rapid such that some elements of intervals are never occupied by non-counterexample "spiral" elements (see discussion above in this section). However, by our method of triples of intervals, we know that each triple of "spiral" elements is mapped to by an exponent 1 as well as by an exponent 2, and if x maps to y via the exponent 1, then x is *less than y*. Thus a descent through a sequence of exponents 1 and 2 that always yield x less than the original y ensures that the rate of forward movement of "spiral"s can be kept small enough so that there does not exist an infinite sequence of intervals each of which has a fixed number of non-counterexamples that is less than the size of the interval. (One such sequence, in descending order of exponents, is 2, 1, 1, 2, 1, 1, 2 ...)

• Show that a countable infinity of intervals beyond I_{26} are filled in.

• Show that the similar filling-in behavior of counterxample "spiral"s leads to a contradiction.

• Show that if each interval in an infinity of triples of intervals is not filled with non-counterexamples, this implies that for each "spiral" s having elements in the triple, there is a lowest level such that for all lower levels, only condition (2) in the proof of Lemma 18.0 applies, i.e., the first element of each "spiral" at the lower levels is always a multiple-of-3. If this can be proved to be an impossibility, then we have a proof of the $3x + 1$ Conjecture.

• Show that there is no difference between the behavior of "spiral"s if counterexamples do not exist, and if counterexamples exist. In particular, assume that counterexamples exist, and let I_k be the last interval that is entirely filled with non-counterexamples. Then show that it is impossible for "spiral" elements in, e.g., the next triple of intervals beyond I_k to behave any differently than they would if counterexamples did not exist. (The reader is invited, as an exercise, to find a way that the behavior of these "spiral" elements can differ in the two cases.)

• Show that a contradiction arises from the fact that, if a counterexample exists, there is a minimum counterexample range element (see next sub-section).

Some Facts About Counterexamples to Keep In Mind

If a counterexample exists, then there is a minimum counterexample range element y_c, and y_c lies in some interval in $S_1 = \{1, 5, 21, 85, 341, ... \}$. All tuples having y_c as first element must have the greater-to-less-than-or-equal-to property. Otherwise there would be another counterexample range element that is less than y_c, contrary to the assumption that y_c is the minimum range element. So y_c must map to y_c' in a single iteration of the $3x + 1$ function via the exponent 1, since only the exponent 1 has the less-to-greater property. On the other hand, each tuple in which y_c is not the first element must have the less-to-greater-than-or-equal-to property. Otherwise there would be another counterexample range element that is less than y_c, contrary to the assumption that y_c is the minimum range element. So y_c must be mapped to by the exponent 2 (and therefore is mapped to by all other even exponents).

Clearly, y_c must be the first element of a "spiral". The triples grouping of "spiral" elements described in the previous sub-section applies to all counterexample "spiral"s. Is there a contradiction awaiting discovery in this fact, and the greater-to-less... and the less-to-greater... properties required by y_c?

Strategy of "Filling-in" of Residue Classes

The top rows of all 2-level tuple-sets are $\{1, 7, 13, 19, ... \}$ and $\{5, 11, 17, 23, ... \}$ (Lemma 3.057). The elements of the first row are mapped to by all even exponents, and the elements of the second by all odd exponents. These facts, and Lemma 15.0, suggest a "filling-in" strategy for tuple-sets that is analogous to the one described above for recursive "spiral"s. For, since, by Lemma 15.0, the parity of exponents mapping to any base element of a "spiral" alternates, this means that any "spiral" in the infinite set of "spirals" whose base element is 1, "fills in" an infinite number of "locations" in the above two rows. For example, the base sequence relative to 1 is $\{1, 5, 21, 85, 341, ... \}$. So 1 and 85 "fill in" the locations 1 and 85 in $\{1, 7, 13, 19, ... \}$. 5 and 341 fill in the locations 5 and 341 in $\{5, 11, 17, 23, ... \}$. Elements of every higher-level "spiral" fill in additional locations in these two rows. Each "spiral" fills in an infinite number of locations in each of the two rows. Lemma 15.85 generalizes Lemma 15.0 to higher-level rows. Thus, informally, we can think of the elements of each "spiral" as "winding" endlessly through ever increasing elements of a sequence of reduced residue classes mod $2 \cdot 3^{i-1}$.

Similar questions regarding this filling-in process arise as for the recursive "spiral"s case. (See also the closely-related "Strategy Based on the Threading of Non-Counterexample Tuples Through Two 2-level Tuple-sets" on page 68.)

Strategy Based on the Threading of Non-Counterexample Tuples Through Two 2-level Tuple-sets

For the purposes of a proof of the 3x + 1 Conjecture, we only need two tuple-sets, namely, the 2-level tuple-sets $T_{\{1\}}$ and $T_{\{2\}}$. We know all the elements of level 1 and level 2 in each of these two tuple-sets by parts (a) and (b) of "Lemma 1.0" on page 11. The set of level-2 elements in both tuple-sets is the set of range elements of the $3x + 1$ function (odd, positive integers that are not multiples-of-3). All multiples-of-3 are in the set of level-1 elements in both tuple-sets. Furthermore, we know were all the multiples-of-3 in level 1 of each tuple-set are, because these are simply every third level-1 element after the first multiple-of-3.

The reason that these two tuple-sets suffice is that each range element is mapped to by an infinite "spiral", and each element of an infinite "spiral" is either a multiple-of-3 or a range element. Each range element is mapped to either by all odd exponents, or by all even exponents. If the first element of the "spiral" is a multiple-of-3, and the "spiral" is mapped to by all *odd* exponents, then the next element of the "spiral" maps to the range element by 2^3. If the first element of the "spiral" is a multiple-of-3, and the "spiral" is mapped to by all *even* exponents, then the next element of the "spiral" maps to the range element by 2^4. Thus we can trace or thread backward (or downward) indefinitely from any range element via only the two named tuple-sets. For example, we can trace backward from 5 (the first element of level 2 in $T_{\{1\}}$) to 3, the first element of level 1 of $T_{\{1\}}$}. But since 3 is a multiple-of-3, we choose 13 instead, since 13 maps to 5 via the exponent 3. We see that 13 is the third element of level 1 in $T_{\{2\}}$. From 13 we go next to 17, which is the third element of level 2 in $T_{\{1\}}$, etc.

It would seem that we have all the makings of an inductive proof of the $3x + 1$ Conjecture, where the basis step would probably be, for some i, the set of range elements less than $2 \cdot 3^{i-1}$ — in other words, the set of level i anchors — that are known, by hand-calculation or computer test to be all non-counterexamples. If we can show that the level-2 elements of just one of our two tuple-sets are all non-counterexamples, then we have a proof of the $3x + 1$ Conjecture, because this contradicts Lemma 5.0 in our paper, "Are We Near a Solution to the $3x + 1$ Problem?" on occampress.com.

Of considerable aid seems to be the ...3, e, o, 3, ... pattern for successive elements of a "spiral", where "3" denotes a multiple-of-3, "e" denotes an element that is mapped to by all even exponents, and "o" denotes an element that is mapped to by all odd exponents.

It is important to realize that in each tracing backward, we are in effect tracing backward though the elements of a tuple. For example, in our tracing backward example above, we traced backward the elements of the tuple <17, 13, 5>, which is associated with the exponent sequence {2, 3}.

We remind the reader that by Lemma 5.0, there is an *infinity* of tuples that are associated with precisely the *same* exponent sequence, and indeed with *each* finite exponent sequence — an infinity of non-counterexample tuples and, if counterexamples exist, an infinity of counterexample tuples. Of course, at this point, we do not know if any of our tuples are counterexample tuples. But we might be able to show the following: if we consider the set of all infinite non-counterexample tuples whose last element is 1 (these infinite tuples are arrived at by descending from the root 1 in the 1-tree), then each range element is included in a sufficiently long tuple that is established by working backward from 1. Since we are only dealing with a total of four levels here — the two level-2 levels and the two level-1 levels of our two tuple-sets — and since the two level-2 levels constitute the set of all range elements, and the two level-1 levels constitute the set of all domain elements (the set of all odd, positive integers), this goal may not be hopeless.

It might be possible to show that, because of Lemma 5.0, if all range elements less than $2 \bullet 3^{35-1}$ are non-counterexamples — which we know that they are as a result of computer tests — in other words, if all 35-level anchor tuples are non-counterexample tuples, then all range elements less than, say, $2 \bullet 3^{70-1}$ are likewise counterexamples, etc. This would give us our inductive proof of the $3x + 1$ Conjecture.

More specifically, we might invoke Lemma 18.0 in our paper, "A Solution to the 3x + 1 Problem" on occampress.com. This Lemma states that for each range element y and for each finite exponent sequence A, there exists an x that maps to y via A possibly followed by a "buffer" exponent. Let the range element be 1, and consider the set of all 35-level exponent sequences A. We know by computer tests that all 35-level anchors, hence all 35-level anchor tuples, are non-counterexample. Let us choose a (necessarily non-counterexample) range element y close to, but less than, $2 \bullet 3^{35-1}$ that, like 1, is mapped to by even exponents. Invoking Lemma 18.0, let us consider all 35-level tuples whose last element is y'. (We will be working backward from y', tracing tuples in reverse order.) The question is, is it possible that there is a range element, e.g., a counterexample range element, that is not an element of any of these tuples?

If we can show that the answer is no, then we have our proof by induction, because we can continue to apply our reasoning to higher and higher ranges of exponents. The reader should keep in mind that all 35-level tuples that are associated with the same exponent sequence, lie in the same 35-level tuple-set, and that"Lemma 1.0" on page 11 tells us how far apart elements of tuples consecutive at level j, where $1 \leq j \leq 35$, are.

The skeptical reader will argue that we are simply asking if there is a way to prove, from tuples in 35-level tuple-sets, that a tuple is a counterexample tuple, and the answer is almost certainly no. However, we would point out that in our case, tuples are imposed upon ("threaded through") another structure, namely, the four levels of our two 2-level tuple-sets. This structure is exceedingly simple. Furthermore, we know the values of all elements in it, by Lemma 1.0.

We cannot refrain from asking the reader the following question, because it directly relates to our first two propoosed proofs of the $3x + 1$ Conjecture (see our paper, "A Solution to the $3x + 1$ Problem", on occampress.com. By computer tests, we know that all range elements up to at least 10^{15} are known to be non-counterexamples. This fact holds whether or not counterexamples exist. The question is: how exactly can there be a difference, in our two tuple-sets, between the case: counterexamples do not exist case and the case: counterexamples exist, given that the distance functions for levels 1 and 2 in each of our two tuple-sets, are the same regardless if counterexamples exist or not? At present, our answer is that there is no difference.

Strategy Based on the Application of "Spiral"s to 2-level Tuple-sets

This strategy is the reduction of the domain of another strategy, namely, of "Strategy of 'Filling-in' of Intervals in the Base Sequence Relative to 1" in the first file of our paper, "The Structure of the $3x + 1$ Function: An Introduction" on occampress.com. In this strategy, we reduce the domain to the two tuple-sets, $T_{\{1\}}$ and $T_{\{2\}}$.

Even though 2-level tuple-sets are a sequential listing of all 2-level tuples, we immediately observe that, although Lemma 5.0 states that if counterexamples exist, then each tuple-set contains an infinity of counterexample tuples and an infinity of non-counterexample tuples, there is apparently no way of knowing *which* tuples in a tuple-set are non-counterexample tuples.

However, the distance function for "spiral"s shows that this observation is not true. It states that if x is an element of a non-counterexample "spiral", then $x' = 4x + 1$ is also an element of the "spiral". Thus we know an infinity of non-counterexamples from the fact that x is a non-counterexample.

A weaker version of this fact is stated in Lemma 3.24.

Lemma 3.24. *Let x be a range element that is a minimum residue of a reduced residue class mod* $2 \cdot 3^{(i-1)}$*, and let*

$$\frac{3x+1}{2^j} = h$$

Then if

$$j \equiv k \ \ mod \ \ (2 \cdot 3^{i-1})$$

there exists an x′ such that

$$\frac{3x'+1}{2^k} = h$$

and furthermore

$$x \equiv x' \ \ mod \ \ (2 \cdot 3^{i-1})$$

The following facts might also lead to a proof of the $3x + 1$ Conjecture.

- The set of first elements of all 2-tuples in the set of all 2-level tuple-sets $T_{\{1\}}$ and $T_{\{2\}}$ is the set of first elements of all "spiral"s.

- The sequence of elements in a "spiral" follows the pattern ...3, *e*, *o*, 3, ..., where "3" denotes a multiple-of-3, "e" denotes the element is mapped by even exponents only, "o" denotes the element is mapped to by odd exponents only.

The distance between successive elements x, x' of a "spiral" is given by $x' = 4x + 1$.

- The set of all range elements of the $3x + 1$ function is the union of the two sets {1, 7, 13, 19, ... } and {5, 11, 17, 23, ... }. The first set is the set of range elements that are mapped to, in one iteration of the $3x + 1$ function, by even exponents, and thus is the set of second elements of all 2-tuples in the tuple-set $T_{\{2\}}$. The second set is the set of range elements that are mapped to, in one iteration of the $3x + 1$ function, by odd exponents, and thus is the set of second elements of all 2-tuples in the tuple-set $T_{\{1\}}$.

- The sequence of elements in a "spiral" therefore follows an infinite path through the tuples in the tuple-sets $T_{\{1\}}$ and $T_{\{2\}}$. A portion of this path is:

  Element of {1, 7, 13, 19, ... }, then (larger) element of {5, 11, 17, 23, ... }, then (larger) multiple-of-3 (not the first element of a 2-tuple in $T_{\{1\}}$ or $T_{\{2\}}$ unless it is the first element of a "spiral"). (No multiple-of-3 is a range element, hence no multiple-of-3 is the second element of a 2-tuple in either of the tuple-sets $T_{\{1\}}$ and $T_{\{2\}}$.) Then (larger) element of {1, 7, 13, 19, ... }, etc.

  But now each range element in the path is the second element of a 2-tuple, the first element of which is the first element of another, different "spiral"! And so another infinite path is created. Each range element in that path is the second element of a 2-tuple, the first element of which is the first element of another, different "spiral"! Clearly, since there is an infinite number of range elements in each "spiral", this process never stops.

- The elements of the "spiral" that maps to 1 in one iteration of the $3x + 1$ function, namely, the "spiral", {1, 5, 21, 85, 341, ... }, are present in the set of tuples in $T_{\{1\}}$ and $T_{\{2\}}$. And thus we see that our present strategy is simply the "filling-in" strategy referred to at the start of this section, but here with the domain of application reduced (without loss of generality) to merely the two tuple-sets $T_{\{1\}}$ and $T_{\{2\}}$. The reader should keep in mind that each "spiral" has an element in a countable infinity of *successive* intervals in the "spiral" {1, 5, 21, 85, 341, ... }. (This is established in the section referenced at the start of this section.)

Let $T_{\{1\}(1)}$ denote the set of *first* elements of all tuples in the tuple-set $T_{\{1\}}$. Let $T_{\{1\}(2)}$ denote the set of *second* elements of all tuples in the tuple-set $T_{\{1\}}$. And similarly for $T_{\{2\}(1)}$ and $T_{\{2\}(2)}$. An example of the process we described above then is the following. We begin with the "spiral" {3, 13, 53, 213, ... }.

$3 \in T_{\{1\}(1)}$
$13 \in T_{\{2\}(2)}$;
$53 \in T_{\{2\}(1)}$;
$213 \in T_{\{1\}(1)}$

- We know, by computer test, that all odd positive integers up to at least 10^{15}.are non-counterexamples. This means that we know, because of what we have said regarding "spiral"s, that many[1] *countable infinities* of odd, positive integers are non-counterexamples, and we know the elements of each of these countable infinities, via the above facts.

---

1. We cannot say "10^{15} countable infinities" because if x, $4x + 1$, $4(4x + 1) + 1$, etc. are all less than 10^{15}, then these numbers define only one countable infinity of non-counterexamples.

- If one asks, "How many (non-counterexample) 'spiral's are there as a result of the 10^{15} non-counterexamples?", the answer is, "As many as there are elements of the two range element sets {1, 7, 13, 19, ... } and {5, 11, 17, 23, ... } up to 10^{15}".

If we can show that, beginning with the known 10^{15} non-counterexamples, the process we have described via our paths, above, is sufficient to prove that all elements of the 2-level sets {1, 7, 13, 19, ... } and {5, 11, 17, 23, ... }are non-counterexamples, then we will have a proof of the $3x + 1$ Conjecture. The following is worthy of consideration.

1. Consider the "spiral" {1, 5, 21, 85, 341, ... }. Let I_{i+} denote the ith element of the "spiral", plus all odd, positive integers up to but not including the (i+1)th element of the "spiral". Thus, for example, I_{2+} = {5, 7, 9, 11, 13, 15, 17, 19}.

2. We assert without proof that the number $|I_{i+}|$ of elements in I_{i+} is 2^{2i-1}. Thus, for example, $|I_{2+}| = 8$. We further assert without proof that the number of range elements in I_{i+} is about $2/3|I_{i+}|$, because about 2/3 of odd, positive integers are range elements, since only multiples-of-3 are not.

3. Let I_{k+} denote the largest interval that is known, by computer test, to contain solely non-counterexample elements. Consider interval $I_{(k+1)+}$. The total number of range elements in $I_{(k+1)+}$ is about $2/3|I_{(k+1)+}|$, or $2^2(2/3)|I_{k+}|$, as the reader can check.

(*Proof*: $2/3|I_{(k+1)+}| - 2/3|I_{k+}| = 2/3(2^{2(k+1)-1}) -$
$2/3(2^{2k-1}) = 2/3(2^{2k-1}(2^2 - 1) = 2(2^{2k-1})$.□

4. We know that there is one element of each of $|I_{k+}|$ "spiral"s in $I_{(k+1)+}$. About $2/3|I_{i+}|$ are range elements. n order to fill $I_{(k+1)+}$ with the maximum amount of range elements, we need to add $2|I_{k+}|$ range elements to $I_{(k+1)+}$. (*Proof*: $2/3|I_{(k+1)+}| - 2/3|I_{k+}| = 2/3(2^{2(k+1)-1}) - 2/3(2^{2k-1}) = 2/3(2^{2k-1}(2^2 - 1) = 2(2^{2k-1})$.□) So we need to add $2|I_{i+}|$ range elements to the $|I_{k+}|$ elements that are already in $I_{(k+1)+}$.

Each of these range elements y is mapped to by a "spiral". The first three "spiral" elements either map to y via the exponents 1, 3, 5 or via the exponents 2, 4, 6. Regardless of the parity of exponents, The possible patterns for the first three elements of each "spiral" are: (1) 3, e, o, (2) e, o, 3, and (3) o, 3, e, as explained above in this section.

Our task is to show that in the recursive descent of "spiral"s, we must eventually add a sufficient number of range elements to make a total of $2^2(2/3)|I_{k+}|$.

4. We begin with the $2/3|I_{i+}|$ range elements that we know are in $I_{(k+1)+}$. About 2/3 of those further range elements. And about 2/3 of those 2/3 are still further range elements. Etc. So in $I_{(k+1)+}$ we have more than $(2/3)|I_{k+}|$ + $(4/9)|I_{k+}| + (12/27)I_{k+} + (36/81)|I_{k+}| = (130/81)\ |I_{k+}| \approx (1.6)|I_{k+}|$ range elements. But we need $2|I_{k+}|$ range elements to fill $I_{(k+1)+}$ with range elements, so we do not have enough. However

lim, as $k \to \infty$, of $(2/3)|I_{k+}| + (2/3)^2|I_{k+}| + (2/3)^3|I_{k+}| + (2/3)^4|I_{k+}| + ... = |I_{k+}| \dfrac{(2/3)^k - 1}{2/3 - 1} - 1 =$ $2|I_{k+}|$.

This is the number of range elements that we need to fill $I_{(k+1)+}$ with range elements. However, there are two problems: first, the need for an infinity of descending steps in our recursions by 2/3 is not realistic, and, second, we must recognize that the 2/3 factor is only approximate — at best an average. If we can overcome these problems, then perhaps a proof of the $3x + 1$ Conjecture might be in sight.

We encourage the reader to ask the question that underlies our proposed proofs of the $3x + 1$ Conjecture (see “A Solution to the $3x + 1$ Problem” on occampress.com) — namely, “How exactly does the different behavior that is required of a counterexample, actually occur following the initial set of non-counterexample tuples?”

See also, “Strategy of Filling-in of Residue Classes” in the first file of our paper, “The Structure of the $3x + 1$ Function: An Introduction” on the web site occampress.com.

Strategy of Using a Topology Defined on “Spiral”s

It is natural to wonder if defining an appropriate topology on “spiral”s or their elements or finite paths in “spiral”s might lead us to a proof of the $3x + 1$ Conjecture. We might begin by taking as our set of points the set of all range elements, and then, for each range element y, defining a neighborhood of y as the set of all range elements mapping to y in $\leq n$ iterations, where n is a non-negative integer.

Unfortunately, the neighborhood system thus defined fails one of the conditions for a topological space, namely the condition:

If U is a neighborhood of y, and U $\subset$ V, then V is a neighborhood of y.

For, let $y = 1$, and let U be a neighborhood of 1. Let V = {U $\cup$ {z}}, where z is a counterexample. Then clearly U $\subset$ V but, by definition, V is not a neighborhood of U.

The condition is not violated, of course, if we take as our set of points, the set of range elements mapping to a given range element, y, e.g., $y = 1$. But in this case, there is a separate topology for the set of range elements mapping to 1, and for each connected set of counterexamples, where a connected set of counterexamples has the property that each element maps to another element of the set, or is mapped to by one or more other elements of the set.

We believe that an investigation of topologies defined on “spiral”s or their elements would be worthwhile.

Strategy of the Boundary Between Non-Counterexamples and Counterexamples

The motivation for this strategy is the simple fact that whether or not a counterexample exists, a portion of the infinite set of “spiral”s relative to the base element 1, remains the same.

We begin with the observation that, if a counterexample exists, then the set of all intervals defined by each “spiral” in the infinite set of “spiral”s relative to the base element 1, contains an *infinity* of counterexamples. The reason is that the elements of each “spiral” and the intervals between them constitute all odd, positive integers $\geq$ the first element of the “spiral”. Since, if a counterexample exists, an infinity of counterexamples exists, the observation follows.

Therefore, if a counterexample exists, in each "spiral" in the infinite set of "spiral"s relative to the base element 1, there is a first interval, and a first element of that interval, that is a counterexample.

Suppose we mark with black all "spiral" elements in the above infinite set, and all interval elements in each "spiral", that map to 1. We mark with red all counterexample elements in all "spiral" intervals. Call the set of elements marked in black, *B*. We then ask how it is possible for counterexamples to exist at all, given that beginning with any "spiral" element in *B*, it is not possible to tell, by applying successive inverses of the $3x + 1$ function, whether or not a counterexample exists. Putting it another way, we can in principle represent the infinite set of "spiral"s with a diagram, defining some appropriate scaling factor for the distances between "spiral" elements and base elements. The portion of the diagram representing the set *B* will be exactly the same whether or not a counterexample exists. We ask how exactly does the remainder of the diagram differ for the two cases? In short, where do the two cases — (1) no counterexamples exist, (2) counterexamples exist — "begin to diverge" from *B*?

Generalizations of the 3*x* + 1 Function

Numerous generalizations of the $3x + 1$ function have appeared in the literature. Here we give only two because it appears, on the basis of limited examination, that the tuple-sets structure, including the distance functions, apply to them.

For further details, see Appendix C, “$3x + 1$ - like” Functions, in the first part of our paper, “Are We Near a Solution to the $3x + 1$ Problem?” on occampress.com.

The 3*x* – 1 Function

Here, division in each iteration is by

$$2^{ord(3x - 1)}.$$

The negative of the range elements of this function are the range elements of the $3x + 1$ function applied to the odd, negative integers.

It is well-known that at least three cycles exist in this function. They involve 1, 5, and 17.

Description of Tuple-sets for the 3x –1 Function

<1, 1>
<5, 7>
<9, 13>
<13, 19>
<17, 25>
...

<7, 5>
<15, 11>
<23, 17>
<31, 23>
<39, 29>
...

<1, 1, 1>
<9, 13, 19>
<17, 25, 37>

The 3*x* + 3*^k^* Function

Here, $k \geq 0$. Each k defines a separate function. The 0 case of course gives us our familiar $3x + 1$ function. Division in each iteration is by

$$2^{ord(3x + 3^k)}.$$

As far as we know, this class of functions was first defined in 1993 by Barry Brent (email 6/27/02). The paper is accessible on Brent’s web site, www.home.earthlink.net/~barryb0/.

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