Are We Near a Solution to the 3*x* + 1 Problem?

A Discussion of a Promising Strategy

by

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Abstract

We present a possible strategy for solving the $3x + 1$ Problem, which asks if repeated iterations of the function $C(x) = (3x + 1)/(2^a)$ always terminate in 1 Here x is an odd, positive integer, and a is the largest positive integer such that the denominator divides the numerator. The strategy begins by partitioning the set of all finite sequences of iterations of C (each sequence being represented by a *tuple*) into *tuple-sets*. If a tuple is associated with the sequence $A = \{a_2, a_3, ..., a_i\}$ of exponents of 2, where $i \geq 2$, then the tuple is an element of the tuple-set T_A. We show that if counterexamples exist, each tuple-set contains an infinity of counterexample tuples and an infinity of non-counterexample tuples. Also that each range element of C, including, for example, 1, is mapped to by every finite exponent sequence. But the first tuple in a tuple-set (the so-called *anchor* tuple) cannot be both a counterexample tuple and a non-counterexample tuple, and so our strategy is aimed at showing that the existence of counterexamples implies the contrary, and so gives us a proof of the Conjecture by contradiction.

Key words: $3x + 1$ Problem, $3n + 1$ Problem, Syracuse Problem, Ulam's Problem, Collatz Conjecture, computational number theory, proof of termination of programs, recursive function theory

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Introduction

Statement of Problem

For x an odd, positive integer, set

$$C(x) = \frac{3x+1}{2^{ord_2(3x+1)}}$$

where $ord_2(3x + 1)$ is the largest exponent of 2 such that the denominator divides the numerator. Thus, for example, C(17) = 13, C(13) = 5, C(5) = 1. The $3x + 1$ Problem, also known as the $3n$ + 1 Problem, the Syracuse Problem, Ulam's Problem, the Collatz Conjecture, Kakutani's Problem, and Hasse's Algorithm, asks if repeated iterations of C always terminate at 1. The conjecture that they do is hereafter called the $3x + 1$ Conjecture. We call C the $3x + 1$ function; note that $C(x)$ is by definition odd.

Other equivalent formulations of the $3x + 1$ problem are given in the literature; we base our formulation on the C function because, as we shall see, it brings out certain structures that are not otherwise evident.

Summary of Research on the Problem

As stated in [Lagarias 1985], "The exact origin of the $3x + 1$ problem is obscure. It has circulated by word of mouth in the mathematical community for many years. The problem is traditionally credited to Lothar Collatz, at the University of Hamburg. In his student days in the 1930's, stimulated by the lectures of Edmund Landau, Oskar Petron, and Issai Schur, he became interested in number-theoretic functions. His interest in graph theory led him to the idea of representing such number-theoretic functions as directed graphs, and questions about the structure of such graphs are tied to the behavior of iterates of such functions... In the last ten years [that is, 1975-1985] the problem has forsaken its underground existence by appearing in various forms as a problem in books and journals..."

As far as we have been able to determine, our approach to a solution of the Problem is original. We have been unable to find papers in the literature that suggest that other researchers have had the same basic idea as ours, namely, that which is described in "Abstract" on page 2 and, in more detail, in the first two paragraphs under "Tuple-Sets" on page 5.

Summary of Solution Strategy

The strategy begins with a partitioning of the set of all finite sequences of iterations of C into "tuple-sets". Each such sequence is represented by a tuple, and is associated with its corresponding exponent sequence. Thus, for example, the sequence <7, 11, 17, 13> is associated with the exponent sequence $A = \{1, 1, 2\}$, because 7 maps to 11(in one iteration of C) via the exponent 1, 11 maps to 17 (in one iteration of C) via the exponent 1, and 17 maps to 13 (in one iteration of C) via the exponent 2. The tuple-set T_A contains all tuples associated with A, plus all tuples associated with each prefix of A. There are always an infinity of tuples in a tuple-set.

We then show that if counterexamples exist, each tuple-set contains an infinity of counterexample tuples *and* an infinity of non-counterexample tuples[1]. Furthermore, in each i-level tuple-

1. "Lemma 5.0: Statement and Proof" on page 39

set, $i \geq 2$, and for each exponent a_{i+1}, there is an infinity of non-counterexample tuples that have an extension via a_{i+1}[1]. If counterexamples exist, the same holds for counterexamples in the tuple-set.

We also show that each range element of C, including, for example, 1, is mapped to by each finite exponent sequence.

Next we show that every odd, positive integer must eventually become an element of the unique first i-level tuple (the so-called *anchor* tuple) in an i-level tuple-set, for some i, and, of course, remain an element of all extensions of that tuple[2]. An anchor tuple is a non-counterexample tuple or a counterexample tuple, but not both.

Our strategy consists in deriving a contradiction from these facts. Informally, our strategy attempts to show that there is not enough "room" for counterexample anchor tuples in the set of all tuple-sets. If there are no counterexample anchor tuples, then there are no counterexamples.

On the Format of this Paper

Because the tuple-set structure lies at the heart of our solution to the Problem, and thus is referred to in most proofs of lemmas, and because defining this structure requires defining quite a few terms, we have placed all these definitions at the beginning of our exposition.

To enhance ease and rapidity of understanding, we merely state lemmas in the course of the exposition. All proofs are given in Appendix A. The page number of each proof is given in the text. Lemmas are numbered consecutively, beginning with 1.0., in the order in which they appear in the exposition. But this means that lemmas that are not mentioned in the exposition have higher numbers than the numbers of the lemmas of the proofs they support.

Referenced equations in each lemma are numbered n.1, n.2, etc., where n is the number of the lemma.

1. "Lemma 3.0: Statement and Proof" on page 37
2. "Lemma 8.0: Statement and Proof" on page 41

Tuple-Sets

In the first part of this paper, we describe a structure called *tuple-sets* that underlies all finite sequences of iterations of the $3x + 1$ function, C. We have placed virtually all definitions in this first part of the paper because the terms defined are used repeatedly in the lemmas and proofs given later.

A tuple-set can be briefly, and informally, described as follows. (A formal definition is given under "Tuple-set" on page 7.) Consider the sequence of two iterations of C: $C(17) = 13$ (via the exponent 2 in the definition of C) followed by $C(13) = 5$ (via the exponent 3 in the definition of C). This sequence can be represented by the tuple <17, 13, 5>. The tuple-set T_A defined by the 2-level exponent sequence $A = \{2, 3\}$ contains: the tuple <17, 13>, and the tuple <17, 13, 5>. But in addition it contains all other tuples that are generated by the exponent sequences $\{2\}$ and $\{2, 3\}$ — in other words, all other tuples that are associated with "approximations" to A. For example, the tuples <33, 25> and <81, 61, 23> are in T_A, because they too are generated by the exponent sequences $\{2\}$ and $\{2, 3\}$, respectively, as the reader can verify.

We then show that each i-level tuple-set, where $i \geq 2$, has a unique first i-level tuple (called an *anchor* tuple) that (like all tuples) must be either a non-counterexample tuple or a counterexample tuple, but cannot be both. In the second part of this paper, we show how a basic result — that if counterexamples exist, then every tuple-set contains an infinity of counterexample tuples and an infinity of non-counterxample tuples — enables us to prove that there are no counterexample anchor tuples, hence no counterexamples.

We now proceed with our definitions.

Iteration

An *iteration* takes an odd, positive integer, x, to another odd, positive integer, y, via one application of the $3x + 1$ function, C. Thus, in one iteration C takes 17 to 13 because $C(17) = 13$.

Trajectory

A *trajectory* (sometimes called an *orbit*) is a sequence of one or more successive iterations of C, that is, if the sequence is finite,

$$(C^k(x))_{k \geq 0} = (x, C(x), C^2(x), \ldots C^k(x))$$

or, if the sequence is infinite,

$$(C^\infty(x)) = (x, C(x), C^2(x), \ldots)$$

The last element of the finite sequence need not be 1 and it need not be an infinity of successive 1's in the case of an infinite sequence.

A trajectory or orbit is the same as a *tuple*, which is defined below.

Non-Counterexample and Counterexample

If x is the first element of an infinite tuple <x, ..., 1, 1, 1, ... > , then x is called a *non-counterexample*. Otherwise, x is called a *counterexample*. Thus, a counterexample never yields 1 under repeated iterations of the $3x + 1$ function.

Exponent

If $C(x) = y$, with $y = (3x + 1)/2^a$, we say that *x maps under iteration to y* (or *x maps directly to y*) *via the exponent a*, and that *a is the exponent associated with x*. We sometimes speak of a as *mapping directly to y*. We sometimes omit the word *directly* when context makes clear that it is implied. The sequence $\{a_2, a_3, ..., a_i\}$, where $a_2, a_3, ..., a_i$ are the exponents associated with x, $C(x), ..., C^{(i-2)}(x)$ respectively, is called an *admissible vector* in [Wirsching 1998]. We call the sequence an *exponent sequence*. We define the function $e(x)$ to be the exponent associated with x. We sometimes refer to y as a *range element*. It is easily shown that y cannot be a multiple of 3 (see “Lemma 10.0: Statement and Proof” on page 41). An element x of the domain of the $3x + 1$ function, whether multiple of 3 or not, we sometimes refer to as a *domain element*.

Clearly, an exponent is a positive integer.

Tuple

A *tuple* is a sequence of one or more successive iterations of C, that is, if the sequence is finite,

$$(C^k(x))_{k \geq 0} = (x, C(x), C^2(x), \ldots C^k(x))$$

or, if the sequence is infinite,

$$(C^{\infty}(x)) = (x, C(x), C^2(x), \ldots)$$

A finite sequence is not required to end with a 1, and an infinite sequence is not required to end with an infinity of successive 1's. If an infinite sequence does not end with an infinity of successive 1’s, then it consists of counterexamples to the $3x + 1$ Conjecture.

A finite tuple is denoted[1] <$x, y, y', ..., y^{(n)}$>. For example, <5, 1> and <11, 17, 13> are finite tuples. An infinite tuple, which represents an infinite trajectory, is denoted <x, y, y', ... >. For example, <5, 1, 1, 1, ... > and <11, 17, 13, 5, 1, 1, 1, ... > are infinite tuples.

Let t = <$x, y, y', ..., y^{(n)}$> be a finite tuple. Then the tuple t' = <$x, y, y', ..., y^{(n)}, y^{(n+1)}$> is an *extension* of t. An extension of an extension of t we likewise call an extension of t, etc. By definition of the function C, every tuple has an infinite number of extensions. In the case of a sequence of iterations of C that eventually yield 1, the corresponding infinite tuple is <x, y, y', ..., 1, 1, 1, ...>. A tuple consisting of an infinite number of extensions is an *infinite tuple*. We denote an infinite tuple by $\bar{t}$.

1. In a tuple, “$x^{(n)}$”, “$y^{(n)}$”, etc., denotes x with n primes, y with n primes, etc.

Clearly, since the domain of C consists of the odd, positive integers, every odd, positive integer is the first element of an infinite tuple.

If $\overline{t}$ is an infinite tuple, we denote the first i levels of $\overline{t}$ (that is, the first i elements of $\overline{t}$), by $\overline{t}(i)$, and we call $\overline{t}(i)$ a *prefix* of $\overline{t}$. For example, if $\overline{t}$ = <17, 13, 5, 1, 1, 1, ... >, then $\overline{t}(1) = 17$, and $\overline{t}(4)$ = <17, 13, 5, 1>. Thus every finite tuple is a prefix of an infinite tuple and every prefix of an infinite tuple is a finite tuple. The term *tuple* standing alone, without the qualifier "infinite", denotes a finite tuple, that is, the prefix of an infinite tuple, unless context clearly indicates the reference is to an infinite tuple.

In the literature on the $3x + 1$ Problem, tuples are sometimes called "trajectories" or "orbits".

Each tuple element except, possibly, the first, is an odd, positive integer that is not a multiple of 3. The element is odd by definition of the $3x + 1$ function, C, and is not a multiple of 3 by "Lemma 10.0: Statement and Proof" on page 41.

Exponent Sequence Associated With a Tuple

As we established under "Exponent" on page 6, associated with every finite sequence of iterations of the function C — hence with every tuple — is an exponent sequence. We speak of the exponent sequence *associated with* a finite tuple. If t is a tuple, then we denote the exponent sequence associated with t by $A(t)$. Thus, for example, if t = <17, 13, 5, 1> then $A(t) = \{2, 3, 4\}$ because 17 maps directly to 13 via the exponent 2, 13 maps directly to 5 via the exponent 3, and 5 maps directly to 1 via the exponent 4.

Extension of an Exponent Sequence

Let $A = \{a_2, a_3, ..., a_i\}$ be a finite sequence of exponents, where $i \geq 2$. Then an exponent sequence $A' = \{a_2, a_3, ..., a_i, a_{i+1}\}$ is an *extension* of A. An extension of A' is also an extension of A, etc.

Tuple-set

(The reader might find it helpful to refer to Fig. 1 in this sub-section while reading the following.)

Let $A = \{a_2, a_3, ..., a_i\}$ be a finite sequence of exponents, where $i \geq 2$. The *tuple-set* T_A consists of all and only the following tuples:

all tuples <x> such that x does not map to an odd, positive integer via a_2;

all tuples <x, y> such that x maps to y via a_2 (that is, $e(x) = a_2$) but y does not map to an odd, positive integer via a_3;

all tuples <x, y, y'> such that x maps to y via a_2 (that is, $e(x) = a_2$) and y maps to y' via a_3 (that is, $e(y) = a_3$) , but y' does not map to an odd, positive integer via a_4;

...

all tuples <$x, y, y', ..., y^{(i-3)}, y^{(i-2)}$> such that x maps to y via a_2 (that is, $e(x) = a_2$) and y maps to y' via a_3 (that is, $e(y) = a_3$) and ... and $y^{(i-3)}$ maps to $y^{(i-2)}$ via the exponent a_i (that is, $e(y^{(i-3)}) = a_i$). (The longest tuple in an i-level tuple-set has i elements.)

If $A = \{a_2, a_3, ..., a_i\}$ is a finite exponent sequence, then of an i-level tuple t in the tuple-set T_A, we say that *t is generated by* the exponent sequence A and that A is *associated with t*. Finally, we say that the tuple-set T_A *is generated by* the sequence A. To review: for each tuple, we speak of the exponent sequence *associated with* it; for each exponent sequence, we speak of the tuple or tuple-set it *generates*.

As an example of (part of) a tuple-set: in Fig. 1, where $A = \{a_2, a_3, a_4\} = \{1, 1, 2\}$ and where we adopt the convention of orienting tuples vertically on the page, the tuple-set T_A includes:

the tuple <1>, because $e(1) \neq a_2$;
the tuple <3, 5> , because $e(3) = a_2 = 1$, but $e(5) = 4 \neq a_3 = 1$;
the tuple <15, 23, 35>, because $e(15) = a_2 = 1$, and $e(23) = a_3 = 1$, but $e(35) = 1 \neq a_4 = 2$.

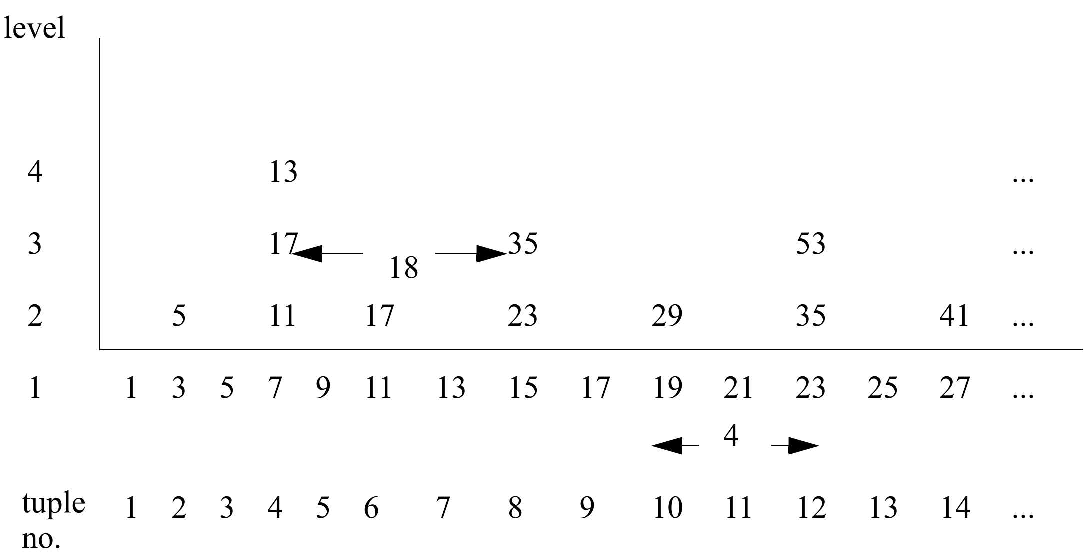


Fig. 1. Part of the tuple-set T_A associated with the sequence $A = \{1, 1, 2\}$

The numbers 18 and 4 between the arrows are values of the distance functions established by Lemma 1.0 (see “Lemma 1.0: Statement and Proof” on page 32).

In each i-level tuple-set T_A, where $i \geq 2$, for each odd, positive integer x there exists a tuple whose first element is x. The tuple may be one-level (<x>), or two-level (<x, y>), or ... or i-level (<$x, y, y', ..., y^{(i-3)}, y^{(i-2)}$>). Thus each tuple-set is non-empty.

Lemma 4.0 (see “Lemma 4.0: Statement and Proof” on page 39) establishes that a tuple-set T_A exists for each exponent sequence A.

Note: *our proofs will almost always involve only i-level tuples in i-level tuple-sets. We have included j-level tuples, where j < i, in our definition of tuple-set because we feel that these tuples are necessary to fully describe the structure of tuple-sets. If the level of a tuple in an i-level tuple-set is not specified, the reader should assume that the tuple is i-level.*

Ordering of Tuples in a Tuple-set

Tuples in a tuple-set T_A are linearly ordered by the natural order of their first elements. We denote a specific tuple in a tuple-set by $t_{(r)}$, where $r \geq 1$. If T_A is an i-level tuple-set, where $i \geq 2$, we denote the jth element of $t_{(r)}$ (if it exists in T_A) by $t_{(r)(j)}$, where $1 \leq j \leq i$.

The reader may find it helpful to imagine an i-level tuple-set, where $i \geq 2$, as a "picket fence" infinite to the right, with the tuples serving as the pickets, as suggested by Fig. 1 under "Tuple-set" on page 7.

Level in a Tuple-set

A *level j* in a tuple-set is defined as follows. If $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, is a finite sequence of exponents, the subscript j in a_j, $2 \leq j \leq i$, denotes the *level j* in T_A. Subscripts of exponents in an exponent sequence are numbered beginning with 2 instead of with 1 so that the last subscript then indicates the number of levels in the corresponding tuple-set. Thus, for example, if $A = \{a_2\}$, then T_A is a 2-level tuple-set; if $A = \{a_2, a_3\}$, T_A is a 3-level tuple-set, etc. Level 1 is then the level containing the set of all possible tuple first elements {1, 3, 5, 7, ... } in T_A, that is, the set of odd, positive integers. Thus, for example in the tuple <17, 13, 5, 1>, 17 is at level 1, 13 is at level 2, 5 is at level 3, and 1 is at level 4. We denote the element at level j in the nth tuple in a i-level tuple-set, where $i \geq 2$, by $t_{(n)(j)}$, where $1 \leq j \leq i$. (The element at level j is the jth element in the tuple.)

If a tuple has an element at level j, but none at level $j + 1$, we refer to the tuple as a *j-level tuple*. If the tuple also has an element at level $j + 1$, we sometimes refer to the tuple as a ($\geq j$)-level tuple. The longest tuple in a tuple-set generated by an i-level exponent sequence is an i-level tuple.

In the case that $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, we refer to A as an *i-level exponent sequence*. An i-level exponent sequence consists of (i - 1) exponents.

Tuples Consecutive at Level j

Tuples *consecutive at level j*, $j \geq 2$, are defined as follows. Let t_k, t_n be ($\geq j$)-tuples in some i-level T_A, where $i \geq 2$. If there is no ($\geq j$)-tuple between t_k and t_n, we say that t_k and t_n are *tuples consecutive at level j*. Here, "between" means relative to the natural linear ordering of tuples based on their first elements.

Thus, for example, in Fig. 1, the tuples numbered 4 and 8 are consecutive at level 3.

Extension of a Tuple-set

Let T_A be a tuple-set, where $A = \{a_2, a_3, ..., a_i\}$. Then a tuple-set $T_{A'}$, where $A' = \{a_2, a_3, ..., a_i, a_{i+1}\}$ is an *extension* of T_A. A proof that there exists such an extension for each exponent a_{i+1} is given in Lemma 3.0 (see "Lemma 3.0: Statement and Proof" on page 37).

Tuple-sets and Infinite Tuples

Tuples in a tuple-set are oriented vertically in accordance with our convention (see "Tuple-set" on page 7). Each tuple is a prefix of an infinite tuple (see "Tuple" on page 6). Therefore the infinite tuples whose prefixes constitute the finite tuples in a tuple-set, are likewise oriented vertically.

The infinite tuples having prefixes in a tuple-set thus occupy a single, vertical plane P_A that is infinite in the upward direction and to the right.

If T_A is an i-level tuple-set, where $i \geq 2$, then *each tuple-set that is an extension of* T_A *is contained, as a set of prefixes, in the set of infinite tuples whose i-level prefixes constitute the tuples in* T_A. Putting it another way, each tuple-set that is an extension of T_A — each tuple in each such tuple-set — is contained in the single, vertical plane P_A .

Distance Functions on Tuple-sets

Lemma 1.0

(a) Let $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, *be a sequence of exponents, and let* t_k, t_n *be tuples consecutive at level i in* T_A. *Then d(i, i), the distance between* t_k *and* t_n *at level i, is defined to be the absolute value of the difference between the level i elements of* t_k *and* t_n, *that is, it is defined to be* $|t_{k(i)} - t_{n(i)}|$, *and is given by*:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) Let t_k, t_n *be tuples consecutive at level i in* T_A. *Then d(1, i), the distance between* t_k *and* t_n *at level* 1*, is defined to be the absolute value of the difference between the level 1 elements of* t_k *and* t_n*, that is, it is defined to be* $|t_{k(1)} - t_{n(1)}|$*, and is given by:*

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3})\dots(2^{a_i})$$

Thus, in Fig. 1 under "Tuple-set" on page 7, the distance $d(3, 3)$ between $t_{8(3)} = 35$ and $t_{4(3)} = 17$ is $2 \cdot 3^{(3-1)} = 18$. The distance $d(1, 2)$ between $t_{12(1)} = 23$ and $t_{10(1)} = 19$ is $2 \cdot 2^1 = 4$.

Proof: see "Lemma 1.0: Statement and Proof" on page 32.

Remarks About the Distance Functions

(1) Strictly speaking, we should include the sequence A of exponents as arguments of $d(1, i)$, $d(i, i)$, but this notation would be cumbersome and, since typically this sequence is known, unnecessary.

(2) The distance functions make clear that, for each finite sequence of exponents, there exists an infinity of tuples produced by that sequence. (The equivalent of this statement is made in [Wirsching 1998] (p. 48).) The following table shows the distance relationships for $(i - j)$-level elements of tuples consecutive at level $(i - j)$ in an i-level tuple-set, where $0 \leq j \leq (i - 1)$. The distances are easily proved using Lemma 1.0. (An example is given following the table.) We only use the distances at levels 1 and i in this paper.

Table 1: Distances between elements of tuples t_k, t_n, consecutive at level i

Level	Distance between $(i-j)$-level elements of tuples consecutive at level $(i-j)$, where $0 \le j \le (i-1)$
i	$2 \cdot 3^{i-1}$
$i-1$	$2 \cdot 3^{i-2} \cdot 2^{a_i}$
$i-2$	$2 \cdot 3^{i-3} \cdot 2^{a_{i-1}} 2^{a_i}$
$i-3$	$2 \cdot 3^{i-4} \cdot 2^{a_{i-2}} 2^{a_{i-1}} 2^{a_i}$
...	...
2	$2 \cdot 3 \cdot 2^{a_3} \ldots 2^{a_{i-1}} 2^{a_i}$
1	$2 \cdot 2^{a_2} 2^{a_3} \ldots 2^{a_{i-1}} 2^{a_i}$

For example, let x be an element at level $(i-1)$ of an i-level tuple. Then, by the table, the element at level $(i-1)$ in the next i-level tuple (that is, in the next tuple consecutive at level $(i-1)$) =

$(x + 2 \cdot 3^{i-2} \cdot 2^{a_i})$, and so it must be the case that

$$\frac{3(x + 2 \cdot 3^{i-2} \cdot 2^{a_i}) + 1}{2^{a_i}} = \frac{3x + 1}{2^{a_i}} + 2 \cdot 3^{i-1}$$

which, as the reader can check, is indeed the case.

(3) Lemma 1.0 makes clear that no two i-level tuples in an i-level tuple-set have the same last element. In fact, the values of the last elements of i-level tuples in an i-level tuple-set always increase as one proceeds along the sequence of i-level tuples.

We now state the two lemmas that are required for our proof that tuple-sets exist as defined.

Every Possible 2-Level Tuple-set Exists

Lemma 2.0

For each exponent a_2, a tuple-set T_A, where $A = \{a_2\}$, exists.

Proof: See "Lemma 2.0: Statement and Proof" on page 37.

Every Possible Extension of Each *i*-Level Tuple-set Exists
Lemma 3.0
Each i-level tuple-set T_A, where $A = \{a_2, a_3, ..., a_i\}$ and $i \geq 2$, has an extension via each exponent a_{i+1}.

Proof: See "Lemma 3.0: Statement and Proof" on page 37.

Proof That Tuple-sets Exist as Defined
Lemma 4.0
For each exponent sequence $A = \{a_2, a_3, a_4, ..., a_i\}$, where $i \geq 2$, there exists a tuple-set T_A.

Proof: See "Lemma 4.0: Statement and Proof" on page 39.

Lemmas 2.0, 3.0 and 4.0 establish, as part of their proofs, that there are an infinite number of tuples in each tuple-set. A plausible question at this point is: Why should there be? The answer is given in the next section.

Why There Are An Infinite Number of Tuples in Each Tuple-set
Every exponent sequence — that is, every sequence of positive integers — generates an *i*-level tuple-set ("Lemma 4.0: Statement and Proof" on page 39), where $i \geq 2$. The last element (that is, the *i*-level element) of each tuple maps directly to one and only one odd, positive integer via one and only one exponent. Consider the tuple-set T_A generated by the exponent sequence $A = \{a_2, a_3, a_4, ..., a_i\}$ where $i \geq 2$. T_A has an extension for *each* positive integer a_{i+1} ("Lemma 3.0: Statement and Proof" on page 37). But since the last element of each tuple in T_A maps directly to one and only one odd positive integer, and since by Lemma 2.0 (see "Lemma 3.0: Statement and Proof" on page 37) each tuple-set $T_{A'}$, $A' = \{a_0, a_1, a_2, ..., a_i, a_{i+1}\}$, likewise has an extension for each positive integer a_{i+2}, etc., it follows that, for *each* a_i, there exists an *infinity* of tuples in T_A whose last elements directly map to their respective odd, positive integers *via* a_i. In short, the reason there are an infinite number of tuples in each *i*-level tuple-set is that (1) each *i*-level tuple-set has an infinity of extensions, namely, one for each exponent a_{i+1}, but (2) each tuple maps directly to one and only one odd, positive integer via one and only one exponent.

Thus, in each *i*-level tuple-set T_A, where $i \geq 2$, the countable infinity of *i*-level non-counterexample tuples consists of:

an infinity that have an extension via the exponent 1, and
an infinity that have an extension via the exponent 2, and
an infinity that have an extension via the exponent 3, and
...
If counterexamples exist, the same is true for *i*-level counterexample tuples.

On Non-Counterexample and Counterexample Tuples in a Tuple-set

Lemma 5.0

Assume a counterexample exists. Then for all i ≥ 2, each i-level tuple-set contains an infinity of i-level counterexample tuples and an infinity of i-level non-counterexample tuples.

Proof: see “Lemma 5.0: Statement and Proof” on page 39.

Remark

This lemma establishes that there is no way to distinguish counterexamples from non-counterexamples on the basis of the *finite exponent sequences* associated with each. Of course, if a non-trivial cycle exists, then an infinite tuple $<x_1, x_2, ..., x_1, x_2, ..., x_1, x_2, ... >$ exists, and thus the finite tuple $<x_1, x_2, ..., x_1>$ immediately tells us that a counterexample exists. But there is no requirement that a counterexample be the source of a non-trivial cycle. A counterexample can simply give rise to an infinite tuple in which no element recurs, and which has no element = 1.

To repeat: there is no way of telling from a *finite exponent sequence* that it is associated with a counterexample. For example, the sequence $\{a_2, a_3, ..., a_2, a_3, ..., a_2, a_3, ... \}$, in which $\{a_2, a_3, ..., a_2\}$ is repeated, say, a trillion times, does not imply the existence of a counterexample cycle.

We are now at the final stage of our preparation for the proof of the 3*x* + 1 Conjecture. This stage is concerned with the first *i*-level tuple in an *i*-level tuple-set. This tuple is called the *anchor tuple* of the tuple-set.

How the Existence of Counterexamples Changes the Set of All Tuple-sets

Consider the set of all tuple-sets in the two cases that (1) there are no counterexamples and (2) that counterexamples exist. It is natural to say that there are “no differences”, because if *x* is an element of a tuple, then *x* maps to a certain *y* in one iteration of the 3*x* + 1 function, and this *y* is the same whether or not counterexamples exist. If (contrary to fact) counterexamples and only counterexamples were negative numbers, then the set of all tuple-sets if no counterexamnples existed would be different (no negative numbers in tuples) from the set of all sets of tuple-sets if counterexamples existed (negative numbers in some tuples).

As we have pointed out on several occasions in this paper, there is no way of distinguishing counterexamples locally, meaning, by examining a single odd, postive integer, or even by examining the odd, positive integers produced by several iterations of the 3*x* + 1 function (the shortest cycle, if a cycle exists, would be thousands of elements long).

When we consider the structure of the inverse of the 3*x* + 1 function (see “Section 2. Recursive ‘Spiral’s”, in the first file of our paper “The Structure of the 3x + 1 Function: an Introduction” (www.occampress.com)) we see that there definitely *is* a difference in this structure if counterexamples exist as opposed to if counterexamples do not exist. Specifically, if counterexamples do not exist, then there is no infinite set of “spiral”s whose set of elements is disjoint from the set of elements in the infinite set of “spiral”s having base element 1. If counterexamples exist, on the other hand, then there exists at least one infinite set of “spiral”s whose set of elements is disjoint from the set of elements in the infinite set of “spiral”s having base element 1.

For a long time, we did not realize that exactly the same kind of difference holds for the set of all tuple-sets, namely, that if counterexamples do not exist, then the elements of all tuples in all tuple-sets are connected in the sense that for each element in each tuple, we can proceed through extensions of that tuple until we arrive at 1, and then from 1 we can proceed "backwards" through some other tuple until we arrive at any pre-selected element in another tuple.

If counterexamples exist, this is not possible. In that case, we can partition the set of tuples in the set of all tuple-sets into a set of (partial) tuple-sets whose tuples contain only non-counterexamples, and one or more other (partial) tuple-sets whose tuples contain only counterexamples.i

Infinite Exponent Sequences <u>Not</u> Associated With Counterexamples

Lemma 5.5.

Let a be a finite exponent sequence such that if x maps to y via a, then y > x. Then there does not exist a counterexample x such that the infinite tuple <x, ... > is associated with the exponent sequence {a, a, a, ... }.

Proof: See "Lemma 5.5: Statement and Proof" on page 40.

Anchor and Anchor Tuple

Since tuples in a tuple-set are linearly ordered by the natural order of their first elements, in every i-level tuple-set, where $i \geq 2$, there is a unique first i-level tuple, which we call the *anchor tuple* of the tuple-set. The last element, that is, the i-level element, of the anchor tuple we call the *anchor* of the anchor tuple, sometimes referring to it as the *i-level anchor.*

Each anchor tuple element (like the elements of all tuples) is an odd, positive integer that is not a multiple of 3. The element is odd by definition of the $3x + 1$ function, C, and is not a multiple of 3 by "Lemma 10.0: Statement and Proof" on page 41.

Lemma 6.0

Let t be the i-level anchor tuple in an i-level tuple-set, where $i \geq 2$. *Then the last element y of t, that is, the i-level element of t (which is the anchor), is a number* less than $2 \cdot 3^{(i-1)}$.

Proof: see "Lemma 6.0: Statement and Proof" on page 40.

Definition of "Reduced Residue Class" and of "Complete Set of Reduced Residue Classes"

If a residue class mod m is such that each element of the class is relatively prime to m, then we call the class a *reduced residue class mod m*. Thus, for example, the residue class mod 6 whose minimum element is 5 is a reduced residue class mod 6. The set of all reduced residue classes mod m we call a *complete set of reduced residue classes mod m*.

Lemma 7.0

The set of i-level elements of all i-level tuples in an i-level tuple-set is a complete set of reduced residue classes modulo $2 \cdot 3^{(i-1)}$.

Proof: see "Lemma 7.0: Statement and Proof" on page 40.

Mark

Lemma 8.0

For each odd, positive integer x there exists a minimum $i = i_0$ *such that for each* $i \geq i_0$, *x is the first element of the first i-level tuple in some i-level tuple-set, that is, x is the first element of an i-level anchor tuple in some i-level tuple-set. In terms of infinite tuples, this lemma states: if x is an odd, positive integer, then in the infinite tuple* $\bar{t} = \langle x, y, y', \ldots \rangle$, *there exists a minimum level* i_0 *such that:*

$\bar{t}(i_0)$ is the i_0-level anchor tuple in an i_0-level tuple-set;
$\bar{t}(i_0 + 1)$ is the $(i_0 + 1)$-level anchor tuple in an $(i_0 + 1)$-level tuple-set;
$\bar{t}(i_0 + 2)$ is the $(i_0 + 2)$-level anchor tuple in an $(i_0 + 2)$-level tuple-set;
etc.

Proof: see “Lemma 8.0: Statement and Proof” on page 41.

Remark

To describe the infinite sequence of anchor tuples in the lemma, we sometimes say, informally, “Once an anchor tuple, always an anchor tuple”.

We call the level i_0 in Lemma 8.0 the *mark* of the infinite tuple $\bar{t}$. We denote the mark i_0 by *m*. We write $m(\bar{t})$ to denote the mark of $\bar{t}$, and we write $\bar{t}(m)$ to denote the prefix (that is, finite tuple) corresponding to the mark *m*. This prefix is an anchor tuple.

For example, the mark of the infinite tuple $\langle 3, 5, 1, 1, 1, 1, \ldots \rangle$ is at level 2 (namely, at 5) because 5 is the first element of the tuple that is less than $2 \cdot 3^{(i-1)}$ for some $i \geq 2$. Specifically, for $i = 2$, $2 \cdot 3^{(i-1)} = 6$, and $5 < 6$. As another example, consider the infinite tuple $\langle 433, 325, 61, 23, 35, \ldots, 1, 1, 1, 1, \ldots \rangle$. The mark is not at 325 (level 2) because for level 2, $2 \cdot 3^{(i-1)} = 6$ and 325 is not less than 6. The mark is not at 61 (level 3) because for level 3, $2 \cdot 3^{(i-1)} = 18$ and 61 is not less than 18. The mark *is* at 23 (level 4) because for level 4, $2 \cdot 3^{(i-1)} = 54$ and 23 *is* less than 54.

Infinite Tuples, Marks, and Tuple-sets

We here summarize the pertinent facts concerning infinite tuples, marks, and tuple-sets, because it is crucial, for an understanding of our proofs of the 3*x* + 1 Conjecture, that the reader understand these facts and their relationships.

By definition, an *i*-level tuple-set T_A, where $i \geq 2$, *includes* all *i*-level tuples *t* such that $A(t) = A$, that is, such that the exponent sequence associated with *t* is *A*. We emphasize *includes* because, by definition of “tuple-set”, the tuple-set also includes 1-level, 2-level, 3-level, ..., $(i-1)$-level tuples (see “Tuple-set” on page 7). Another way of saying what we have just said regarding *i*-level tuples is: a tuple-set T_A, where $i \geq 2$, includes all prefixes $\bar{t}(i)$ of infinite tuples $\bar{t}$ such that $A(\bar{t}(i)) = A$. Thus by abuse of language we may say that a tuple-set consists of a set of infinite tuples.

At this point it is appropriate that we describe the relationship between *successive* prefixes of an infinite tuple $\bar{t}$ (counterexample or non-counterexample) and the tuple-sets in which the prefixes appear. Let $\overline{t} = <x_1, x_2, x_3, x_4, ... >$ and let $\{a_2, a_3, a_4, a_5, ... \}$ be the associated exponents. That is,

x_1 maps to x_2 in one iteration of the $3x + 1$ function via a_2;
x_2 maps to x_3 via one iteration of the $3x + 1$ function via a_3;
etc.

Then, by definition of *tuple-set*:
in each tuple-set T_A determined by the exponent sequence $A = \{b_2, b_3, b_4, b_5, ... b_i\}$ such that $b_2 \neq a_2$, the tuple $<x_1>$ is an element;
in each tuple-set T_A determined by the exponent sequence $A = \{b_2, b_3, b_4, b_5, ... b_i\}$ such that $b_2 = a_2$, but $b_3 \neq a_3$, the tuple $<x_1, x_2>$ is an element;
in each tuple-set T_A determined by the exponent sequence $A = \{b_2, b_3, b_4, b_5, ... b_i\}$ such that $b_2 = a_2$, $b_3 = a_3$, but $b_4 \neq a_4$ the tuple $<x_1, x_2, x_3>$ is an element;
...
in the one tuple-set T_A determined by the exponent sequence $A = \{b_2, b_3, b_4, b_5, ... b_i\}$ such that $b_2 = a_2$, $b_3 = a_3$, $b_4 = a_4$, ..., $b_i = a_i$ the tuple $<x_1, x_2, x_3, ..., x_i>$ is an element;

Let $\overline{t}$ be an infinite tuple. It has a mark, m. Each prefix $\overline{t}\,(m + j)$ of $\overline{t}$, where $j \geq 0$, is an anchor tuple. But then, by abuse of language, we allow ourselves to say that each prefix $\overline{t}\,(i)$, where $i \geq 2$, is *a prefix of an anchor tuple* (namely, the anchor tuple $\overline{t}\,(m + j)$). Thus each prefix $\overline{t}\,(i)$, where $i \geq 2$, is a prefix of an *infinity o*f anchor tuples.

Each infinite tuple $\overline{t}$ is an independent entity. By this we mean that an infinite tuple $\overline{t}$ is determined solely by its first element. Thus, informally, an infinite tuple does not somehow "acquire" properties depending on the tuple-set in which it has a prefix.

In an i-level tuple-set there is exactly one infinite tuple with a mark that is less than or equal to i , namely, the infinite tuple whose prefix is the anchor tuple. All other infinite tuples having i-level prefixes in the tuple-set must have marks greater than i (otherwise there would be two or more anchor tuples in a tuple-set, which is impossible). It may well be the case, however, that an $(i - j)$-level tuple (prefix), where $1 \leq j \leq (i - 1)$, in the tuple-set has a mark! The following is an example:

The infinite tuple $\overline{t} = <7, 11, 17, 13, 5, 1, 1, 1, ... >$ has its mark at level 3 (namely at 17) because 17 is the first element of the tuple that is less than $2 \cdot 3^{(i-1)}$ for some $i \geq 2$. Here, $i = 3$, so $2 \cdot 3^{(i-1)} = 18$, and $17 < 18$. So $<7, 11, 17> = \overline{t}(3)$ is an anchor tuple: specifically, it is the anchor tuple of the tuple-set T_A, where $A = \{1, 1\}$ (7 maps to 11 via the exponent 1; 11 maps to 17 via the exponent 1). By our rule (see under "Mark" on page 15) expressed informally as "once an anchor tuple, always an anchor tuple", we know that $<7, 11, 17, 13> = \overline{t}(4)$ is also an anchor tuple: specifically, it is the anchor tuple of the 4-level tuple-set $T_{A'}$, where $A' = \{1, 1, 2\}$ (7 maps to 11 via the exponent 1, 11 maps to 17 via the exponent 1, 17 maps to 13 via the exponent 2).

But $<7, 11, 17> = \overline{t}(3)$ is also present in the 4-level tuple-set $T_{A''}$, where $A'' = \{1, 1, 1\}$ The reason is that, since 17 maps to 13 via the exponent 2, not via the exponent 1, the tuple $<7, 11, 17>$ is associated with merely an "approximation", namely $\{1, 1\}$, to the exponent sequence $\{1, 1, 1\}$. But therefore, by definition of "tuple-set" (see under "Tuple-set" on page 7), it belongs in the tuple-set $T_{A''}$.

We conclude our preparation for the proof of the $3x + 1$ Conjecture with the definition of "sufficiently long extension of a tuple" and "sufficiently long extension of an exponent sequence".

"Sufficiently Long" Extensions of Tuples and Exponent Sequences
"Bottom Up" Sufficiently Long Extensions

We begin with two definitions. First we recall that each infinite tuple has a mark m that denotes the smallest prefix of the tuple that is an anchor tuple (see "Mark" on page 15).

Definition of "Sufficiently Long" Extension of a Tuple

Definition: Let $\bar{t}$ be an infinite tuple with mark m. Let $\bar{t}(i)$ be a prefix of $\bar{t}$, where $i < m$. Then there exists an extension $\bar{t}(i + j)$ of $\bar{t}(i)$, where $m = i + j$. We say that $\bar{t}(i + j)$ is a *sufficiently long extension of* $\bar{t}(i)$ *that is an anchor tuple*. (All longer extensions are likewise anchor tuples, by our rule, "once an anchor tuple, always an anchor tuple".)

It follows (trivially) that:

For each tuple (that is, for each prefix of an infinite tuple) there exists a sufficiently long extension of the tuple that is an anchor tuple.

Definition of a "Sufficiently Long" Extension of an Exponent Sequence

Definition: Let $\bar{t}$ be a *non-counterexample* infinite tuple with mark m. Let $\bar{t}(i)$ be a prefix of $\bar{t}$, where $i < m$. Let $A(\bar{t}(i))$ denote the exponent sequence associated with $\bar{t}(i)$. Let the extension $\bar{t}(i + j)$ of $\bar{t}(i)$ be a sufficiently long extension of $\bar{t}(i)$ that is an anchor tuple. Then we say that $A(\bar{t}(i + j))$ *is an extension of* $A(\bar{t}(i))$ *that is sufficiently long to be associated with a non-counterexample anchor tuple*.

An Erroneous Objection to the Definition

Several readers have challenged the definition of a "sufficiently long" extension of an exponent sequence with the following argument. Let us imagine, they say, a "demon" who presents us with an i-level exponent sequence, A, where $i \geq 2$. The demon has before him all the non-counterexample infinite tuples $\bar{t}_{nc}$ having i-level prefixes that are associated with the exponent sequence A. In other words, he has before him all the non-counterexample infinite tuples $\bar{t}_{nc}$ whose prefixes constitute all the i-level non-counterexample tuples in the tuple-set T_A. He now proceeds to concatenate exponents onto A, taking care that, as soon as the resulting exponent sequence equals $A(\bar{t}_{nc} + (m - 1))$ for some infinite non-counterexample tuple $\bar{t}_{nc}$ whose mark is m, the next exponent in his sequence will make the resulting exponent sequence *not* equal to $A(\bar{t}_{nc}(m))$. He repeats this indefinitely. It is clear, then, that his exponent sequence will never be that of a non-counterexample anchor tuple.

The error in this objection is that the demon is not creating a sequence of exponent sequences that are associated with a sequence of extensions of a single tuple. Rather, he is in effect switching tuples in order to create his sequence of exponent sequences.

Complete Sets of Tuples

Definition of a "Complete" Set of Tuples

Let S be a set of i-level tuples, where $i \geq 2$. Then we say that S is *complete* if S is associated with the set of all i-level exponent sequences. Otherwise, we say that S is *incomplete*.

Lemma 8.5

Assume counterexamples exist. Let $\bar{t}_{nc}$, $\bar{t}_c$ be non-counterexample and counterexample infinite tuples, respectively, with marks m_{nc}, m_c respectively.
Then for all levels $i \geq max(m_{nc}, m_c) = i_0$, $A(\bar{t}_{nc}(i)) \neq A(\bar{t}_c(i))$, where max(u, v) denotes the maximum of u, v, and A(t) denotes the exponent sequence associated with the tuple t.

Proof: Assume the contrary. Then for some $i \geq i_0$, $A(\bar{t}_{nc}(i)) = A(\bar{t}_c(i))$, which implies that a tuple-set exists having both a non-counterexample and a counterexample anchor tuple, which is impossible. □

Lemma 8.7

If counterexamples do not exist, then
(a) *For each $i \geq 2$, the set of i-level non-counterexample anchor tuples is complete.*

(b) *Each non-counterexample infinite tuple has a prefix, namely, that determined by its mark, such that that prefix, and all larger prefixes, are elements of* ***complete*** *sets of non-counterexample anchor tuples.*

If counterexamples exist, then

(c) F*or each $i \geq$ some i_0, the set of i-level non-counterexample anchor tuples is incomplete, so that a complete set of i-level non-counterexample tuples must include tuples other than anchor tuples.*
(d) *Each non-counterexample infinite tuple has a prefix, namely, that determined by its mark, such that that prefix, and all larger prefixes, are elements of* ***incomplete*** *sets of non-counterexample anchor tuples.*

Proof

(a) Follows trivially from the fact that if counterexamples do not exist, all tuples in all tuple-sets are non-counterexample tuples.

(b) Follows trivially from the fact that the mark determines the smallest prefix of an infinite tuple that is an anchor tuple.

(c) By "Lemma 8.5" on page 18, if counterexamples exist, then for all $i \geq max(m_{nc}, m_c) = i_0$, there exist i-level exponent sequences with which i-level anchor tuples are not associated. These are the exponent sequences with which i-level *counterexample* anchor tuples are associated. But by "Lemma 5.0" on page 13, each i-level tuple-set, regardless whether the anchor tuple is non-counterexample or counterexample, contains an infinity of non-counterexample tuples and an infinity of counterexample tuples. Thus to obtain a complete set of i-level non-counterexample tuples, it is necessary to include a non-counterexample tuple from each tuple-set having a counterexample anchor tuple.

(d) Follows directly from "Lemma 8.5" on page 18. □

Challenging Questions About Anchor Tuples and Tuple-sets

Regardless of the success of the proof strategies described in this paper, and of the implementations of some of these strategies that are given in the paper, "A Solution to the $3x + 1$ Problem" on the website www.occampress.com, this research will not be completed until the questions described in this section are satisfactorily answered. They lie at the heart of the tantalizing difficulty of discovering valid proofs of the $3x + 1$ Conjecture.

Question 1

One question, "Why are there an infinite number of tuples in each tuple-set?" we believe has been satifactorily answered in the section "Why There Are An Infinite Number of Tuples in Each Tuple-set" on page 12.

Question 2

Another question arises from an error in one of our early attempts at a proof of the Conjecture. We had made the following argument: if counterexamples exist, then beginning at some level $i_0 \geq 2$, there must be both non-counterexample and counterexample anchor tuples. But since for all $i \geq 2$ the set of all i-level anchor tuples must be associated with the set of all i-level exponent sequences, this means that some i-level exponent sequences, where $i \geq i_0$, will not be associated with non-counterexample anchor tuples (these exponent sequences will be "missing" from the set of exponent sequences associated with i-level non-counterexample anchor tuples), and similarly for counterexample anchor tuples. Furthermore this fact holds for all levels greater than i . But then, we argued, this contradicts "Lemma 5.0" on page 13, hence we have our proof.

Readers pointed out that Lemma 5.0 states that, if counterexamples exist, each tuple-*set* contains a countable infinity of non-counterexample anchor tuples and a countable infinity of counterexample anchor tuples, so the "missing" exponent sequences are not really missing. An infinity of non-counterexample tuples are associated with them, and similarly for the "missing" exponent sequences for counterexample tuples.

So our question is: "What is the difference between the sequence of exponent sequences associated with the sequence of extensions of an anchor tuple, and the sequence of exponent sequences associated with other tuples in the corresponding sequence of tuple-set extensions?"

One answer is the following: the sequence of exponent sequences associated with the sequence of extensions of an anchor tuple are all associated with extensions of *one* tuple, namely, the anchor tuple. But the sequence of exponent sequences associated with other tuples in the corresponding sequence of tuple-set extensions are *not* all associated with extensions of one tuple. That is, in order for the tuples in a sequence of tuple-set extensions always to be associated with the sequence of anchor tuple extensions, it is necessary that some tuples "fall away" and that the remaining ones have the required extensions. In some of our papers, we refer to this phenomenon as the "pushing away" phenomenon, because tuples whose exponent sequence matches that of the anchor tuple, are always farther and farther away (as measured by the difference between first elements) from the anchor tuple.

Thus, an arbitrarily long exponent sequence can only be associated with the arbitrarily long extension of *one* anchor tuple, not with more than one tuple (anchor or non-anchor).

Question 3

In Question 2, we pointed out that, if counterexamples exist, then for all levels greater than or equal to some minimum level i_0, there will be both non-counterexample and counterexample anchor tuples. We would like to get a clearer understanding of the implications of this fact.

We begin with the case that counterexamples do not exist. We ask (and this is Question 3), "What would happen if we removed just one anchor tuple (necessarily a non-counterexample anchor tuple) from the set of all tuple-sets?" (We know from our discussion in Question 2 that at least one non-counterexample anchor tuple (in fact an infinity) would be removed from the set of all tuple-sets if counterexamples existed.)

To remove one non-counterexample anchor tuple is to remove one non-counterexample infinite tuple $\bar{t}_{nc}$. But since each element of each infinite tuple except possibly the first element is a range element, then by "Lemma 13.0: Statement and Proof" on page 43 each element is mapped to by an infinity of odd, positive integers, and so on, recursively. And indeed, as the reader can confirm by checking Fig. 4 in "Section 2. Recursive 'Spiral'" in the first file of our paper, "The Structure of the 3*x* + 1 Function: An Introduction" on the web site www.occampress.com, it appears that if we remove just one non-counterexample infinite tuple, and all tuples having a last element that is a range element in $\bar{t}_{nc}$, and all tuples having a last element that is a range element in each of these tuples, and ..., *we remove all non-counterexample tuples,* because one of the elements in each non-counterexample infinite tuple is 1.

If the reader argues that we are not justified in removing all tuples having a last element that is a range element in $\bar{t}_{nc}$, then we must ask what becomes of these tuples if $\bar{t}_{nc}$ is replaced by a counterexample infinite tuple?

Of course, if, in fact, the removal of just one non-counterexample anchor tuple would constitute the removal of all non-counterexample anchor tuples, then it would seem that we have a proof of the 3*x* + 1 Conjecture, since the removal of all those tuples would contradict "Lemma 5.0" on page 13.

Another way of answering our question is this: if counterexamples exist, there are nevertheless non-counterexample infinite tuples having prefixes that are anchor tuples. Each non-counterexample ultimately contains 1. Therefore the set of all odd, positive integers that map to 1 must be present, eventually, as anchors. But this is precisely the case if no counterexamples exist. In short, it does not seem possible for there to be counterexample anchor tuples.

Question 4

What might be called the Big Question is the following: "Can a graphical relationship be shown between tuple-sets and recursive 'spiral's?" (The latter is the structure of the function's inverse, and is described in "Section 2. Recursive 'Spiral's" in the first file of the paper, "The Structure of the 3x + 1 Function: An Introduction", on the website www.occampress.com.) So far, the effort to answer the question in the positive has proved tantalizingly difficult, despite the fact that an infinite set of such "spiral"s exists at each element of each tuple whose level is greater than 1, the tuple element being the base element of the infinite set. In particular, we have been unable to find a relationship between the location of the *i*'th element of a tuple-set (where the element is known to be a non-counterexample) and the location of the element in the infinite set whose base element is 1. We will welcome hearing from readers.

Recursive "Spiral"s

Recursive "Spiral"s are a graphical description of the inverse of the $3x + 1$ function. They are defined and described in "Section 2. Recursive 'Spiral's" in the first file of the paper, "The Structure of the 3x + 1 Function: An Introduction", on the website www.occampress.com. The proof of the following Lemma will provide an introduction to them.

Motivation for Lemma 8.8

The odd, positive integer 13 maps to 1, as the reader can verify. We ask: if the $3x + 1$ Conjecture were proved false tomorrow, would 13 map to 1 thereafter? We reply yes. Let y be any odd, positve integer that is known to map to 1. We ask: if the $3x + 1$ Conjecture were proved false tomorrow, would y map to 1 thereafter? Again we reply yes. So it seem plausible that exactly one set J of odd, positive integers maps to 1, regardless whether counterexamples exist or not. This is certainly a counter-intuitive statement, but not, we believe, an invalid one. It simply means that the $3x + 1$ Conjecture can be expressed as: Are there any odd, positive integers besides those that map to 1? If the answer is yes, then counterexamples exist. If the answer is no, then counterexamples do not exist.

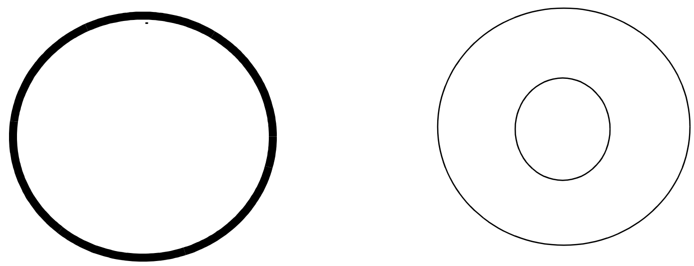
We can express in a similar way the question whether there are any odd perfect numbers. Let P denote the set of perfect numbers. (These are numbers that are equal to the sum of their proper factors. Thus, for example, $6 = 3 + 2 + 1$ is a perfect number, as is $28 = 14 + 7 + 4 + 2 + 1$.) Let P_E denote the subset of P consisting of even perfect numbers, and let P_O denote the subset of P consisting of odd perfect numbers. Then the question, Do odd, perfect numbers exist? (the answer is not yet known) can be expressed as, Are there any perfect numbers besides those that are in P_E? If the answer is yes, then odd, perfect numbers exist. If the answer is no, then odd, perfect numbers do not exist. In either case, observe that the following statement is true: There is exactly one set of even, perfect numbers, regardless whether odd, perfect numbers exist or not.

(The equivalent of the $3x + 1$ function in the perfect number case is a function f that, for the positive integer n, returns "yes" if n is a perfect number, "no" otherwise. It is sufficient if the program that implements f does so by simply determining the proper factors of n, then adding them and determining if the result is n. Clearly, f cannot be Euler's well-known formula for even perfect numbers, $2^{k-1}(2^k - 1)$, where $2^k - 1$ is a Mersenne prime, because the formula returns only even perfect numbers.)

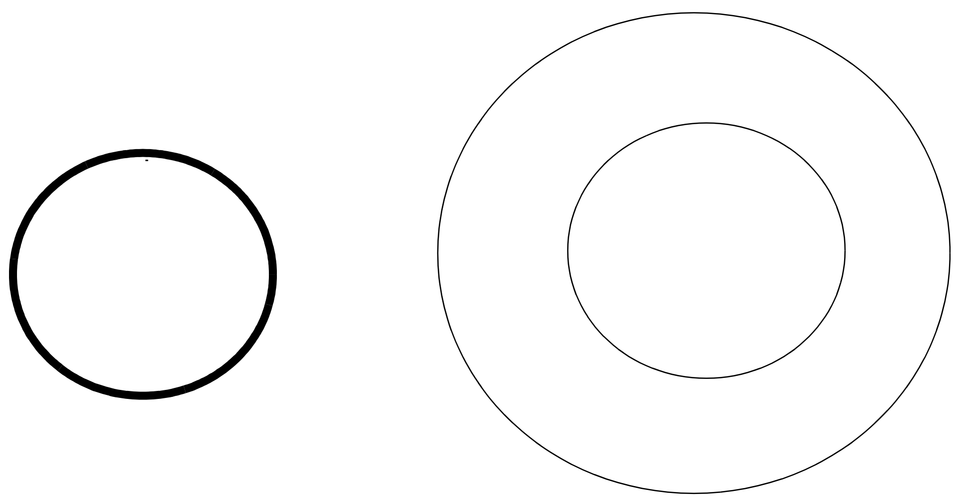
There are two ways to regard the set of odd, positive integers that map to 1. The most natural (and, as we will show, the wrong way) is to regard the set as one of two, different sets, as represented by the following diagram. If counterexamples do not exist, then the set of odd, positive integers that map to 1 is the same as the set of odd, positive integers. This is represented by the circle on the left, the bold-face indicating that the set of odd, positive integers that map to 1 is superimposed on the set of odd, positive integers.

If counterexamples do not exist, then the set of odd, positive integers that map to 1 is a proper subset of the set of odd, positive integers. This is represented by the circles on the right, the smaller circle representing the set of odd, positive integers that map to 1.

Note! The diagram is intended to be a *conceptual* aid. Our convention is that if a circle has a radius smaller than another circle, then the set represented by the smaller circle is a proper subset of the set represented by the larger circle. Obviously, the set of odd, positive integers, and the set of odd, positive integers that map to 1 (whether or not counterexamples exist) are all of the same "size", since they are all countably infinite sets.



The second way to regard the set of odd, positive integers that map to 1 is much less natural, and is in fact counter-intuitive. But it is the correct way, as Lemma 8.8 shows. It is represented by the following diagram



Here, the circle on the left, and the smaller circle on the right, represent the set of odd, positive integers that map to 1. Since they are both of the same diameter, they represent the same set. If counterexamples do not exist, then the set of odd, positive integers that map to 1 is the set of odd, positive integers, the bold-face indicating that the sets are the same. If counterexamples exist, then the set of odd, positive integers must be a superset of the set of odd, positive integers that map to 1, and this is indicated by the pair of circles on the right.

We can express this second (and, we believe, correct) way of regarding the set of odd, positive integers that map to 1, in set-theoretic language as follows.

(1) There is exactly one set, J, of odd, positive integers that map to 1, regardless whether counterexamples exist or not.

(2) Let S_1 denote the singleton set containing the set of all odd, positive integers. Let S_2 denote the set containing all proper subsets of the odd, positive integers. Then if counterexamples do not exist, $J \in S_1$; if counterexamples exist, then $J \in S_2$.

Remark

Some readers regard Lemma 8.8 as false. One reason seems to be that they confuse the statement of this Lemma with the statement, "The *range* of the $3x + 1$ function is the same regardless whether counterexamples exist or not.." Now this statement is clearly false, because if counterexamples do not exist, then the range of the $3x + 1$ function is $\{1\}$. If they do exist, then the range is a larger set that contains 1.

Lemma 8.8

Exactly one set J of odd, positive integers maps to 1, regardless whether counterexamples exist or not. In other words:

If counterexamples exist, then the set of odd, positive integers that map to 1 is J.
If counterexamples do not exist, then the set of odd, positive integers that map to 1 is J.

Proof:

The set J =

$\{$odd, positive integers $y \mid y$ maps to 1 in *one* iteration of the $3x + 1$ function$\} \cup$
$\{$odd, positive integers $y \mid y$ maps to 1 in *two* iterations of the $3x + 1$ function$\} \cup$
$\{$odd, positive integers $y \mid y$ maps to 1 in *three* iterations of the $3x + 1$ function$\} \cup$
...

The set of odd, positive integers that map to 1 in one iteration of the $3x + 1$ function is $\{1, 5, 21, 85, 341, ...\}$. This set is called a "spiral" in "Section 2. Recursive 'Spiral's" in the first file of the paper "The Structure of the $3x + 1$ Function: An Introduction" on the web site www.occampress.com. In that Section it is shown that:

If x is an element of the "spiral", then $4x + 1$ is the next element;
The "spiral" contains a countable infinity of multiples of 3. These cannot be range elements of the $3x + 1$ function (by "Lemma 10.0: Statement and Proof" on page 41), that is, cannot be mapped to;
The "spiral" also contains a countable infinity of range elements of the function: each in turn is mapped to by another "spiral", which yields, recursively, the set of odd, positive integers that map to 1 in two, three, four, ... iterations.

It is therefore clear that no odd, positive integer can be added to or removed from a "spiral". Hence the set J is unique, regardless whether counterexamples exist or not. .□

Remark 1

The set J is an infinite set of recursive "spiral"s whose base element is 1. These infinite sets are defined in "Section 2. Recursive 'Spiral's" in the first file of the paper "The Structure of the $3x + 1$ Function: An Introduction" on the web site www.occampress.com. The following is a diagram of part of J :

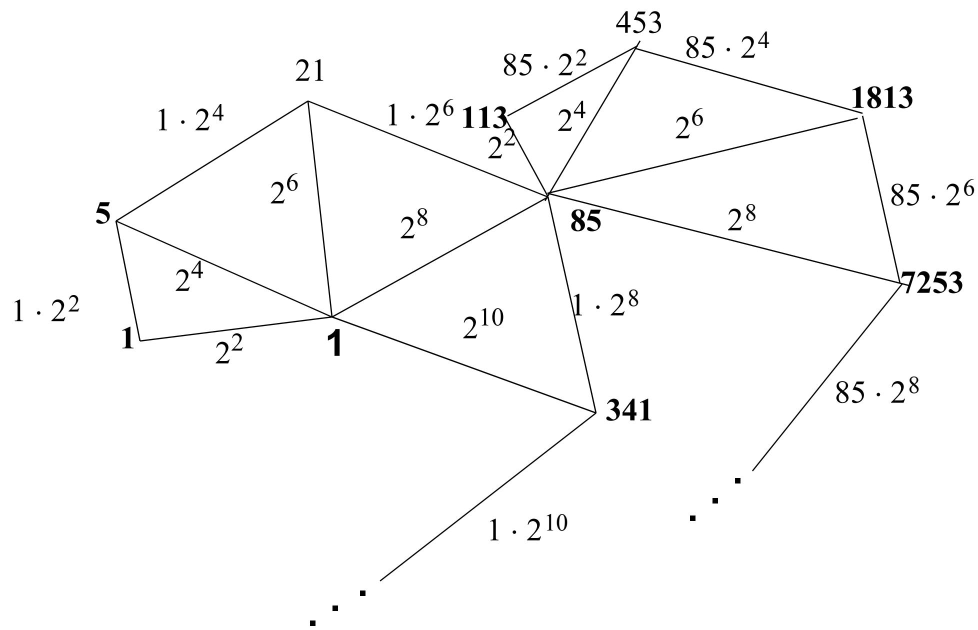


Fig. 4. Recursive "spirals" structure of odd, positive integers that map to 1.
Bold-faced numbers are range elements (21 and 453 are multiples of 3, hence not range elements). Partial "spirals" surrounding the base elements 1 and 85 are shown. The line connecting 1813 to 85 is marked with a 2^6 because $(3 \bullet 1813 + 1)/ 2^6 = 85$. The line connecting 453 to 1813 is marked $85 \bullet 2^4$ because $453 + 85 \bullet 2^4 = 1813$. The exponents of 2 are not even in all "spiral"s, of course. For example, the "spiral" of numbers (not shown) mapping to 341 has odd exponents.

It is easily shown that
{odd, positive integers $y \mid y$ maps to 1 in *one* iteration of the $3x + 1$ function} =
{1, 5, 21, 85, 341, ... }.
This set is a recursive "spiral". Lemma 11.0 in the "...Introduction" paper, states that if y is an element of a "spiral", the next element is $4y + 1$.

It is also easily shown that each recursive "spiral" contains an infinity of range elements and an infinity of non-range-elements (multiples-of-3). **End of Remark**

We see, therefore, that the $3x + 1$ Problem can be expressed as follows: a set J of odd, positive integers maps to 1, regardless whether counterexamples exist or not. Obviously, the set of odd, positive integers is the same, regardless whether counterexamples exist or not. So the question is: Are there any other odd, positive integers (namely, counterexamples) in the set of odd, positive integers besides those that map to 1? This is certainly a counter-intuitive expression of the Problem. Initially, at least, it is natural for us to assume that, if counterexamples exist, then some of the odd, positive integers that map to 1 if counterexamples do not exist, "become" counterexamples if counterexamples exist. But that is not correct.

Strategies to Prove the 3*x* + 1 Conjecture
Strategies Based on Tuple-sets
Strategies Based on Idea There is "Not Enough Room" for Counterexamples
Motivation for Strategy

Informally, our strategy is to show that, if counterexamples exist, there is (informally) not enough "room" in the set of all anchor tuples for all the non-counterexample anchor tuples *and* for all the counterexample anchor tuples that are required by "Lemma 5.0: Statement and Proof" on page 39. The reason we use the phrase "not enough 'room'" is as follows.

If counterexamples do not exist, then every anchor tuple is a non-counterexample tuple. These anchor tuples are associated with the set of all finite exponent sequences. Furthermore, there are no redundancies in the set of all tuple-sets: in particular, there is exactly one anchor tuple in each tuple-set. Now if counterexamples exist, there must somehow be both non-counterexample *and* counterexample anchor tuples. And yet, as in the no counterexamples case, the set of all non-counterexample anchor tuples must be associated with the set of all finite exponent sequences. But now, in addition, the same must be true of counterexample anchor tuples, of where there are an infinity. It doesn't seem possible to us that such a state of affairs can exist — there doesn't seem to be enough "room" for all those counterexample anchor tuples, given that, if counterexamples exist, there still is only one anchor tuple in each tuple-set.

Another argument for our strategy is the following: no new tuples are added to any tuple-set if counterexamples exist: regardless whether counterexamples exist the set of first elements of all tuples in a given *i*-level tuple-set T_A is the same.

The proof that shows most clearly that there is "not enough room" for counterexample tuples is given under "Proof of the 3*x* + 1 Conjecture" in the paper "A Solution to the 3*x* + 1 Problem" on the web site www.occampress.com.

We believe our strategy is supported by the following facts:

(1) *Whether or not counterexamples exist, the structure of all tuple-sets remains unchanged. There are no "redundancies" in the set of all tuple-sets.* That is:

(1.1) Whether or not counterexamples exist, for each finite exponent sequence A, there exists exactly one tuple-set T_A. The distance functions between elements of tuples in a tuple-set ("Lemma 1.0" on page 10) remain unchanged whether or not counterexamples exist.

(1.2) Each infinite tuple, whether non-counterexample or counterexample, has a mark denoting the lowest level at which a prefix of the infinite tuple is an anchor tuple. All higher-level prefixes are likewise anchor tuples.

(1.3) In each tuple-set, there exists exactly one anchor tuple, regardless whether or not counterexamples exist. The anchor tuple (like all tuples) is either a non-counterexample tuple or a counterexample tuple, but not both. Another way of saying this, and one that we will often use, is the following:

Let $\bar{t}_{nc}$ be a non-counterexample infinite tuple with mark m_{nc}, and let $\bar{t}_c$ be a counterexample infinite tuple with mark m_c. Let $A(t)$ denote the exponent sequence associated with tuple (prefix) t. Then if, for some level i it is the case that m_{nc} and m_c are both less than or equal to i, it cannot be the case that $A(\bar{t}_{nc}(i)) = \mathrm{A}(\bar{t}_c(i))$, because that would mean that the tuple-set T_A, where $A = A(\bar{t}_{nc}(i)) = \mathrm{A}(\bar{t}_c(i))$ has two anchor tuples, which is impossible.

(2) *In each tuple-set, there is a countable infinity of non-counterexample tuples. If counterexamples exist, there is also a countable infinity of counterexample tuples* (“Lemma 5.0” on page 13).

(But clearly, the countable infinity of non-counterexample tuples is not the same in the case that counterexamples exist and in the case that they do not exist.)

(3) *No two infinite tuples are associated with the same infinite exponent sequence*, regardless whether the tuples are both non-counterexample, both counterexample, or one is non-counterexample and the other is counterexample. *Proof*: two identical infinite sequences and the fact of marks would imply that for an infinity of consecutive levels, namely all those greater than the maximum of the marks for a pair of non-counterexample infinite tuples, there were two anchor tuples in the same tuple-set, a contradiction. □

(4) *If counterexamples exist, then beginning at some level $i_0 \geq 2$, some i_0-level tuple-sets have non-counterexample anchor tuples, others have counterexample anchor tuples.* This means the following: let $A(t)$ denote the exponent sequence associated with tuple (prefix) t. Let $\{A(\bar{t}_{nc}(i_0))\}$ denote the set of exponent sequences associated with i_0-level prefixes of non-counterexample infinite tuples such that these prefixes are i_0-level anchor tuples. And similarly for $\{A(\bar{t}_c(i_0))\}$. Then $\{A(\bar{t}_{nc}(i_0))\} \cap \{A(\bar{t}_c(i_0))\} = \varnothing$, because otherwise there would be a non-counterexample anchor tuple and a counterexample anchor tuple in the same tuple-set. We say that the set of i_0-level non-counterexample anchor tuples is *incomplete*, by which we mean that the set of i_0-level exponent sequences associated with the set of i_0-level non-counterexample anchor tuples is not the set of all i_0-level exponent sequences. And similarly for the set of i_0-level counterexample anchor tuples. But it is the case for all levels $i_0 + j$ that $\{A(\bar{t}_{nc}(i_0))\} \cup \{A(\bar{t}_c(i_0))\}$ equals the set of all $(i_0 + j)$-level exponent sequences, where $j \geq 0$.

Some Possible Implementations of the Strategy

In previous versions of this and our other $3x + 1$ papers, we have sometimes called the implementations “approaches”. We will continue to add implementations to this section as we complete their writing and checking.

Final version(s) of what we regard as the best implementations can be found in the paper, “A Solution to the $3x + 1$ Problem” on the web site www.occampress.com.

Among the possible implementations are the following:

Show That There Are No Finite Marks In Counterexample Infinite Tuples, An Impossibility

That is, show that, by a reiterative argument, that counterexample marks are “pushed up” without limit. This is equivalent to showing that counterexample tuples are always tuples in tuple-sets having non-counterexample anchor tuples. But if no counterexample is an element of a counterexample anchor tuple, then no counterexample is less than $2 \cdot 3^{(i-1)}$ for any i. Hence there are no counterexamples.

This is a particularly tantalizing implementation. We know that, by "Lemma 5.0" on page 13, and by the section, "Infinite Tuples, Marks, and Tuple-sets" on page 15, that for each i-level exponent sequence A, where $i \geq 2$, the set of i-level prefixes of all non-counterexample infinite tuples is complete. Furthermore, each non-counterexample infinite tuple is an infinity of anchor tuples.

Let $\bar{t}_c$ be a counterexample infinite tuple. It has a mark m_c. But m_c must be greater than $m_{nc} + j$ for each $j \geq 0$, if $A(\bar{t}_{nc}(m_{nc} + j) = A(\bar{t}_c(m_{nc} + j)$. Here $A(\bar{t}(j))$ denotes the exponent sequence associated with the prefix $\bar{t}(j)$ of an infinite tuple $\bar{t}$, and m_{nc} is the mark of $\bar{t}_{nc}$. So m_c is pushed up in all these cases. Does the fact that the set of i-level prefixes of all non-counterexample infinite tuples is complete, allow us to conclude that the mark of $\bar{t}_c$ is pushed up without limit? A Yes answer is given in "First Proof" in the paper, "A Solution to the 3x + 1 Problem" on the web site www.occampress.com.

Show That Non-counterexample and Counterexample Infinite Tuples Have Same Exponent Sequences, An Impossibility

The impossibility is explained in (3) in the previous section.

Show That The Existence Of Counterexamples Implies That No Prefix Of A Non-counterexample Infinite Tuple Is Associated With A Certain Finite Exponent Sequence

This is a contradiction to "Lemma 5.0" on page 13.

Show That a Certain "Completeness" Property of Infinite Tuples Makes Counterexamples Impossible

Let $\bar{t}_c$ be a counterexample infinite tuple, and let $\bar{t}_{nc}$ be a non-counterexample infinite tuple. *Definition*: a set of j-level prefixes of infinite tuples is *complete* if the set is associated with the set of all j-level exponent sequences. Then by "Lemma 5.0" on page 13 we know that:

(1)

The set $\{\bar{t}_c(2)\}$ of all 2-level prefixes of all infinite tuples in $\{\bar{t}_c\}$ is complete.
The set $\{\bar{t}_c(3)\}$ of all 3-level prefixes of all infinite tuples in $\{\bar{t}_c\}$ is complete.
The set $\{\bar{t}_c(4)\}$ of all 4-level prefixes of all infinite tuples in $\{\bar{t}_c\}$ is complete.
...

(2)

The set $\{\bar{t}_{nc}(2)\}$ of all 2-level prefixes of all infinite tuples in $\{\bar{t}_{nc}\}$ is complete.
The set $\{\bar{t}_{nc}(3)\}$ of all 3-level prefixes of all infinite tuples in $\{\bar{t}_{nc}\}$ is complete.
The set $\{\bar{t}_{nc}(4)\}$ of all 4-level prefixes of all infinite tuples in $\{\bar{t}_{nc}\}$ is complete.
...

We emphasize that the statements in (1) and (2) concern *prefixes* of infinite tuples. A prefix of an infinite tuple is not necessarily an anchor tuple, although it is a prefix of an infinity of anchor tuples.

The best implementation of a strategy based on (1) and (2) that we have been able to discover is set forth in “First Proof” in “A Solution to the 3x + 1 Problem” on occampress.com. However, we offer the following thoughts, which might lead to other proofs.

Recall that if t is a prefix of an infinite tuple (that is, if t is a finite tuple), then we denote the exponent sequence associated with t by $A(t)$. We can now make the following statements:

Let $\bar{t}_{nc}$ be a fixed non-counterexample infinite tuple with mark m_{nc}. We now consider all pairs $< \bar{t}_{nc}, \bar{t}_c >$, where $\bar{t}_c$ is any counterexample infinite tuple. The following statements hold:

If $A(\bar{t}_{nc}(2)) = A(\bar{t}_c(2))$, and $m_{nc} > 2$, and $A(\bar{t}_{nc}(3)) \neq A(\bar{t}_c(3)$, then m_c can have any value.
If $A(\bar{t}_{nc}(3)) = A(\bar{t}_c(3))$, and $m_{nc} > 3$, and $A(\bar{t}_{nc}(4)) \neq A(\bar{t}_c(4)$, then m_c can have any value.
If $A(\bar{t}_{nc}(4)) = A(\bar{t}_c(4))$, and $m_{nc} > 4$, and $A(\bar{t}_{nc}(5)) \neq A(\bar{t}_c(5)$, then m_c can have any value.
...
If $A(\bar{t}_{nc}(m_{nc})) = A(\bar{t}_c(m_{nc}))$, then m_c must be $> m_{nc}$.
If $A(\bar{t}_{nc}(m_{nc} + 1)) = A(\bar{t}_c(m_{nc} + 1))$, then m_c must be $> m_{nc} + 1$.
...

A corresponding set of statements holds if we fix $\bar{t}_c$ and then consider all pairs $< \bar{t}_c, \bar{t}_{nc} >$, where where $\bar{t}_{nc}$ is any counterexample infinite tuple.

Does this give us the basis for a contradiction?

Another approach based on (1) and (2) is the following: raise the level i, beginning at $i = 2$, through successive levels 2, 3, 4, ... and consider the properties of $\{\bar{t}_c(i)\}$ and of $\{\bar{t}_{nc}(i)\}$, keeping in mind:

(3) that each of these sets is an (infinite) set of prefixes of anchor tuples. In other words, each of these sets is an (infinite) set of prefixes of first tuples in tuple-sets, and

(4) that each of these sets is complete, and

(5) that each i-level tuple-set has exactly one first i-level tuple (the anchor tuple), and

(6) that beyond a minimum level i_0 there are both non-counterexample and counterexample anchor tuples, and that the set of each of these is incomplete (otherwise, there would be two anchor tuples in some tuple-set).

One conclusion we can draw from these facts can be expressed informally as: each non-counterexample infinite tuple $\bar{t}_{nc}$ is eventually an element of an incomplete set, and similarly for each counterexample infinite tuple $\bar{t}_c$. Formally: for each pair $< \bar{t}_{nc}, \bar{t}_c >$ there exists a level which is equal to the maximum of the marks of $\bar{t}_{nc}$, $\bar{t}_c$ such that, for all greater levels i, $A(\bar{t}_{nc}(i)) \neq A(\bar{t}_c(i))$.

However, this fact does not contradict (1) or (2), because as i increases, there is always, by “Lemma 5.0” on page 13, a residue of complete counterexample prefixes and a residue of complete non-counterexample prefixes.

It is worth investigating where “Lemma 18.0: Statement and Proof” on page 47 can give us a contradiction despite this fact. That lemma states that, for each range element y (for example, the range element 1), and for each exponent sequence A, there exists an x that maps to y via A possibly followed by a single buffer exponent.

Begin by Considering the Set of All Tuple-sets if Counterexamples Do Not Exist, and Show That It is Impossible for Counterexamples to Exist

Whether or not counterexamples exist, certain infinite tuples remain the same: for example, the tuple $\bar{t}$ = <11, 17, 13, 5, 1, 1, 1, ... > In fact, whether or not counterexamples exist, the set of odd, positive integers that directly or indirectly map to 1 is the same (the laws of arithmetic are not subject to the truth or falsity of the 3*x* + 1 Conjecture). But then each 1 in a non-counterexample infinite tuple such as $\bar{t}$, is mapped to, directly or indirectly, by the same set of odd, positive integers. This is, of course, true in the case that there are no counterexamples, and in that case all anchor tuples are non-counterexample tuples. But it must then also be true in the case that there are counterexamples, and thus, there is no "room" for counterexample infinite tuples.

Implementations That Compare the Case That Counterxamples Exist and the Case That Counterexamples Do Not Exist

An example of this type of implementation is to show that the contents of all tuples in all tuple-sets are unchanged by the existence of counterexamples, which is clearly a contradiction. A formal implementation of this strategy is "Second Proof" in the paper, "A Solution to the 3*x* + 1 Problem" on the web site www.occampress.com.

Objections to the comparison of the two cases have fallen into several categories. We give these now, along with our replies to these objections.

Replies to Objections to These Implementations

Objection (1) The word "counterexample" is not legitimate in a lemma or in a proof connected with the 3*x* + 1 Problem

Only one reader has made this objection. The reader asserted that one can only show directly, without proof by contradiction, that each odd, positive integers maps to 1, or does not map to 1.

Reply: We believe that most researchers working on the 3*x* + 1 Problem would regard this objection as extreme.

Objection (2) The phrase "If counterexamples exist" can only be used in a proof by contradiction

Reply: No reader that we know of has objected to "Lemma 5.0" on page 13, which begins with a sentence that is the equivalent of "If counterexamples exist" , namely, "Assume a counterexample exists." No reader that we know of has objected to "Lemma 8.7" on page 18 which uses the phrase explicitly.

Objection (3): The phrases "If counterexamples exist..." and "If counterexamples do not exist..." cannot legitimately be used together; therefore any proof that contains these two phrases is fallacious.

Reply (1): What does "used together" mean? In the same sentence? In the same paragraph? In successive paragraphs? In the same paper? In different papers by the same author?

Reply (2): These two phrases are routinely used, without objection, in discourse concerning the 3*x* + 1 Problem — for example, in statements along the lines: "If counterexamples exist, then the range of the $3x + 1$ function is a proper subset of the odd, positive integers, whereas if counterexamples do not exist, the range is the entire set of odd, positve integers."

Reply (3): Our "Lemma 8.7" on page 18 uses the two phrases in its statement, and thus far we know of no objections from readers.

Reply (4): Several lemmas in this paper begin with the phrase "If counterexamples exist, then ..." and have an implicit complement, "If counterexamples do not exist, then ..." An example of such a lemma is "Lemma 5.0" on page 13.

Objection (4): The phrase "Whether or not counterexamples exist..." is meaningless or at best ambiguous; therefore any proof that contains this phrase is fallacious.

Reply: This phrase is equivalent to the two phrases in the previous Objection. We can only point to numerous examples in which the phrase is neither meaningless nor ambiguous. Among them:

"Whether or not odd perfect numbers exist, 6 and 28 are even perfect numbers."
"Whether or not the Riemann Conjecture is true, Fermat's Last Theorem is true."
"Whether or not counterexamples to the $3x + 1$ Conjecture exist, 13 maps to 1."

Objection (5): The proof falsely proves that counterexamples do not exist in the 3x – 1 Problem, where counterexamples are known to exist.

Reply: see "Appendix B — $3x + C$ Functions" in the paper, "A Solution to the $3x + 1$ Problem" on the web site www.occampress.com.

Remark

We believe that Objections (3) and (4) are rooted in the false belief that if two contradictory cases are compared, this implies the simultaneous existence of both cases.

We will welcome further comments from readers..

Strategies Based On Topology

These are described in the paper, "The Structure of the 3x + 1 Function: An Introduction", www.occampress.com, in the section "Strategy of Using a Topology Defined on Tuples or Tuple-sets".

Strategies Based on Recursive "Spiral"s

One of these utilizes the fact that whether or not counterexamples exist, the set of odd, positive integers that directly or indirectly map to 1 is the same (the laws of arithmetic are not subject to the truth or falsity of the $3x + 1$ Conjecture). See "Recursive "Spiral"s" on page 21.

Others of these strategies are discussed in "Section 2. Recursive 'Spiral's" in the paper, "The Structure of the $3x + 1$ Function: An Introduction" on the web site www.occampress.com.

References

[1] Jeff Lagarias, The $3x + 1$ Problem and Its Generalizations, *American Mathematical Monthly*, **93** (1985), 3-23.

[2] Günther J. Wirsching, *The Dynamical System Generated by the 3n + 1 Function*, Springer-Verlag, Berlin, Germany, 1998.

Appendix A — Statement and Proof of Each Lemma

Lemma 1.0: Statement and Proof

(a) Let $A = \{a_2, a_3, ..., a_i\}$, *where* $i \geq 2$, *be a sequence of exponents, and let* t_k, t_n *be tuples consecutive at level i in* T_A. *Then d(i, i), the distance between* t_k *and* t_n *at level i, is defined to be the absolute value of the difference between the level i elements of* t_k *and* t_n, *that is, is defined to be* $|t_{k(i)} - t_{n(i)}|$, *and is given by*:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) Let t_k, t_n *be tuples consecutive at level i in* T_A. *Then d(1, i), the distance between* t_k *and* t_n *at level* 1, *is defined to be the absolute value of the difference between the level 1 elements of* t_k *and* t_n, *that is, is defined to be* $|t_{k(1)} - t_{n(1)}|$, *and is given by:*

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3})\dots(2^{a_i})$$

Thus, in Fig. 1 in the section "Tuple-set" on page 7, the distance $d(3, 3)$ between $t_{8(3)} = 35$ and $t_{4(3)} = 17$ is $2 \cdot 3^{(3-1)} = 18$. The distance $d(1, 2)$ between $t_{12(1)} = 23$ and $t_{10(1)} = 19$ is $2 \cdot 2^1 = 4$.

Proof:

The proof is by induction.

Proof of Basis Step for Parts (a) and (b) of Lemma 1.0:

Let t_r and t_s be the first and second 2-level tuples, in the standard linear ordering of tuples based on their first elements, that are consecutive at level $i = 2$ in T_A. (See Fig. 2 (1).)

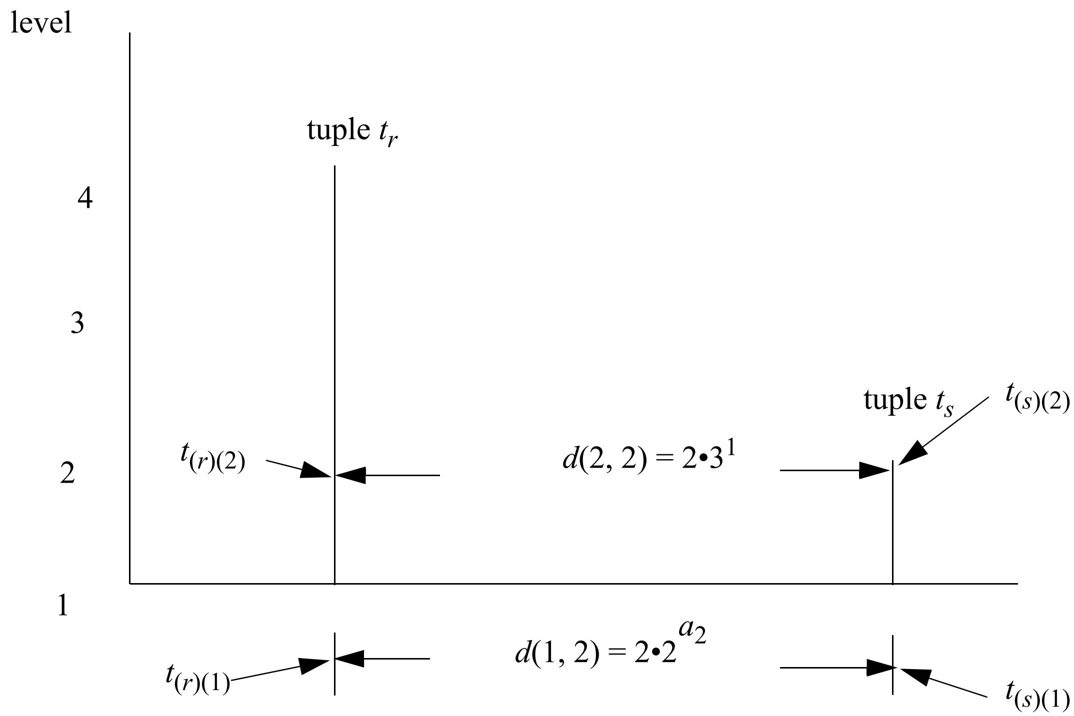


Fig. 2 (1). Illustration for proof of Basis Step of Lemma 1.0.

Then we have:

$$\frac{3t_{(r)(1)} + 1}{2^{a_2}} = t_{(r)(2)} \tag{1.1}$$

and since, by definition of $d(1, 2)$,

$$t_{(s)(1)} = t_{(r)(1)} + d(1, 2)$$

we have:

$$\frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}} = t_{(s)(2)} \tag{1.2}$$

Therefore, since, by definition of $d(i, i)$,

$$t_{(r)(2)} + d(2, 2) = t_{(s)(2)}$$

we can write, from (1.1) and (1.2):

$$\frac{3t_{(r)(1)} + 1}{2^{a_2}} + d(2, 2) = \frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}}$$

By elementary algebra, this yields:

$$2^{a_2} d(2, 2) = 3 \cdot d(1, 2)$$

Now $d(2, 2)$ must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level i, it must be the smallest such even number, whence it follows that $d(2, 2)$ must = $3 \cdot 2$, and necessarily

$$d(1, 2) = 2 \bullet 2^{a_2}$$

A similar argument establishes that $d(2, 2)$ and $d(1,2)$ have the above values for every other pair of tuples consecutive at level 2.

Thus we have our proof of the Basis Step for parts (a) and (b) of Lemma 1.0.

Proof of Induction Step for Parts (a) and (b) of Lemma 1.0

Assume the Lemma is true for all levels j, $2 \leq j \leq i$.

Let t_r, t_s be tuples consecutive at level i, and let t_r, t_f be tuples consecutive at level i +1. (See Fig. 2 (2).)

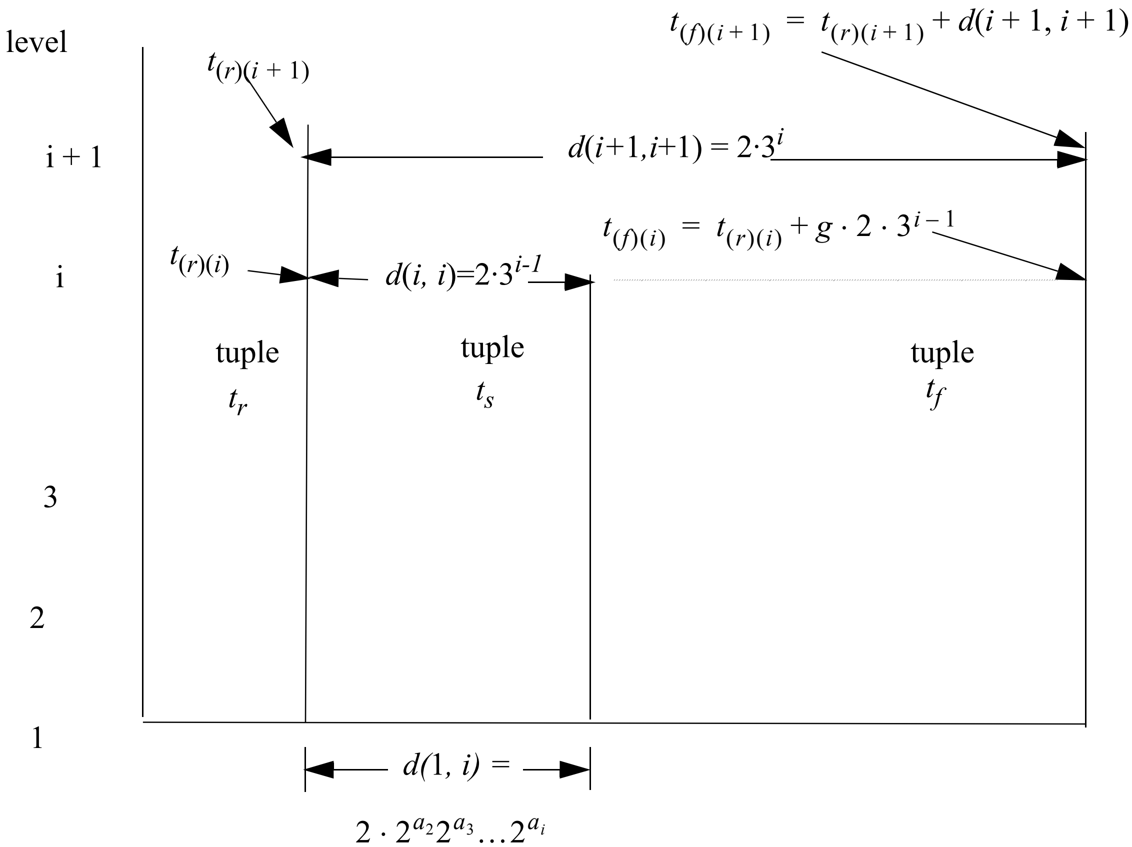


Fig. 2 (2). Illustration for proof of Induction Step of Lemma 1.0.

Then we have:

$$\frac{3t_{(r)(i)} + 1}{2^{a_{i+1}}} = t_{(r)(i+1)}$$

and since, by definition of $d(i, i)$,

$$t_{(f)(i)} = t_{(r)(i)} + g \cdot d(i, i)$$

for some $g \geq 1$, we have:

$$\frac{3(t_{(r)(i)} + g \cdot d(i, i)) + 1}{2^{a_{i+1}}} = t_{(f)(i+1)}$$

Thus, since

$$t_{(r)(i+1)} + d(i+1, i+1) = t_{(f)(i+1)}$$

we can write:

$$\frac{3t_{t_{(r)(i)}} + 1}{2^{a_{i+1}}} + d(i+1, i+1) = \frac{3(t_{t_{(r)(i)}} + gd(i, i)) + 1}{2^{a_{i+1}}}$$

This yields, by elementary algebra:

$$2^{a_{(i+1)}} d(i+1, i+1) = 3 \cdot gd(i, i)$$

As in the proof of the Basis Step, $d(i+1, i+1)$ must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level $i+1$, it must be the smallest such even number. Thus $d(i+1, i+1) = 3 \cdot d(i, i)$, and

$$g \cdot d(i, i) = 2^{a_{i+1}} d(i, i) \quad .$$

Hence

$$g = 2^{a_{i+1}}$$

Now g is the number of tuples consecutive at level i that must be "traversed" to get from $t_{(r)}$ to $t_{(f)}$. By inductive hypothesis, d(1, i) for *each* pair of these tuples is:

$$d(1, i) = 2 \cdot 2^{a_2} \cdot 2^{a_3} \cdot \ \ldots \cdot 2^{a_i}$$

hence, since

$$g = 2^{a_{i+1}}$$

we have

$$d(1, i+1) = d(1, i) \cdot 2^{a_{i+1}}.$$

A similar argument establishes that $d(i+1, i+1)$ and $d(1, i+1)$ have the above values for every pair of tuples consecutive at level $i+1$.

Thus we have our proof of the Induction Step for parts (a) and (b) of Lemma 1.0. The proof of Lemma 1.0 is completed. □

Lemma 2.0: Statement and Proof

For each exponent a_2, a tuple-set T_A, where $A = \{a_2\}$, exists.

Proof:

By Lemma 13.0 (see "Lemma 13.0: Statement and Proof" on page 43) we know that each range element is mapped to by all exponents of one parity only. Then since 5 is mapped to by 3 via the exponent 1, we know that 5 is mapped to by all odd exponents. Since 1 is mapped to by 1 via the exponent 2, we know that 1 is mapped to by all even exponents. Both 1 and 5 are level-2 anchors, since each is less than $2 \cdot 3^{2-1} = 6$. Therefore each tuple $<x, 5>$, where x maps to 5 via the odd exponent a_2 is the anchor tuple of a tuple-set, and each tuple $<x', 1>$, where x maps to 1 via the even exponent a_2', is the anchor tuple of a tuple-set. The result follows by Lemma 1.0 (a) and (b) (see "Lemma 1.0: Statement and Proof" on page 32), which assures us of an infinite number of tuples in each 2-level tuple-set. □

Lemma 3.0: Statement and Proof

Each i-level tuple-set, where $i \geq 2$, can be extended by each even or odd exponent a_{i+1}.

Proof:

By Lemma 2.0 (see "Lemma 2.0: Statement and Proof" on page 37), for each exponent a_2, a tuple-set $T_{A'}$, where $A' = \{a_2\}$, exists. So we show that for each exponent $a_2 = a_{i+1}$, the sequence of first elements of all tuples in $T_{A'}$ has at least one element in common with the sequence of i-level elements in T_A.

The sequence of i-level elements in the i-level tuple-set T_A is given by

$$2 \cdot 3^{i-1}k + y \tag{3.1}$$

where $k \geq 0$ and y is an i-level anchor, that is, y is an odd, positive integer that is less than or equal to, and relatively prime to, $2 \cdot 3^{(i-1)}$.

The sequence of 1-level elements of $T_{A'}$ is given by

$$\frac{2^{a_2}y' - 1}{3} + j2 \cdot 2^{a_2} \tag{3.2}$$

where $y' = 1$ or 5 is a 2-level anchor and $j \geq 0$ (see "Lemma 1.0: Statement and Proof" on page 32). Specifically, y' is 1 if $a_2 = a_{i+1}$ is even, and y' is 5 if $a_2 = a_{i+1}$ is odd. The left-hand term of (3.2) gives the value of the first element x of the level-1 sequence of $T_{A'}$ because

$$\frac{3x+1}{2^{a_2}} = y'$$

and an anchor, namely, y', is the smallest i-level element (in this case 2-level element) of an i-level tuple-set. The right-hand term of (3.2) is j times the difference between successive first elements of $T_{A'}$ (see "Lemma 1.0: Statement and Proof" on page 32).

Setting (3.1) equal to (3.2), we must prove that a solution j, k exists to the equation

$$2 \cdot 3^{i-1}k + y = \frac{2^{a_2}y' - 1}{3} + j2 \cdot 2^{a_2}$$

Multiplying through by 3, then dividing through by 2, which we can do since $3y + 1$ is even, we get

$$3^i k + \frac{3y+1}{2} = 2^{a_2 - 1}y' + 3j2^{a_2}$$

Rearranging terms, we have

$$3^i k - 3j2^{a_2} = -\frac{3y+1}{2} + 2^{a_2 - 1}y' \tag{3.3}$$

or

$$3(3^{i-1}k - j2^{a_2}) = -\frac{3y+1}{2} + 2^{a_2 - 1}y'$$

The right-hand side of the equation must be a multiple of 3, and so we can divide both sides by 3 and write:

$$3^{i-1}k - 2^{a_2}j = U$$

This is an equation of the form

$$au + bv = c$$

and a basic fact of Diophantine Equations states that such an equation has a solution u, v if and only if (a, b) divides c. In our case,

$$(3^{i-1}, 2^{a_2}) = 1$$

and so (3.3) has a solution j, k.

Lemma 1.0 (see “Lemma 1.0: Statement and Proof” on page 32) then assures us of an infinity of i-level elements in T_A that have extensions via the exponent $a_{2} = a_{i+1}$, thus creating the tuple-set $T_{A''}$, where A″ = $\{a_2, a_3, ..., a_i, a_{i+1}\}$. □

Lemma 4.0: Statement and Proof

For each exponent sequence $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, there exists a tuple-set T_A generated by A.

Proof:

The proof is by induction.

Basis Step:

By Lemma 2.0 (see “Lemma 2.0: Statement and Proof” on page 37) we know that there is a 2-level tuple-set for each exponent a_2.

Inductive Step:

Assume the Lemma is true for all j-level exponent sequences $2 \leq j \leq i$. But then by Lemma 3.0 (see “Lemma 3.0: Statement and Proof” on page 37) it is true for all tuple-sets generated by $(i + 1)$-level exponent sequences. .□

Lemma 5.0: Statement and Proof

Assume a counterexample exists. Then for all $i \geq 2$, each i-level tuple-set contains an infinity of i-level counterexample tuples and an infinity of i-level non-counterexample tuples.

Proof:

1. Assume counterexamples exist. Then:

There is a countable infinity of non-counterexample range elements.

Proof: Each non-counterexample maps to a range element, by definition of *range element*.

Each range element is mapped to by an infinity of elements (“Lemma 13.0: Statement and Proof” on page 43). A countable infinity of these are range elements (proof of “Lemma 18.0: Statement and Proof” on page 47).

There is a countable infinity of counterexample range elements.

Proof: same as for non-counterexample case.

2. For each finite exponent sequence A, and for each range element y, non-counterexample or counterexample, there is an x that maps to y via A possibly followed by a buffer exponent ("Lemma 18.0: Statement and Proof" on page 47). The presence of the buffer exponent does not change the fact that x is the first element of a tuple associated with the exponent A.. □

Lemma 5.5: Statement and Proof

Let a be a finite exponent sequence such that if x maps to y via a, then y > x. Then there does not exist a counterexample x such that the infinite tuple <x, ... > is associated with the exponent sequence {a, a, a, ... }.

Proof:

Assume the contrary. Then there exists a counterexample x such that x is the first element of the infinite tuple <x, ... > that is associated with the exponent sequence $\{a, a, a, ...\}$. But x maps to y via a, and by hypothesis $y > x$, so y is the first element of the infinite tuple <y, ... > and this infinite tuple is likewise associated with the exponent sequence $\{a, a, a, ...\}$. Therefore, in the infinite sequence of tuple-set extensions associated with the infinite sequence $\{a\}$, $\{a, a\}$, $\{a, a, a\}$, ... of exponent sequences, there must occur an i-level tuple-set T_A in which <x, ... > and <y, ... > have i-level prefixes that are tuples consecutive at level i. The infinite tuples <x, ... > and <y, ... > have $(i + j)$-level prefixes in all j-level extensions of T_A. But since x and y are the same for all these prefixes, the level 1 distance function defined by part (b) of "Lemma 1.0" on page 10 is violated, and this contradiction gives us our proof. □

Lemma 6.0: Statement and Proof

Let t be the i-level anchor tuple in an i-level tuple-set, where $i \geq 2$. *Then the last element y of t, that is, the i-level element of t (which is the anchor), is a number* less than $2 \cdot 3^{(i-1)}$.

Proof:

By definition of *i-level anchor tuple*, t is the first i-level tuple in an i-level tuple-set. Hence there are no i-level tuples to the left of t under our convention for ordering tuples from left to right in a tuple-set. By the distance function defined in Lemma 1.0 (a), the distance between the last elements of consecutive i-level tuples in an i-level tuple-set is $2 \cdot 3^{(i-1)}$. But if there is no i-level tuple to the left of t, it follows that the last element y of t must be less than $2 \cdot 3^{(i-1)}$. □

Lemma 7.0: Statement and Proof

The set of i-level elements of all i-level tuples in an i-level tuple-set is a complete set of reduced residue classes modulo $2 \cdot 3^{(i-1)}$.

Proof:

Let T_A be an i-level tuple-set. Since the first i-level tuple t in T_A is an anchor tuple, the last element y of t is an anchor. By Lemma 6.0 (see "Lemma 6.0: Statement and Proof" on page 40), $y < 2 \cdot 3^{i-1}$. Since the range of the $3x + 1$ function C is the set of odd, positive integers not divisible by

3 (see “Lemma 10.0: Statement and Proof” on page 41), the result follows by part (a) of Lemma 1.0 (see “Lemma 1.0: Statement and Proof” on page 32), which states that the distance between i-level elements of successive i-level tuples in an i-level tuple-set is $2 \cdot 3^{i-1}$. □

Lemma 8.0: Statement and Proof

For each odd, positive integer x there exists a minimum $i = i_0$ such that for each $i \geq i_0$, x is the first element of the first i-level tuple in some i-level tuple-set, that is, x is the first element of an i-level anchor tuple in some i-level tuple-set. In terms of infinite tuples, this lemma states: if x is an odd, positive integer, then in the infinite tuple $\bar{t} = <x, y, y', ... >$, there exists a minimum level i_0 such that:

$\bar{t}(i_0)$ is the i_0-level anchor tuple in an i_0-level tuple-set;
$\bar{t}(i_0 + 1)$ is the $(i_0 + 1)$-level anchor tuple in an $(i_0 + 1)$-level tuple-set;
$\bar{t}(i_0 + 2)$ is the $(i_0 + 2)$-level anchor tuple in an $(i_0 + 2)$-level tuple-set;
etc.

Proof:

Let x be an odd, positive integer. Then x is the first element of an infinite tuple $\bar{t} = <x, y, ... >$. With each increment of i, $i \geq 2$, the element of $\bar{t}$ at level i increases by at most a factor of 2, since for all exponents except 1, $C(y) < y$, and for exponent 1, $C(y) \leq 2y$. However, with each increment of i, $2 \cdot 3^{(i-1)}$ increases by a factor of 3. Therefore, a level $i = i_0$ must eventually be reached such that the element y' of $\bar{t}$ at level i is less than $2 \cdot 3^{(i-1)}$. But then by definition y' is an anchor, and hence the prefix $<x, y, ..., y'>$ is an anchor tuple. By our rule, “once an anchor tuple, always an anchor tuple” (see under “Mark” on page 15), the final part of our result follows. □

Lemma 10.0: Statement and Proof

No multiple of 3 is a range element.

Proof :

If

$$\frac{3x+1}{2^a} = 3m$$

then $1 \equiv 0 \bmod 3$, which is false. □

Lemma 11.0: Statement and Proof

Each odd, positive integer (except a multiple of 3) is mapped to by a multiple of 3 in one iteration of the 3x + 1 function.

Proof:

Since the domain of the $3x + 1$ function is the odd, positive integers, the only relevant generators are $3(2k + 1)$, $k \geq 0$. We show that, for each odd, positive integer y not a multiple of 3, there exists a k and an a such that

$$y = \frac{(3(3(2k+1))+1)}{2^a} , \qquad (11.1)$$

where a is necessarily the largest such a, since y is assumed odd.

Rewriting (11.1), we have:

$$y2^{a-1} - 5 = 9k . \qquad (11.2)$$

Without loss of generality, we can let $y \equiv r \bmod 18$, where r is one of 1, 5, 7, 11, 13, or 17 (since y is odd and not a multiple of 3, these values of r cover all possibilities mod 18). Or, in other words, for some q, r, $y = 18q + r$. Then, from (11.2) we can write:

$$18(2^{a-1})q + (2^{a-1})r - 5 = 9k . \qquad (11.3)$$

Since the first term on the lefthand side is a multiple of 9, $(2^{a-1})r - 5$ must also be if the equation is to hold. We can thus construct the following table. (Certain larger a also serve equally well, but those given suffice for purposes of this proof.)

Table 2: Values of *r*, *a*, for Proof of Lemma

r	a	$(2^{a-1})r - 5$
1	6	27
5	1	0
7	2	9
11	5	171
13	4	99
17	3	63

Given q and r (hence y), we can use r to look up a in the table, and then solve (11.3) for integral k, thus producing the multiple of 3 that maps to y in one iteration of the $3x + 1$ function. □

Lemma 12.0: Statement and Proof

For each range element y there exists an infinity of x that map directly to y. Specifically,

If

$$\frac{3x+1}{2^a} = y$$

Then, for all n ≥ 1,

$$\frac{3(x + (2^{a+2(0)} + 2^{a+2(1)} + \ldots + 2^{a+2(n-1)})y) + 1}{2^{a+2(n)}} = y$$

Proof:

The proof is a matter of straightforward algebra.

From the antecedent, we have:

$$x = \frac{2^a y - 1}{3}$$

Substituting into the left-hand side of the consequent, multiplying the term in parentheses by 3, cancelling two 1's, and factoring out $(2^a)(y)$ yields:

$$\frac{2^a y(1 + 3(2^0 + 2^2 + 2^4 + \ldots + 2^{2(n-1)}))}{2^{a+2(n)}}$$

The 2^as cancel, the term (1 + 3(...)) is easily shown to equal $2^{2(n)}$, the $2^{2(n)}$ in numerator and denominator cancel, and we are left with y, which gives us our result. □

Remark

Lemma 12.0 and Lemma 11.0 (see "Lemma 11.0: Statement and Proof" on page 41) imply that if a counterexample exists, then there is an infinity of counterexamples.

Lemma 13.0: Statement and Proof

Each range element y is mapped to, in one iteration of the 3x + 1 function, by all exponents of one parity only.

The following proof is an edited version of a proof by Sanjai Gupta. Any errors it contains are entirely our own.

Proof:

Fix a range element y, and suppose that x maps to y via the exponent a. Now a is either even or odd, hence $a = 2n + h$, where h is either 0 or 1. Since $y = (3x + 1)/2^a$, it follows that $(2^a)y = 3x+1$. Reduce the equation mod 3, and we get $(2^h)y \equiv 1 \bmod 3$, by the following reasoning: $(2^a)y$

$\equiv 1 \bmod 3$ implies $(2^{2n+h})y \equiv 1 \bmod 3$ implies $2^{2n} 2^h y \equiv 1 \bmod 3$ implies $2^h y \equiv 1 \pmod 3$ because $2^{2n} = 4^n \equiv 1 \bmod 3$.

Since y is fixed, either $y \equiv 1$ or $y \equiv 2 \bmod 3$. (We know that y, a range element, is not a multiple of 3 by Lemma 10.0 (see "Lemma 10.0: Statement and Proof" on page 41)). If $y \equiv 1 \bmod 3$, then we have $2^h(1) \equiv 1 \bmod 3$, which implies that h must be 0. If $y \equiv 2 \bmod 3$, then we have $(2^h)(2) \equiv 1 \bmod 3$, implying that h must be 1, which proves the Lemma. □

Lemma 14.0: Statement and Proof

There exists an explicit construction of the tuple-set whose exponent sequence is associated with a given tuple.

Proof:

Let x be the first element of a tuple and let $\{a_2, a_3, ..., a_{n+1}\}$ be the sequence of exponents associated with the first n extensions of the tuple <x>. The last element of the tuple is given by:

$$\frac{3^n x + r}{2^a}$$

where

$$a = \sum_{i=2}^{n} a_i$$

The term r is most easily calculated by iterating from $x = 0$, then multiplying by the appropriate power of 2, as shown in the table at the end of this proof. We want the integral x that produce odd outputs:

$$\frac{3^n x + r}{2^a} = 2k + 1$$

which gives

$$3^n x - 2^{a+1} k = 2^a - r$$

This is a standard linear Diophantine equation. Since $(3^n, 2^{a+1}) = 1$, and 1 divides the right-hand side of the equation, the equation has a solution. One solution is:

$$x_0 = (-(2^a - r))\left(\frac{2^{2 \cdot 3^{n-1} \cdot (a+1)} - 1}{3^n}\right)$$

$$k_0 = (-(2^a - r))(2^{(2 \cdot 3^{n-1} - 1)(a+1)})$$

Note that the ratio in the expression for x_0 is an integer because

$$2^{2 \cdot 3^{n-1}} \equiv 1 \quad mod \quad 3^n$$

The general solution is:

$$x = x_0 + t \cdot (-2^{a+1})$$

$$k = k_0 - t \cdot 3^n$$

where t ranges over the integers. Thus, the x's are the inputs that iterate with the specified exponents and

$$2k + 1 = 2k_0 - t \cdot 2 \cdot 3^n + 1$$

are the outputs.

Table 3: Successive values of *n*, the *x* term, and *r* in proof of Lemma 14.0

n	***x* term**	***r***	**level of tuple element yielded, i.e., *i* in a_i**
1	3^1x	1	2
2	3^2x	$3^1 + 2^{a_2}$	3
3	3^3x	$3^2 + 3^1 2^{a_2} + 2^{a_2} 2^{a_3}$	4
4	3^4x	$3^3 + 3^2 2^{a_2} + 3^1 2^{a_2} 2^{a_3} + 2^{a_2} 2^{a_3} 2^{a_4}$	5

Table 3: Successive values of *n*, the *x* term, and *r* in proof of Lemma 14.0

n	***x* term**	***r***	**level of tuple element yielded, i.e., *i* in a_i**
. . .	. . .	. . .	. . .

□

Lemma 15.0: Statement and Proof

For each range element y, and for each finite sum a of exponents, a domain element x exists that maps to y via a sum a' that contains a.

Proof:

We are looking for an x such that the sequence of iterations represented by

$$\frac{3^n x + r}{2^a}$$

where n, a, and r are defined as in Lemma 14.0 (see “Lemma 14.0: Statement and Proof” on page 44), lead to a computation that ends with y. n, a, and r are determined by the exponent sequence we want. There also has to be an optional buffer iteration between the above and y, for example, to allow for parity constraints on the exponent leading to y (see “Lemma 12.0: Statement and Proof” on page 42). Thus, for example, if y is mapped to by even exponents, and our exponent sequence a ends with an odd exponent, then there must be a buffer exponent following the sequence a. So, we want

$$\frac{3\left(\frac{3^n x + r}{2^a}\right) + 1}{2^j} = y$$

or

$$\frac{3^{n+1} x + 3r + 2^a}{2^{a+j}} = y$$

which gives

$$3^{n+1} x = (2^a y) 2^j - 3r - 2^a \tag{15.1}$$

or

$$(2^a y)2^j \equiv 3r + 2^a \ \ mod \ \ 3^{n+1}$$

We are looking for x and j. Since y is a range element, it cannot be a multiple of 3 (see "Lemma 10.0: Statement and Proof" on page 41). Therefore $2^a y$ is relatively prime to 3^{n+1}, as is $3r + 2^a$. Since 2^j, where $j \geq 0$, is an element of a reduced residue class mod 3^{n+1}, the congruence is solvable. Hence we can find j, and then, from (15.1), x. □

Remarks

The result would hold for each finite number of buffer exponents following the exponent sum a, since they do not change the fact that a tuple generating each exponent sequence whose sum is a is guaranteed by the proof.

A recursive proof of the Lemma is possible because the set of odd, positive integers mapping to a range element y in one iteration of the $3x + 1$ function C includes an infinite subset each element of which is mapped to by an infinity of even exponents, and an infinite subset each element of which is mapped to by an infinity of odd exponents. (See "Lemma 13.0: Statement and Proof" on page 43, and Lemma 15.0, p. 57, in our paper, "The Structure of the $3x + 1$ Function: An Introduction" on the web site www.occampress.com).

Lemma 18.0: Statement and Proof

Let y be a range element of the 3x + 1 function. Then for each finite exponent sequence A, there exists an x that maps to y via A possibly followed by a "buffer" exponent. (If y is mapped to by even exponents, and our exponent sequence A ends with an odd exponent, then there must be a "buffer" exponent following A, and similarly if y is mapped to by odd exponents and A ends with an even exponent.)

Proof:

1. Each range element y is mapped to by all exponents of one parity ("Lemma 13.0: Statement and Proof" on page 43).

2. Each range element y is mapped to by a multiple of 3 ("Lemma 11.0: Statement and Proof" on page 41).

Each range element is mapped to by an infinity of range elements ("Lemma 11.0: Statement and Proof" on page 41).

3. Let y be a range element and let $S = \{s_1, s_2, s_3, ... \}$ be the set of all odd, positive integers that map to y in one iteration of the $3x + 1$ function. Furthermore, let the s_i be in increasing order of magnitude. It is easily shown that $s_{i+1} = 4s_i + 1$.

(In Fig. 18, $y = 13$, $S = \{17, 69, 277, 1109, ... \}$

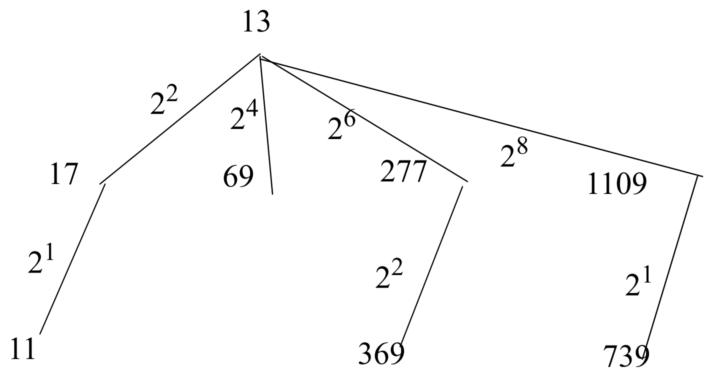


Fig. 18

4. If s_i is a multiple of 3, then $4s_i$ +1 is mapped to, in one iteration of the $3x + 1$ function, by all exponents of even parity.

To prove this, we need only show that x is an integer in the equation

$$4(3u) + 1 = \frac{3x+1}{2^2}$$

Multiplying through by 2^2 and collecting terms we get

$$(48u) + 4 = 3x + 1$$

and clearly x is an integer.

5. If s_j is mapped to by all even exponents, then $4s_j + 1$ is mapped to, in one iteration of the $3x + 1$ function, by all exponents of odd parity.

(The proof is by an algebraic argument similar to that in step 4.)

6. If s_k is mapped to by all odd exponents, then $4s_k + 1$ is a multiple of 3.

(The proof is by an algebraic argument similar to that in step 4..)

7. The Lemma follows by an inductive argument that we now describe.

Let *y* be a range element. It is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. As a consequence of steps 1 through 6, we can represent an infinite sub-sequence of the sequence by

...3, 2, 1, 3, 2, 1, ...

where
"3" means "this odd, positive integer is a multiple of 3 and therefore is not mapped to by any odd, positive integer";
"2" means "this odd, positive integer is mapped to by all even exponents";
"1" means "this odd, positive integers is mapped to by all odd exponents".

Each type "2" and type "1" odd, positive integer is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. We can represent an infinite sub-sequence of the sequence by

...3, 2, 1, 3, 2, 1, ...

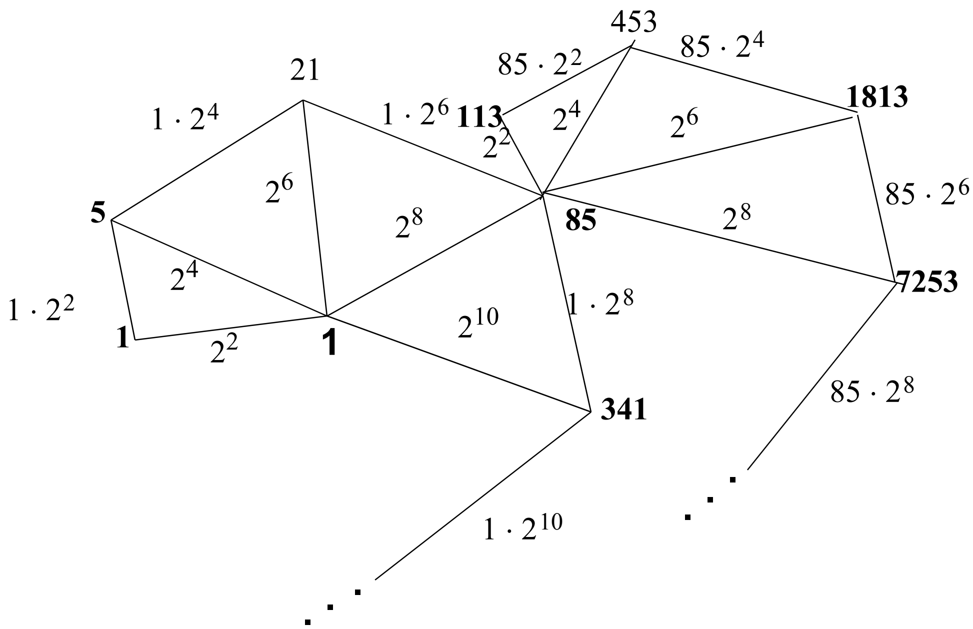
where each integer has the same meaning as above.

Temporarily ignoring the case in which a buffer exponent is needed, it should now be clear that, for each range element *y*, and for each finite sequence of exponents *B*, we can find a finite path down through the infinitary tree we have just established, starting at the root *y*. The path will end in an odd, positive integer *x*. Let *A* denote the path *B* taken in reverse order. Then we have our result for the non-buffer-exponent case. The buffer-exponent case follows from the fact that the buffer exponent is one among an infinity of exponents of one parity. Thus *y* is mapped to by an infinite sequence of odd, positive integers. We then simply apply the above argument.. □

Apppendix B — Analysis of a Failed Strategy

In early 2009 we attempted to prove the $3x + 1$ Conjecture using the inverse of the $3x + 1$ function — specifically, the inverse of 1. Our motivation was as follows:

It dawned on us that all odd, positive integers that are known to map to 1 — namely, 1, 3, 5, 7, 9, 11, ..., up to about $5.76 \cdot 10^{18}$, by computer test[1] — map to 1 regardless if counterexamples exist or not. We then thought of the structure of the set of all odd, positive integers that are inverses of 1, a structure we have elsewhere called the "infinite set of recursive 'spiral's whose base element is 1." (See "Section 2. Recursive 'Spiral's" in the first file of the paper "The Structure of the $3x + 1$ Function: An Introduction" on the web site www.occampress.com.) Following is a diagram of part of this structure.



Recursive "spirals" structure of computations produced by the $3x + 1$ function. Bold-faced numbers are range elements (21 and 453 are multiples of 3, hence not range elements). Partial "spirals" surrounding the base elements 1 and 85 are shown. The line connecting 1813 to 85 is marked with a 2^6 because $(3 \bullet 1813 + 1)/ 2^6 = 85$. The line connecting 453 to 1813 is marked $85 \bullet 2^4$ because $453 + 85 \bullet 2^4 = 1813$. The exponents of 2 are not always even, of course. The "spiral" of numbers (not shown) mapping to 341 has odd exponents.

It is easily shown that {all odd, positive integers that map to 1 in *one* iteration of the $3x + 1$ function} = {1, 5, 21, 85, 341, ... }. This set is a recursive "spiral". Lemma 11.0 in the above-referenced "...Introduction" paper, states that if y is an element of a "spiral", the next element is $4y + 1$.

It is also easily shown that each recursive "spiral" contains an infinity of range elements and an infinity of multiples-of-3.

1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All odd, positive integers to at least $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$ have been tested and found to be non-counterexamples.

We then defined the set J from this diagram as follows:

Let J denote

{all odd, positive integers that map to 1 in *one* iteration of the $3x + 1$ function} $\cup$
{all odd, positive integers that map to 1 in *two* iterations of the $3x + 1$ function} $\cup$
{all odd, positive integers that map to 1 in *three* iterations of the $3x + 1$ function} $\cup$
...

We stated that

(1) Each element of J maps to 1 regardless whether counterexamples exist,

or, in other words,

(2) J is the same set regardless whether counterexamples exist.

Our justification was that the contrary would imply that the laws of arithmetic — in particular, those governing the elements of each "spiral" in the above structure — were sensitive to the truth or falsity of the $3x + 1$ Conjecture, which is absurd.

However, statements (1) and (2) drew strong criticism from virtually all readers. Many declared the statements were meaningless. But we persisted, and eventually arrived at the following argument:

Let V = the set of odd, positive integers, and let C = the set of counterexamples. Then (1) implies:

$J \cup C = V = J$, and therefore C is empty. Hence we have a proof of the $3x + 1$ Conjecture.

We received many objections to this argument, most of which we didn't understand. Then Jonathan Kilgallin sent us the following counterargument, which we consider irrefutable. It is that our argument can be applied equally to the $3x – 1$ function. But there we know that counterexamples exist (5 and 7 form an infinite cycle, and thus are counterexamples). Therefore our argument is invalid.

We feel it is important to understand the fault in our reasoning even apart from the $3x – 1$ counterargument. The fault rests in our confusing of domains of discourse.

Case I. Let our domain of discourse = W = the set of odd, positive integers that map to 1 under the $3x + 1$ function. Then (1) and (2) hold, and we can legitimately write:

$J \cup C = W = J$,

because, in fact, there are no counterexamples in W, hence $C = \varnothing$.

Case II. Now let our domain of discourse = V = the set of odd, positive integers. Then, although (1) and (2) hold, and we can write

$$J \cup C = V,$$

it is not necessarily true that

$$V = J.$$

We will not know until we have a proof or disproof of the $3x + 1$ Conjecture.

However, we emphasize that (1) and (2) hold in both cases. In Case I, J has only one value in the domain of discourse, in Case II, J has two possible values in the domain of discourse. Just as we might know that an equation has one solution, x, but we do not know, until we solve for x, if x is real or complex.

Perhaps a better way to understand the counterintuitive fact that J is a single, fixed set in both cases is as follows. Informally, let J be "placed" inside the set of odd, positive integers. If there is any "space" left over (if there are any members outside of J), then counterexamples exist, otherwise, not.