

A Solution to the $3x + 1$ Problem

by

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Abstract

We present a solution to the $3x + 1$ Problem, which asks if repeated iterations of the function $C(x) = (3x + 1)/(2^a)$ always terminate in 1. Here x is an odd, positive integer, and a is the largest positive integer such that the denominator divides the numerator. The solution begins by partitioning the set of all finite sequences of iterations of C (each sequence being represented by a *tuple*) into *tuple-sets*. If a tuple is associated with the sequence $A = \{a_2, a_3, \dots, a_i\}$ of exponents of 2, where $i \geq 2$, then the tuple is a member of the tuple-set T_A . We show that if counterexamples exist, each tuple-set contains an infinity of counterexample tuples and an infinity of non-counterexample tuples. Since the first i -level tuple in an i -level tuple-set (the so-called *anchor* tuple) cannot be both a counterexample tuple and a non-counterexample tuple, we show that counterexample tuples, hence counterexamples, do not exist.

Introduction

Statement of Problem

For x an odd, positive integer, set

$$C(x) = \frac{3x + 1}{2^{\text{ord}_2(3x+1)}}$$

where $\text{ord}_2(3x + 1)$ is the largest exponent of 2 such that the denominator divides the numerator. Thus, for example, $C(17) = 13$, $C(13) = 5$, $C(5) = 1$. The $3x + 1$ Problem, also known as the $3n + 1$ Problem, the Syracuse Problem, Ulam's Problem, the Collatz Conjecture, Kakutani's Problem, and Hasse's Algorithm, asks if repeated iterations of C always terminate at 1. The conjecture that they do is hereafter called the $3x + 1$ Conjecture. We call C the $3x + 1$ function; note that $C(x)$ is by definition odd.

Other equivalent formulations of the $3x + 1$ Problem are given in the literature; we base our formulation on the C function because, as we shall see, it brings out certain structures that are not otherwise evident.

Summary of Research on the Problem

As stated in (Lagarias 1985), "The exact origin of the $3x + 1$ problem is obscure. It has circulated by word of mouth in the mathematical community for many years. The problem is traditionally credited to Lothar Collatz, at the University of Hamburg. In his student days in the 1930's, stimulated by the lectures of Edmund Landau, Oskar Petron, and Issai Schur, he became interested in number-theoretic functions. His interest in graph theory led him to the idea of representing such number-theoretic functions as directed graphs, and questions about the structure of such graphs are tied to the behavior of iterates of such functions. In the last ten years [that is, 1975-1985] the problem has forsaken its underground existence by appearing in various forms as a problem in books and journals..."

As far as we have been able to determine, our approach to a solution of the Problem is original. We have been unable to find papers in the literature that suggest that other researchers have had the same basic idea as ours, namely, that which is described in "Abstract" on page 2 and in the next section, and, in more detail, in the first two paragraphs under "Tuple-Sets" on page 5.

Brief Description of Solution Strategy

We begin with a partitioning of the set of all finite sequences of iterations of C into "tuple-sets". Each tuple-set T_A consists of the set of all tuples that are associated with a finite exponent sequence A and its prefixes. Thus, for example, the tuple $\langle 7, 11, 17, 13 \rangle$ is associated with the exponent sequence $A = \{1, 1, 2\}$, because 7 maps to 11 (in one iteration of C) via the exponent 1, 11 maps to 17 (in one iteration of C) via the exponent 1, and 17 maps to 13 (in one iteration of C) via the exponent 2. The tuple-set T_A contains all tuples associated with A , plus all tuples associated with each prefix of A . There are always an infinity of tuples in a tuple-set.

Our first proof of the Conjecture follows from the fact that if counterexamples exist, each tuple-set contains an infinity of counterexample tuples *and* an infinity of non-counterexample tuples¹. We then show² that if an odd, positive integer exists, it must be an element of the first i -level tuple in some i -level tuple-set.

These two facts enable us to show that no counterexample is ever an element of such a tuple, and this gives us our proof of the Conjecture.

Our second proof shows that each tuple-set is unchanged by the presence of counterexamples. This contradicts a basic lemma³ and thus gives us our proof of the Conjecture.

On the Format of this Paper

Because the tuple-set structure lies at the heart of our solution to the Problem, and thus is referred to in most proofs of lemmas, and because defining this structure requires defining quite a few terms, we have placed all these definitions at the beginning of our exposition.

To enhance ease and rapidity of understanding, we merely state lemmas in the course of the exposition. All proofs are given in “Appendix A — Statement and Proof of Each Lemma” on page 25. The page number of each proof is given in the text. Lemmas are numbered consecutively, beginning with 1.0., in the order in which they appear in the exposition. But this means that lemmas that are not mentioned in the exposition have higher numbers than the numbers of the lemmas of the proofs they support.

Referenced equations in each lemma are numbered $n.1$, $n.2$, etc., where n is the number of the lemma.

1. “Lemma 5.0” on page 13
2. “Lemma 8.0” on page 14
3. “Lemma 8.7” on page 16

Tuple-Sets

In the first part of this paper, we describe a structure called *tuple-sets* that underlies all finite sequences of iterations of the $3x + 1$ function, C . We have placed virtually all definitions in this first part of the paper because the terms defined are used repeatedly in the lemmas and proofs given later.

A tuple-set can be briefly, and informally, described as follows. (A formal definition is given under “Tuple-set” on page 7.) Consider the sequence of two iterations of C : $C(17) = 13$ (via the exponent 2 in the definition of C) followed by $C(13) = 5$ (via the exponent 3 in the definition of C). This sequence can be represented by the tuple $\langle 17, 13, 5 \rangle$. The tuple-set T_A defined by the 2-level exponent sequence $A = \{2, 3\}$ contains: the tuple $\langle 17, 13 \rangle$, and the tuple $\langle 17, 13, 5 \rangle$. But in addition it contains all other tuples that are determined by the exponent sequences $\{2\}$ and $\{2, 3\}$ — in other words, all other tuples that are associated with “approximations” to A . For example, the tuples $\langle 33, 25 \rangle$ and $\langle 81, 61, 23 \rangle$ are in T_A , because they too are determined by the exponent sequences $\{2\}$ and $\{2, 3\}$, respectively, as the reader can verify.

We then show that each i -level tuple-set, where $i \geq 2$, has a unique first i -level tuple (called an *anchor* tuple) that (like all tuples) must be either a non-counterexample tuple or a counterexample tuple, but cannot be both. In the second part of this paper, we show how a basic result — that if counterexamples exist, then every tuple-set contains an infinity of counterexample tuples and an infinity of non-counterexample tuples — enables us to prove that there are no counterexample anchor tuples, hence no counterexamples.

We now proceed with our definitions.

Iteration

An *iteration* takes an odd, positive integer, x , to another odd, positive integer, y , via one application of the $3x + 1$ function, C . Thus, in one iteration C takes 17 to 13 because $C(17) = 13$.

Tuple

A *tuple* is a sequence of zero or more successive iterations of C , that is, if the sequence is finite,

$$(C^k(x))_{k \geq 0} = (x, C(x), C^2(x), \dots, C^k(x))$$

or, if the sequence is infinite,

$$(C^\infty(x)) = (x, C(x), C^2(x), \dots)$$

A finite sequence is not required to end with a 1, and an infinite sequence is not required to end with an infinity of successive 1's. If an infinite sequence does not end with an infinity of successive 1's, then it consists of counterexamples to the $3x + 1$ Conjecture.

A finite tuple is denoted¹ $\langle x, y, y', \dots, y^{(n)} \rangle$. For example, $\langle 5, 1 \rangle$ and $\langle 11, 17, 13 \rangle$ are finite tuples. An infinite tuple is denoted $\langle x, y, y', \dots \rangle$. For example, $\langle 5, 1, 1, 1, \dots \rangle$ and $\langle 11, 17, 13, 5, 1, 1, 1, \dots \rangle$ are infinite tuples.

Let $t = \langle x, y, y', \dots, y^{(n)} \rangle$ be a finite tuple. Then the tuple $t' = \langle x, y, y', \dots, y^{(n)}, y^{(n+1)} \rangle$ is an *extension* of t . An extension of an extension of t we likewise call an extension of t , etc. By definition of the function C , every tuple has an infinite number of extensions. In the case of a sequence of iterations of C that eventually yield 1, the corresponding infinite tuple is $\langle x, y, y', \dots, 1, 1, 1, \dots \rangle$. A tuple consisting of an infinite number of extensions is an *infinite tuple*. We denote an infinite tuple by \bar{t} .

Clearly, since the domain of C consists of the odd, positive integers, each odd, positive integer is the first element of an infinite tuple.

If \bar{t} is an infinite tuple, we denote the first i levels of \bar{t} (that is, the first i elements of \bar{t}), by $\bar{t}(i)$, and we call $\bar{t}(i)$ a *prefix* of \bar{t} . For example, if $\bar{t} = \langle 17, 13, 5, 1, 1, 1, \dots \rangle$, then $\bar{t}(1) = 17$, and $\bar{t}(4) = \langle 17, 13, 5, 1 \rangle$. Thus every finite tuple is a prefix of an infinite tuple and every prefix of an infinite tuple is a finite tuple. The term *tuple* standing alone, without the qualifier “infinite”, denotes a finite tuple, that is, the prefix of an infinite tuple, unless context clearly indicates the reference is to an infinite tuple.

In the literature on the $3x + 1$ Problem, tuples are sometimes called “trajectories” or “orbits”.

Each tuple element except, possibly, the first, is an odd, positive integer that is not a multiple of 3. The element is odd by definition of the $3x + 1$ function, C , and is not a multiple of 3 by “Lemma 9.0: Statement and Proof” on page 33.

Non-Counterexample and Counterexample

If x is the first element of an infinite tuple $\langle x, \dots, 1, 1, 1, \dots \rangle$, then x is called a *non-counterexample*. Otherwise, x is called a *counterexample*. Thus, a counterexample never yields 1 under repeated iterations of the $3x + 1$ function.

Exponent

If $C(x) = y$, with $y = (3x + 1)/2^a$, we say that x *maps under iteration to y* (or x *maps directly to y*) *via the exponent a* , and that a *is the exponent associated with x* . We sometimes speak of a as *mapping directly to y* . We sometimes omit the word *directly* when context makes clear that it is implied. The sequence $\{a_2, a_3, \dots, a_i\}$, where a_2, a_3, \dots, a_i are the exponents associated with $x, C(x), \dots, C^{(i-2)}(x)$ respectively, is called an *admissible vector* in (Wirsching 1998). We call the sequence an *exponent sequence*. We define the function $e(x)$ to be the exponent associated with x . We sometimes refer to y as a *range element*. It is easily shown that y cannot be a multiple of 3 (see “Lemma 9.0: Statement and Proof” on page 33). An element x of the domain of the $3x + 1$ function, whether multiple of 3 or not, we sometimes refer to as a *domain element*.

Clearly, an exponent is a positive integer.

1. In a tuple, “ $x^{(n)}$ ”, “ $y^{(n)}$ ”, etc., denotes x followed by n prime symbols, y followed by n prime symbols, etc.

Symbols for Exponent Sequences and for Tuples

It is important that the reader keep clearly in mind the symbols we use for two different types of sequences. Curly braces denote exponent sequences. Thus, for example, we write $\{a_2, a_3, \dots, a_i\}$ for an i -level exponent sequence, and $\{a_2, a_3, \dots\}$ for an infinite exponent sequence. Angle brackets denote tuples, that is, the results of iterations of the $3x + 1$ function. Thus, for example, we write $\langle x, y, \dots, z \rangle$ or $\langle x, y, y', \dots, y^{(n)} \rangle$ for finite tuples, and $\langle x, y, y', \dots \rangle$ for infinite tuples.

Exponent Sequence Associated With a Tuple

As we established under “Exponent” on page 6, associated with every finite sequence of iterations of the function C — hence with every tuple — is an exponent sequence. We speak of the exponent sequence *associated with* a finite tuple. If t is a tuple, then we denote the exponent sequence associated with t by $A(t)$. Thus, for example, if $t = \langle 17, 13, 5, 1 \rangle$ then $A(t) = \{2, 3, 4\}$ because 17 maps directly to 13 via the exponent 2, 13 maps directly to 5 via the exponent 3, and 5 maps directly to 1 via the exponent 4.

Extension of an Exponent Sequence

Let $A = \{a_2, a_3, \dots, a_i\}$ be a finite sequence of exponents, where $i \geq 2$. Then an exponent sequence $A' = \{a_2, a_3, \dots, a_i, a_{i+1}\}$ is an *extension* of A . An extension of A' is also an extension of A , etc.

Tuple-set

(The reader might find it helpful to refer to Fig. 1 in this sub-section while reading the following.)

Let $A = \{a_2, a_3, \dots, a_i\}$ be a finite sequence of exponents, where $i \geq 2$. The *tuple-set* T_A consists of all and only the following tuples:

all tuples $\langle x \rangle$ such that x does not map to an odd, positive integer via a_2 ;

all tuples $\langle x, y \rangle$ such that x maps to y via a_2 (that is, $e(x) = a_2$) but y does not map to an odd, positive integer via a_3 ;

all tuples $\langle x, y, y' \rangle$ such that x maps to y via a_2 (that is, $e(x) = a_2$) and y maps to y' via a_3 (that is, $e(y) = a_3$), but y' does not map to an odd, positive integer via a_4 ;

...

all tuples $\langle x, y, y', \dots, y^{(i-3)}, y^{(i-2)} \rangle$ such that x maps to y via a_2 (that is, $e(x) = a_2$) and y maps to y' via a_3 (that is, $e(y) = a_3$) and ... and $y^{(i-3)}$ maps to $y^{(i-2)}$ via the exponent a_i (that is, $e(y^{(i-3)}) = a_i$). (The longest tuple in an i -level tuple-set has i elements.)

There is a countable infinity of tuples in each tuple-set, as we establish in “Proof That Tuple-sets Exist as Defined” on page 12 and “Why There Are An Infinite Number of Tuples in Each Tuple-set” on page 12.

If $A = \{a_2, a_3, \dots, a_i\}$ is a finite exponent sequence, then of an i -level tuple t in the tuple-set T_A , we say that t is *determined* by the exponent sequence A and that A is *associated with* t . Finally, we say that the tuple-set T_A is *determined* by the sequence A . To review: given a tuple, we speak of the exponent sequence *associated with* it; given an exponent sequence, we speak of the tuple or tuple-set it *determines*.

As an example of (part of) a tuple-set: in Fig. 1, where $A = \{a_2, a_3, a_4\} = \{1, 1, 2\}$ and where we adopt the convention of orienting tuples vertically on the page, the tuple-set T_A includes:

- the tuple $\langle 1 \rangle$, because $e(1) \neq a_2$;
- the tuple $\langle 3, 5 \rangle$, because $e(3) = a_2 = 1$, but $e(5) = 4 \neq a_3 = 1$;
- the tuple $\langle 15, 23, 35 \rangle$, because $e(15) = a_2 = 1$, and $e(23) = a_3 = 1$, but $e(35) = 1 \neq a_4 = 2$.

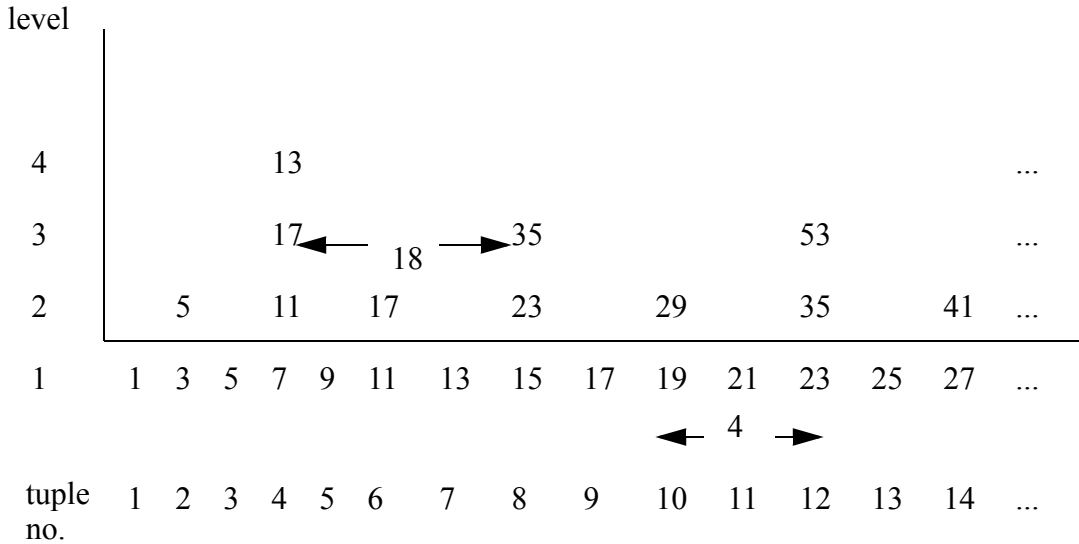


Fig. 1. Part of the tuple-set T_A associated with the sequence $A = \{1, 1, 2\}$

The number 18 between the arrows at level 3 and the number 4 between the arrows at level 1 are the values of the level 3 and level 1 distance functions, respectively, established by Lemma 1.0 (see “Lemma 1.0: Statement and Proof” on page 25).

In each i -level tuple-set T_A , where $i \geq 2$, for each odd, positive integer x there exists a tuple whose first element is x . The tuple may be one-level ($\langle x \rangle$), or two-level ($\langle x, y \rangle$), or ... or i -level ($\langle x, y, y', \dots, y^{(i-3)}, y^{(i-2)} \rangle$). Thus each tuple-set is non-empty.

Lemma 4.0 (see “Lemma 4.0: Statement and Proof” on page 31) establishes that a tuple-set T_A exists for each exponent sequence A .

Note: *our proofs will almost always involve only i -level tuples in i -level tuple-sets. We have included j -level tuples, where $j < i$, in our definition of tuple-set because we feel that these tuples are necessary to fully describe the structure of tuple-sets. If the level of a tuple in an i -level tuple-set is not specified, the reader should assume that the tuple is i -level.*

Ordering of Tuples in a Tuple-set

Tuples in a tuple-set T_A are linearly ordered by the natural order of their first elements. We denote a specific tuple in a tuple-set by $t_{(r)}$, where $r \geq 1$. If T_A is an i -level tuple-set, where $i \geq 2$, we denote the j th element of $t_{(r)}$ (if it exists in T_A) by $t_{(r)(j)}$, where $1 \leq j \leq i$.

The reader may find it helpful to imagine an i -level tuple-set, where $i \geq 2$, as a “picket fence” infinite to the right, with the tuples serving as the pickets, as suggested by Fig. 1 under “Tuple-set” on page 7.

Level in a Tuple-set

A *level* j in a tuple-set is defined as follows. If $A = \{a_2, a_3, \dots, a_i\}$, where $i \geq 2$, is a finite sequence of exponents, for $2 \leq j \leq i$ we shall call a_j the element at level j in the tuple. Subscripts of exponents in an exponent sequence are numbered beginning with 2 instead of with 1 so that the last subscript then indicates the number of levels in the corresponding tuple-set. Thus, for example, if $A = \{a_2\}$, then T_A is a 2-level tuple-set; if $A = \{a_2, a_3\}$, T_A is a 3-level tuple-set, etc. Level 1 is then the level containing the set of all possible tuple first elements $\{1, 3, 5, 7, \dots\}$ in T_A , that is, the set of odd, positive integers. Thus, for example in the tuple $\langle 17, 13, 5, 1 \rangle$, 17 is at level 1, 13 is at level 2, 5 is at level 3, and 1 is at level 4. We denote the element at level j in the tuple $t_{(r)}$ in a i -level tuple-set, where $i \geq 2$, by $t_{(r)(j)}$, where $1 \leq j \leq i$. (The element at level j is the j th element in the tuple.)

If a tuple has an element at level j , but none at level $j + 1$, we refer to the tuple as a *j -level tuple*. If the tuple also has an element at level $j + 1$, we sometimes refer to the tuple as a $(\geq j)$ -level tuple.

In the case that $A = \{a_2, a_3, \dots, a_i\}$, where $i \geq 2$, we refer to A as an *i -level exponent sequence*. An i -level exponent sequence consists of $(i - 1)$ exponents. The longest tuple in a tuple-set determined by an i -level exponent sequence is an i -level tuple.

Tuples Consecutive at Level j

Tuples *consecutive at level* j , $j \geq 2$, are defined as follows. Let $t_{(r)}$, $t_{(s)}$ be $(\geq j)$ -level tuples in some i -level T_A , where $i \geq 2$. If there is no $(\geq j)$ -tuple between $t_{(r)}$ and $t_{(s)}$, we say that $t_{(r)}$ and $t_{(s)}$ are *tuples consecutive at level* j . Here, “between” means relative to the natural linear ordering of tuples based on their first elements.

Thus, for example, in Fig. 1, the tuples numbered 4 and 8 are consecutive at level 3.

Extension of a Tuple-set

Let T_A be a tuple-set, where $A = \{a_2, a_3, \dots, a_i\}$. Then a tuple-set $T_{A'}$, where $A' = \{a_2, a_3, \dots, a_i, a_{i+1}\}$ is an *extension* of T_A . A proof that there exists such an extension for each exponent a_{i+1} is given in Lemma 3.0 (see “Lemma 3.0: Statement and Proof” on page 30).

Tuple-sets and Infinite Tuples

Tuples in a tuple-set are oriented vertically in accordance with our convention (see “Tuple-set” on page 7). Each tuple is a prefix of an infinite tuple (see “Tuple” on page 5). Therefore the infinite tuples whose i -level prefixes constitute the i -level tuples in an i -level tuple-set, are like-

wise oriented vertically, and thus occupy a single, vertical plane P_A that is infinite in the upward direction and to the right.

Let $A = \{a_2, a_3, \dots, a_i\}$ be an i -level exponent sequence. Let T_A be the associated tuple-set. The i -level tuples in T_A are i -level prefixes of infinite tuples. Call the set of these infinite tuples, S . Let $A' = \{a_2, a_3, \dots, a_i, a_{(i+1)}, \dots, a_{(i+k)}\}$ be an extension of A . Then the $(i+k)$ -level tuples in $T_{A'}$ are $(i+k)$ -level prefixes of infinite tuples. These infinite tuples constitute a proper subset of S .

Each tuple-set that is an extension of T_A — each tuple in each such tuple-set — is contained in the single, vertical plane P_A .

Distance Functions on Tuple-sets

Lemma 1.0

(a) Let $A = \{a_2, a_3, \dots, a_i\}$, where $i \geq 2$, be a sequence of exponents, and let $t_{(r)}$, $t_{(s)}$ be tuples consecutive at level i in T_A . Then $d(i, i)$, the distance between $t_{(r)}$ and $t_{(s)}$ at level i , is defined to be the absolute value of the difference between the level i elements of $t_{(r)}$ and $t_{(s)}$, that is, it is defined to be $|t_{(s)(i)} - t_{(r)(i)}|$, and is given by:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) Let $t_{(r)}$, $t_{(s)}$ be tuples consecutive at level i in T_A . Then $d(1, i)$, the distance between $t_{(r)}$ and $t_{(s)}$ at level 1, is defined to be the absolute value of the difference between the level 1 elements of $t_{(r)}$ and $t_{(s)}$, that is, it is defined to be $|t_{(s)(1)} - t_{(r)(1)}|$, and is given by:

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3}) \dots (2^{a_i})$$

Thus, in Fig. 1 under “Tuple-set” on page 7, the distance $d(3, 3)$ between $t_{8(3)} = 35$ and $t_{4(3)} = 17$ is $2 \cdot 3^{(3-1)} = 18$. The distance $d(1, 2)$ between $t_{12(1)} = 23$ and $t_{10(1)} = 19$ is $2 \cdot 2^1 = 4$.

Proof: see “Lemma 1.0: Statement and Proof” on page 25.

Remarks About the Distance Functions

(1) Strictly speaking, we should include the sequence A of exponents as arguments of $d(1, i)$, $d(i, i)$, but this notation would be cumbersome and, since typically this sequence is known, unnecessary.

(2) The distance functions make clear that, for each finite sequence of exponents, there exists an infinity of tuples produced by that sequence. (The equivalent of this statement is made in (Wirsching 1998) (p. 48).) The following table shows the distance relationships for $(i-j)$ -level elements of tuples consecutive at level $(i-j)$ in an i -level tuple-set, where $0 \leq j \leq (i-1)$. The distances are easily proved using Lemma 1.0. (An example is given following the table.) We only use the distances at levels 1 and i in this paper.

Table 1: Distances between elements of tuples $t_{(r)}$, $t_{(s)}$ consecutive at level i

Level	Distance between $(i - j)$ -level elements of tuples consecutive at level $(i - j)$, where $0 \leq j \leq (i - 1)$
i	$2 \cdot 3^{i-1}$
$i - 1$	$2 \cdot 3^{i-2} \cdot 2^{a_i}$
$i - 2$	$2 \cdot 3^{i-3} \cdot 2^{a_{i-1}} 2^{a_i}$
$i - 3$	$2 \cdot 3^{i-4} \cdot 2^{a_{i-2}} 2^{a_{i-1}} 2^{a_i}$
...	...
2	$2 \cdot 3 \cdot 2^{a_3} \dots 2^{a_{i-1}} 2^{a_i}$
1	$2 \cdot 2^{a_2} 2^{a_3} \dots 2^{a_{i-1}} 2^{a_i}$

For example, let x be an element at level $(i - 1)$ of an i -level tuple. Then, by the table, the element at level $(i - 1)$ in the next i -level tuple (that is, in the next tuple consecutive at level $(i - 1)$) = $(x + 2 \cdot 3^{i-2} \cdot 2^{a_i})$, and so it must be the case that

$$\frac{3(x + 2 \cdot 3^{i-2} \cdot 2^{a_i}) + 1}{2^{a_i}} = \frac{3x + 1}{2^{a_i}} + 2 \cdot 3^{i-1}$$

which, as the reader can check, is indeed the case.

(3) Lemma 1.0 makes clear that no two i -level tuples in a given i -level tuple-set have the same last element. In fact, the values of the last elements of i -level tuples in an i -level tuple-set always increase as one proceeds along the sequence of i -level tuples.

We now state the two lemmas that are required for our proof that tuple-sets exist as defined.

Every Possible 2-Level Tuple-set Exists

Lemma 2.0

For each exponent a_2 , a tuple-set T_A , where $A = \{a_2\}$, exists.

Proof: See “Lemma 2.0: Statement and Proof” on page 30.

Every Possible Extension of Each i -Level Tuple-set Exists

Lemma 3.0

Each i -level tuple-set T_A , where $A = \{a_2, a_3, \dots, a_i\}$ and $i \geq 2$, has an extension via each exponent a_{i+1} .

Proof: See “Lemma 3.0: Statement and Proof” on page 30.

Proof That Tuple-sets Exist as Defined

Lemma 4.0

For each exponent sequence $A = \{a_2, a_3, a_4, \dots, a_i\}$, where $i \geq 2$, there exists a tuple-set T_A .

Proof: See “Lemma 4.0: Statement and Proof” on page 31.

Lemmas 2.0, 3.0 and 4.0 establish, as part of their proofs, that there are an infinite number of tuples in each tuple-set. A plausible question at this point is: Why should there be? The answer is given in the next section.

On the Number of Tuple-sets

Lemma 4.5

For each $i \geq 2$, the number of i -level tuple-sets is countably infinite.

Proof:

Each i -level exponent sequence can be regarded as a representation of a positive integer in base 11, with “,” serving as the 11th symbol. (Representations involving “,,,...,” however, that is, involving two or more commas in succession, do not occur.) But since for each positive decimal integer d there is at least one i -level exponent sequence that contains d , and since the number of positive decimal integers is countably infinite, the result follows. \square

Why There Are An Infinite Number of Tuples in Each Tuple-set

Every finite exponent sequence — that is, every finite sequence of positive integers — determines an i -level tuple-set (“Lemma 4.0: Statement and Proof” on page 31), where $i \geq 2$. The last element (that is, the i -level element) of each tuple maps directly to one and only one odd, positive integer via one and only one exponent. Consider the tuple-set T_A determined by the exponent sequence $A = \{a_2, a_3, a_4, \dots, a_i\}$ where $i \geq 2$. T_A has an extension for *each* positive integer a_{i+1} (“Lemma 3.0: Statement and Proof” on page 30). But since the last element of each tuple in T_A maps directly to one and only one odd positive integer, and since by Lemma 3.0 (see “Lemma 3.0: Statement and Proof” on page 30) each tuple-set $T_{A'}$, $A' = \{a_0, a_1, a_2, \dots, a_i, a_{i+1}\}$, likewise has an extension for each positive integer a_{i+2} , etc., it follows that, for *each* a_i , there exists an *infinity* of tuples in T_A whose last elements directly map to their respective odd, positive integers *via* a_i . In short, the reason there are an infinite number of tuples in each i -level tuple-set is that (1) each i -

level tuple-set has an infinity of extensions, namely, one for each exponent a_{i+1} , but (2) each tuple maps directly to one and only one odd, positive integer via one and only one exponent.

Thus, in each i -level tuple-set T_A , where $i \geq 2$, the countable infinity of i -level non-counterexample prefixes consists of:

an infinity that have an extension via the exponent 1, and
an infinity that have an extension via the exponent 2, and
an infinity that have an extension via the exponent 3, and

...

If counterexamples exist, the same is true for i -level counterexample prefixes.

We now come to a lemma that is of crucial importance in our proof of the $3x + 1$ Conjecture.

On Non-Counterexample and Counterexample Tuples in a Tuple-set

Lemma 5.0

Assume a counterexample exists. Then for all $i \geq 2$, each i -level tuple-set contains an infinity of i -level counterexample tuples and an infinity of i -level non-counterexample tuples.

Proof: see “Lemma 5.0: Statement and Proof” on page 32.

Remark 1

This lemma establishes that there is no way to distinguish counterexamples from non-counterexamples on the basis of the *finite exponent sequences* associated with each. Of course, if a non-trivial cycle exists, then an infinite tuple $\langle x_1, x_2, \dots, x_1, x_2, \dots, x_1, x_2, \dots \rangle$ exists, and thus the finite tuple $\langle x_1, x_2, \dots, x_1 \rangle$ immediately tells us that a counterexample exists. But there is no requirement that a counterexample give rise to a non-trivial cycle. A counterexample can simply give rise to an infinite tuple in which no element recurs, and which has no element = 1.

To repeat: there is no way of telling from a *finite exponent sequence* that the sequence is associated with a counterexample. For example, the sequence $\{a_2, a_3, \dots, a_2, a_3, \dots, a_2, a_3, \dots\}$, in which $\{a_2, a_3, \dots, a_2\}$ is repeated, say, a trillion times, does not imply the existence of a counterexample cycle.

Remark 2

Lemma 5.0 implies that the set of all non-counterexample tuples is associated with the set of all finite exponent sequences and, if counterexamples exist, then the set of all counterexample tuples is likewise associated with the set of all finite exponent sequences.

We are now at the final stage of our preparation for the proof of the $3x + 1$ Conjecture. This stage is concerned with the first i -level tuple in an i -level tuple-set. This tuple is called the *anchor tuple* of the tuple-set.

Anchor and Anchor Tuple

Since tuples in a tuple-set are linearly ordered by the natural order of their first elements, in every i -level tuple-set, where $i \geq 2$, there is a unique first i -level tuple, which we call the *anchor*

tuple of the tuple-set. The last element, that is, the i -level element, of the anchor tuple we call the *anchor* of the anchor tuple, sometimes referring to it as the *i -level anchor*. We emphasize that in each i -level tuple-set there is *only one* anchor tuple: it is the first i -level tuple in the tuple-set.

Each element of an anchor tuple except, possibly, the first, (like each element of any tuple) is an odd, positive integer that is not a multiple of 3. The element is odd by definition of the $3x + 1$ function, C , and is not a multiple of 3 by “Lemma 9.0: Statement and Proof” on page 33.

Lemma 6.0

Let t be the i -level anchor tuple in an i -level tuple-set, where $i \geq 2$. Then the last element y of t , that is, the i -level element of t (which is the anchor), is a number less than $2 \cdot 3^{(i-1)}$.

Proof: see “Lemma 6.0: Statement and Proof” on page 32.

Definition of “Reduced Residue Class” and of “Complete Set of Reduced Residue Classes”

If a residue class mod m is such that each element of the class is relatively prime to m , then we call the class a *reduced residue class mod m* . Thus, for example, the residue class mod 6 whose minimum element is 5 is a reduced residue class mod 6. The set of all reduced residue classes mod m we call a *complete set of reduced residue classes mod m* .

Lemma 7.0

The set of i -level elements of all i -level tuples in an i -level tuple-set is a complete set of reduced residue classes modulo $2 \cdot 3^{(i-1)}$.

Proof: see “Lemma 7.0: Statement and Proof” on page 33.

Mark

Lemma 8.0

For each odd, positive integer x there exists a minimum $i = i_0$ such that for each $i \geq i_0$, x is the first element of the first i -level tuple in some i -level tuple-set, that is, x is the first element of an i -level anchor tuple in some i -level tuple-set. In terms of infinite tuples, this lemma states: if x is an odd, positive integer, then in the infinite tuple $\bar{t} = \langle x, y, y', \dots \rangle$, there exists a minimum level i_0 such that:

- $\bar{t}(i_0)$ is the i_0 -level anchor tuple in an i_0 -level tuple-set;
- $\bar{t}(i_0 + 1)$ is the $(i_0 + 1)$ -level anchor tuple in an $(i_0 + 1)$ -level tuple-set;
- $\bar{t}(i_0 + 2)$ is the $(i_0 + 2)$ -level anchor tuple in an $(i_0 + 2)$ -level tuple-set;
- etc.

Proof: see “Lemma 8.0: Statement and Proof” on page 33.

Remark

To describe the infinite sequence of anchor tuples in the lemma, we sometimes say, informally, “Once an anchor tuple, always an anchor tuple”.

We call the level i_0 in Lemma 8.0 the *mark* of the infinite tuple \bar{t} . We denote the mark i_0 by m . We write $m(\bar{t})$ to denote the mark of \bar{t} , and we write $\bar{t}(m)$ to denote the prefix (that is, finite tuple) corresponding to the mark m . This prefix is an anchor tuple.

For example, the mark of the infinite tuple $\langle 3, 5, 1, 1, 1, 1, \dots \rangle$ is at level 2 (namely, at 5) because 5 is the first element of the tuple that is less than $2 \cdot 3^{(i-1)}$ for some $i \geq 2$. Specifically, for $i = 2$, $2 \cdot 3^{(i-1)} = 6$, and $5 < 6$. As another example, consider the infinite tuple $\langle 433, 325, 61, 23, 35, \dots, 1, 1, 1, 1, \dots \rangle$. The mark is not at 325 (level 2) because for level 2, $2 \cdot 3^{(i-1)} = 6$ and 325 is not less than 6. The mark is not at 61 (level 3) because for level 3, $2 \cdot 3^{(i-1)} = 18$ and 61 is not less than 18. The mark is at 23 (level 4) because for level 4, $2 \cdot 3^{(i-1)} = 54$ and 23 is less than 54.

Infinite Tuples, Marks, and Tuple-sets

We here summarize the pertinent facts concerning infinite tuples, marks, and tuple-sets, because they are crucial for an understanding of our proof of the $3x + 1$ Conjecture.

By definition, an i -level tuple-set T_A , where $i \geq 2$, *includes* all i -level tuples t such that $A(t) = A$, that is, such that the exponent sequence associated with t is A . We emphasize *includes* because, by definition of tuple-set, the tuple-set also includes 1-level, 2-level, 3-level, ..., $(i-1)$ -level tuples (see “Tuple-set” on page 7). Another way of saying this is: an i -level tuple-set T_A , where $i \geq 2$, *includes* all prefixes $\bar{t}(i)$ of infinite tuples \bar{t} such that $A(\bar{t}(i)) = A$. Thus by abuse of language we may say that a tuple-set consists of (the prefixes of) a set of infinite tuples.

At this point it is appropriate that we describe the relationship between *successive* prefixes of an infinite tuple \bar{t} (counterexample or non-counterexample) and the tuple-sets in which the prefixes appear. Let $\bar{t} = \langle x_1, x_2, x_3, x_4, \dots \rangle$ and let $\{a_2, a_3, a_4, a_5, \dots\}$ be the associated exponents. That is,

x_1 maps to x_2 in one iteration of the $3x + 1$ function via a_2 ;
 x_2 maps to x_3 via one iteration of the $3x + 1$ function via a_3 ;
 etc.

Then, by definition of *tuple-set*:

in each tuple-set T_A determined by the exponent sequence $A = \{b_2, b_3, b_4, b_5, \dots, b_i\}$ such that $b_2 \neq a_2$, the tuple $\langle x_1 \rangle$ is an element;

in each tuple-set T_A determined by the exponent sequence $A = \{b_2, b_3, b_4, b_5, \dots, b_i\}$ such that $b_2 = a_2$, but $b_3 \neq a_3$, the tuple $\langle x_1, x_2 \rangle$ is an element;

in each tuple-set T_A determined by the exponent sequence $A = \{b_2, b_3, b_4, b_5, \dots, b_i\}$ such that $b_2 = a_2$, $b_3 = a_3$, but $b_4 \neq a_4$ the tuple $\langle x_1, x_2, x_3 \rangle$ is an element;

...

in the one tuple-set T_A determined by the exponent sequence $A = \{b_2, b_3, b_4, b_5, \dots, b_i\}$ such that $b_2 = a_2, b_3 = a_3, b_4 = a_4, \dots, b_i = a_i$ the tuple $\langle x_1, x_2, x_3, \dots, x_i \rangle$ is an element;

Let \bar{t} be an infinite tuple. It has a mark, m . Each prefix $\bar{t}(m+j)$ of \bar{t} , where $j \geq 0$, is an anchor tuple. But then, by abuse of language, we allow ourselves to say that each prefix $\bar{t}(i)$, where $i \geq 2$, is a *prefix of an anchor tuple* (namely, the anchor tuple $\bar{t}(m+j)$). Thus each prefix $\bar{t}(i)$, where $i \geq 2$, is a prefix of an *infinity* of anchor tuples.

Each infinite tuple \bar{t} is an independent entity. By this we mean that an infinite tuple \bar{t} is determined solely by its first element. Thus, informally, an infinite tuple does not somehow “acquire” properties depending on the tuple-set in which it has a prefix.

In an i -level tuple-set there is exactly one infinite tuple with an i -level mark, namely, the infinite tuple whose prefix is the anchor tuple. All other infinite tuples having i -level prefixes in the tuple-set must have marks greater than i (otherwise there would be two or more anchor tuples in a tuple-set, which is impossible). It may well be the case, however, that the mark of the infinite tuple whose i -level prefix is the anchor tuple in an i -level tuple-set is at a level less than i . The following is an example:

The infinite tuple $\bar{t} = \langle 7, 11, 17, 13, 5, 1, 1, 1, \dots \rangle$ has its mark at level 3 (namely at 17) because 17 is the first element of the tuple that is less than $2 \cdot 3^{(i-1)}$ for some $i \geq 2$. Here, $i = 3$, so $2 \cdot 3^{(i-1)} = 18$, and $17 < 18$. So $\langle 7, 11, 17 \rangle = \bar{t}(3)$ is an anchor tuple: specifically, it is the anchor tuple of the tuple-set T_A , where $A = \{1, 1\}$ (7 maps to 11 via the exponent 1; 11 maps to 17 via the exponent 1).

By our rule (see under “Mark” on page 14) expressed informally as “once an anchor tuple, always an anchor tuple”, we know that $\langle 7, 11, 17, 13 \rangle = \bar{t}(4)$ is also an anchor tuple: specifically, it is the anchor tuple of the 4-level tuple-set $T_{A'}$, where $A' = \{1, 1, 2\}$ (7 maps to 11 via the exponent 1, 11 maps to 17 via the exponent 1, 17 maps to 13 via the exponent 2).

But $\langle 7, 11, 17 \rangle = \bar{t}(3)$ is also present in the 4-level tuple-set $T_{A''}$, where $A'' = \{1, 1, 1\}$. The reason is that, since 17 maps to 13 via the exponent 2, not via the exponent 1, the tuple $\langle 7, 11, 17 \rangle$ is associated with merely an “approximation”, namely $\{1, 1\}$, to the exponent sequence $\{1, 1, 1\}$. But therefore, by definition of “tuple-set” (see under “Tuple-set” on page 7), it belongs in the tuple-set $T_{A''}$.

Complete Sets of Tuples

Definition of a “Complete” Set of Tuples

Let S be a set of i -level tuples, where $i \geq 2$. Then we say that S is *complete* if S is associated with the set of all i -level exponent sequences. Otherwise, we say that S is *incomplete*.

Lemma 8.5

Assume counterexamples exist. Let \bar{t}_{nc} , \bar{t}_c be non-counterexample and counterexample infinite tuples, respectively, with marks m_{nc} , m_c respectively.

Then for all levels $i \geq \max(m_{nc}, m_c) = i_0$, $A(\bar{t}_{nc}(i)) \neq A(\bar{t}_c(i))$, where $\max(u, v)$ denotes the maximum of u, v , and $A(t)$ denotes the exponent sequence associated with the tuple t .

Proof: Assume the contrary. Then for some $i \geq i_0$, $A(\bar{t}_{nc}(i)) = A(\bar{t}_c(i))$, which implies that a tuple-set exists having both a non-counterexample and a counterexample anchor tuple, which is impossible. \square

Lemma 8.7

If counterexamples do not exist, then

(a) *For each $i \geq 2$, the set of i -level non-counterexample anchor tuples is complete.*

(b) *Each non-counterexample infinite tuple has a prefix, namely, that determined by its mark, such that that prefix, and all larger prefixes, are elements of **complete** sets of non-counterexample anchor tuples.*

If counterexamples exist, then

(c) *For each $i \geq$ some i_0 , the set of i -level non-counterexample anchor tuples is incomplete, so that a complete set of i -level non-counterexample tuples must include tuples other than anchor tuples.*

(d) *Each non-counterexample infinite tuple has a prefix, namely, that determined by its mark, such that that prefix, and all larger prefixes, are elements of **incomplete** sets of non-counterexample anchor tuples.*

Proof

(a) Follows trivially from the fact that if counterexamples do not exist, all tuples in all tuple-sets are non-counterexample tuples.

(b) Follows trivially from the fact that the mark determines the smallest prefix of an infinite tuple that is an anchor tuple.

(c) By “Lemma 8.5” on page 16, if counterexamples exist, then for all $i \geq \max(m_{nc}, m_c) = i_0$, there exist i -level exponent sequences with which i -level anchor tuples are not associated. These are the exponent sequences with which i -level *counterexample* anchor tuples are associated. But by “Lemma 5.0” on page 13, each i -level tuple-set, regardless whether the anchor tuple is non-counterexample or counterexample, contains an infinity of non-counterexample tuples and an infinity of counterexample tuples. Thus to obtain a complete set of i -level non-counterexample tuples, it is necessary to include a non-counterexample tuple from each tuple-set having a counterexample anchor tuple.

(d) Follows directly from “Lemma 8.5” on page 16. \square

Proof of the $3x + 1$ Conjecture

Theorem. The $3x + 1$ Conjecture is true.

We give three proofs.

First Proof

Before we begin, we remind the reader of several facts that will be of fundamental importance in this proof. Except where otherwise indicated, these facts were established in the section, “Infinite Tuples, Marks, and Tuple-sets” on page 15.

Each infinite tuple \bar{t} , whether counterexample or non-counterexample, consists of an infinity of anchor tuples, namely, of the tuples (prefixes) $\bar{t}(m)$, $\bar{t}(m + 1)$, $\bar{t}(m + 2)$, ..., where m is the mark of \bar{t} . Each prefix $\bar{t}(j)$ of \bar{t} , where $j \geq 2$, is therefore the prefix of an infinity of anchor tuples.

No prefix $\bar{t}(j)$ of \bar{t} , where $2 \leq j < m$, is an anchor tuple. But each such prefix is the prefix of an infinity of anchor tuples, namely, the tuples (prefixes) $\bar{t}(m)$, $\bar{t}(m + 1)$, $\bar{t}(m + 2)$, ..., where m is the mark of \bar{t} .

Each prefix $\bar{t}(j)$ of \bar{t} , where $j \geq 2$, is associated with a finite exponent sequence. (Follows from “Lemma 5.0” on page 13.)

Each i -level tuple-set consists of an infinity of prefixes of infinite tuples, each prefix being a (finite) tuple. The mark of exactly one such infinite tuple, namely, the infinite tuple whose prefix is the anchor tuple of the tuple-set, is $\leq i$. The mark of all the other infinite tuples is greater than i .

We now begin our proof.

1. Assume counterexamples exist. Let $\{\bar{t}_c\}$ denote the set of all counterexample infinite tuples. Let $\{\bar{t}_{nc}\}$ denote the set of all non-counterexample infinite tuples.

2. If a set S of j -level prefixes is associated with the set of all j -level exponent sequences, then we say that S is complete. (See “Complete Sets of Tuples” on page 16.)

By “Lemma 5.0” on page 13 we know that:

(1)

The set $\{\bar{t}_c(2)\}$ of all 2-level prefixes of all infinite tuples in $\{\bar{t}_c\}$ is complete.

The set $\{\bar{t}_c(3)\}$ of all 3-level prefixes of all infinite tuples in $\{\bar{t}_c\}$ is complete.

The set $\{\bar{t}_c(4)\}$ of all 4-level prefixes of all infinite tuples in $\{\bar{t}_c\}$ is complete.

...

(2)

The set $\{\bar{t}_{nc}(2)\}$ of all 2-level prefixes of all infinite tuples in $\{\bar{t}_{nc}\}$ is complete.
 The set $\{\bar{t}_{nc}(3)\}$ of all 3-level prefixes of all infinite tuples in $\{\bar{t}_{nc}\}$ is complete.
 The set $\{\bar{t}_{nc}(4)\}$ of all 4-level prefixes of all infinite tuples in $\{\bar{t}_{nc}\}$ is complete.
 ...

We emphasize that the statements in (1) and (2) concern prefixes of infinite tuples. A prefix of an infinite tuple is not necessarily an anchor tuple, although it is a prefix of an infinity of anchor tuples.

3. Recall that if t is a prefix of an infinite tuple (that is, if t is a finite tuple), then we denote the exponent sequence associated with t by $A(t)$.

Let \bar{t}_c be any counterexample infinite tuple, and let $\bar{t}_c(i)$ be any prefix of \bar{t}_c . Then by (2) there exists a non-counterexample infinite tuple \bar{t}_{nc} and a prefix $\bar{t}_{nc}(i)$ such that $A(\bar{t}_{nc}(i)) = A(\bar{t}_c(i))$. Therefore $\bar{t}_c(i)$ is the prefix of a counterexample tuple in a tuple-set having a non-counterexample anchor tuple.

But since $\bar{t}_c(i)$ is any prefix of any counterexample infinite tuple, we conclude that, if \bar{t}_c is a counterexample infinite tuple, then each prefix of \bar{t}_c is the prefix of a tuple in a tuple-set having a non-counterexample anchor tuple.

It therefore follows that no counterexample tuple is an anchor tuple, or, expressed differently, that there are no finite marks in counterexample infinite tuples, which is impossible. This contradiction gives us our proof of the $3x + 1$ Conjecture. \square

4. Of course, we can, in step 3, interchange “ \bar{t}_c ” and “ \bar{t}_{nc} ”, and related terms, and obtain a proof that there are no finite marks in non-counterexample infinite tuples, which is also impossible. Hence we have a second proof of the Conjecture.

5. Another argument based on (1) and (2) is the following:

Let \bar{t}_c be a counterexample infinite tuple, and let m_c be its mark. Consider any prefix $\bar{t}_c(mc + j)$. Then by (2), there exists a non-counterexample infinite tuple \bar{t}_{nc} such that $A(\bar{t}_{nc}(mc + j)) = A(\bar{t}_c(mc + j))$.

But since all prefixes in (2) are prefixes of anchor tuples, this implies that there exists a non-counterexample anchor tuple such that, among the counterexample tuples in its tuple-set, at least one has a mark that is less than that of the non-counterexample anchor tuple, which is impossible. This contradiction gives us our proof of the $3x + 1$ Conjecture. \square

6. Another argument based on (1) and (2) is the following:

Because no tuple-set can have two anchor tuples, we can state that:

(3)

For each non-counterexample infinite tuple \bar{t}_{nc} , and for each $j \geq 0$, if $A(\bar{t}_{nc}(m_{nc} + j)) = A(\bar{t}_c(m_{nc} + j))$, where m_{nc} is the mark of \bar{t}_{nc} , and \bar{t}_c is a counterexample infinite tuple, then m_c must be greater than m_{nc} , where m_c is the mark of \bar{t}_c .

Observe that, if, for specific \bar{t}_{nc} , and \bar{t}_c , $A(\bar{t}_{nc}(m_{nc} + j)) = A(\bar{t}_c(m_{nc} + j))$ but $A(\bar{t}_{nc}(m_{nc} + j + 1)) \neq A(\bar{t}_c(m_{nc} + j + 1))$, then by the completeness properties of (1) and (2), there must nevertheless be a \bar{t}_{nc}' such that, for some $h \geq 0$, $A(\bar{t}_{nc}'(m_{nc}' + h)) = A(\bar{t}_c(m_{nc}' + h))$, where $m_{nc}' + h = m_{nc} + j + 1$, and m_c must be greater than $m_{nc}' + h$. It is not possible that $m_c \leq m_{nc}' + h$, because that would imply there are two anchor tuples in the same tuple-set.

We conclude that there are no finite marks in counterexample infinite tuples, which is impossible. This contradiction gives us our proof of the $3x + 1$ Conjecture. \square

7. As a useful cross-check on the validity of our proof, we show that it does not apply to what we have called “ $3x + 1$ -like problems” in which counterexamples are known to exist. In particular, we show that our proof does not also apply to the $3x - 1$ function, which has known counterexamples. This we do in “Appendix B — $3x + C$ Functions” on page 45. (If our proof did apply to these problems, then it would have a flaw.)

Second Proof

We emphasize that in this proof, we are *comparing* the set of all tuple-sets if counterexamples exist, and the set of all tuple-sets if counterexamples do not exist. Such a comparison in no way implies an assumption that counterexamples exist and do not exist at the same time, which, of course, would be absurd. A more detailed argument for the legitimacy of comparing the two sets will be found in “Appendix D — On Comparing “Counterexamples Exist” and “Counterexamples Do Not Exist”” on page 50.

1. It is straightforward to show that if an odd, positive integer maps to 1, then it does so regardless whether counterexamples exist or not. (See Lemma 8.8 in “Are We Near a Solution to the $3x + 1$ Problem?” on www.occampress.com.) Thus, for example, 13 maps to 1, and will continue to do so even if the $3x + 1$ Conjecture is proved false.)

2. Consider the set of all two-level tuple-sets. Since the anchor for any such set is either 1 or 5 (these are the only two odd, positive integers relatively prime to $2 \cdot 3^{2-1} = 6$), and since both of these numbers map to 1, we know that the anchor tuple for each two-level tuple-set is non-counterexample.

3. Assume counterexamples exist. Then by “Lemma 5.0” on page 13 we know that each two-level tuple-set has an infinity of non-counterexample two-level tuples and an infinity of two-level

counterexample tuples. Choose any such tuple-set T_A . In the standard linear ordering of two-level tuples, there must be a first level 2 element y_c that is a counterexample. Since the anchor of each 2-level anchor tuple is either 1 or 5, both of which are non-counterexamples, we know that y_c is not an element of the anchor tuple of T_A . So the level 2 element immediately preceding y_c must be a non-counterexample element y_{nc} . If we apply the level-2 distance function $d(2, 2)$ in “Lemma 1.0” on page 10 to y_{nc} , we must get $y_{nc} + d(2, 2) = y_{nc} + 6 = y_c$.

4. But if counterexamples do *not* exist, and we apply the level-2 distance function $d(2, 2)$ to y_{nc} in the same tuple-set T_A , we must get the same value as we did in step 3, namely, $y_{nc} + 6$, since by step 1 y_{nc} is the same as in step 3, and since the distance function does not depend on the existence or non-existence of counterexamples.

We can apply the same argument to each two-level counterexample in T_A , and to all other two-level tuple-sets.

5. Let G denote the set of all tuples in all tuple-sets if counterexamples do not exist. Let $g(A, i, k, j)$ denote the j th element of the k th i -level tuple in the tuple-set T_A in G . Let H denote the set of all tuples in all tuple-sets if counterexamples exist. Let $h(A, i, k, j)$ denote the j th element of the k th i -level tuple in the tuple-set T_A in H .

Thus, for example, let $A = \{3, 4\}$. Then $g(A, 3, 1, 3) = h(A, 3, 1, 3) = 1$. The reason is that:

the tuple $\langle 13, 5, 1 \rangle$ is the anchor tuple of the tuple-set T_A ;
 thus 1 = the 3rd element of the 1st 3-level tuple in the tuple-set, and
 this tuple is present regardless whether or not counterexamples exist (step 1).

Another way of expressing (informally) the third statement is: 13 maps to 1 today, it will continue to map to 1 if the $3x + 1$ Conjecture is proved false tomorrow, and it will continue to map to 1 if the Conjecture is proved true tomorrow.

(Readers who believe the functions g, h are not well-defined should see “Argument that the Functions g, h are Well-defined” on page 51 .)

To say that *there is no difference* between the set of all tuple-sets if counterexamples exist and the set of all tuple-sets if counterexamples do not exist — that is, to say that the set of all tuple-sets is the same whether or not counterexamples exist — is to say:

For all A, i, k, j , $g(A, i, k, j) = h(A, i, k, j)$.

To say that *there is a difference* between the set of all tuple-sets if counterexamples exist and the set of all tuple-sets if counterexamples do not exist is to say:

There exist A, i, k, j such that $g(A, i, k, j) \neq h(A, i, k, j)$.

6. There are two types of counterexample:

(1) The counterexample x is the first element of a non-trivial infinite cycle, that is, there is an infinite tuple $\langle x, y, \dots, z, x, y, \dots, z, x, \dots \rangle$. There can be no such infinite tuple if counterexamples do not exist.

(2) The counterexample x is the first element of an infinite tuple $\langle x, y, \dots \rangle$ such that 1 never appears in the tuple. There can be no such infinite tuple if counterexamples do not exist.

7. Step 4 implies that *there is no difference* between the set of all tuple-sets if counterexamples exist and the set of all tuple-sets if counterexamples do not exist.

Step 6 implies that *there is a difference* between the set of all tuple-sets if counterexamples exist and the set of all tuple-sets if counterexamples do not exist.

This contradiction gives us our proof of the $3x + 1$ Conjecture. \square

8. As a useful cross-check on the validity of our proof, we show that it does not apply to what we have called “ $3x + 1$ -like problems” in which counterexamples are known to exist. In particular, we show that our proof does not also apply to the $3x - 1$ function, which has known counterexamples. This we do in “Appendix B — $3x + C$ Functions” on page 45. (If our proof did apply to these problems, then it would have a flaw.)

Third Proof

1. Assume counterexamples exist. Then there exists a first level $i = i_0$ at which the set of i -level non-counterexample anchor tuples and the set of i -level counterexample anchor tuples are each incomplete (See “Complete Sets of Tuples” on page 16.) If this incompleteness property did not exist, then there would be a tuple-set with two anchor tuples, which is impossible.

2. By “Lemma 18.0: Statement and Proof” on page 42 we know that 1 is mapped to by each exponent sequence A , possibly followed by a buffer exponent. However, the size of the element mapping directly to 1 may be arbitrarily large. For example, the set of odd, positive integers mapping directly to 1 is $\{1, 5, 21, 85, 341, \dots\}$. Thus, for a given A , the last element of the tuple mapping to 1 is any element of this set.

3. Choose any $i \geq i_0$. Recall that if t is a prefix of an infinite tuple (that is, if t is a finite tuple), then we denote the exponent sequence associated with t by $A(t)$.

Now choose the pair of i -level anchors 1 and y_c , where the latter is a counterexample. Then $A(\bar{t}_{nc}(i)) \neq A(\bar{t}_c(i))$, where \bar{t}_{nc} is the non-counterexample infinite tuple such that the last element of $\bar{t}_{nc}(i)$ is 1, and $\bar{t}_c(i)$ is the counterexample infinite tuple such that the last element of $\bar{t}_c(i)$ is y_c .

Let the level j be the smallest level such that, for our two tuples (prefixes), $A(\bar{t}_{nc}(j)) \neq A(\bar{t}_c(j))$. It is straightforward to show that we can always find y_c such that j must be less than i . We are assured that $A(\bar{t}_{nc}(j-1)) = A(\bar{t}_c(j-1))$ because for levels k less than i_0 the set of non-counterexample anchor tuples is complete, and by “Lemma 5.0” on page 13, there is a complete set of counterexample k -level prefixes.

4. Let the exponent sequence $B = A(\bar{t}_c(j) \dots \bar{t}_c(i))$, that is, let B equal the exponent sequence associated with the suffix $\bar{t}_c(j) \dots \bar{t}_c(i)$ of the tuple (prefix) $\bar{t}_c(i)$. By step 3 we know that $B \neq (C = A(\bar{t}_{nc}(j) \dots \bar{t}_{nc}(i)))$. We also know that the inequality of B and C will remain for all higher levels, since the anchors 1 and y_c will be anchors at all higher levels, and since the number that a tuple element maps to does not depend on the level of the element.

Then by step 2 we know that, for some level i , there exists an anchor y_{nc}' such that y_{nc}' is the anchor of an i -level non-counterexample tuple and such that y_{nc}' is mapped to by B . But this contradicts step 3 and thus gives us our proof of the $3x + 1$ Conjecture.

5. As a useful cross-check on the validity of our proof, we show that it does not apply to what we have called “ $3x + 1$ -like problems” in which counterexamples are known to exist. In particular, we show that our proof does not also apply to the $3x - 1$ function, which has known counterexamples. This we do in “Appendix B — $3x + C$ Functions” on page 45. (If our proof did apply to these problems, then it would have a flaw.)

Conclusion

We have proved that the $3x + 1$ Conjecture is true.

References

Lagarias, J., (1985), “The $3x + 1$ Problem and Its Generalizations”, *American Mathematical Monthly*, **93**, 3-23.

Wirsching, Günther J.. *The Dynamical System Generated by the $3n + 1$ Function*, Springer-Verlag, Berlin, Germany, 1998.

Appendix A — Statement and Proof of Each Lemma

Lemma 1.0: Statement and Proof

(a) Let $A = \{a_2, a_3, \dots, a_i\}$, where $i \geq 2$, be a sequence of exponents, and let $t_{(r)}$, $t_{(s)}$ be tuples consecutive at level i in T_A . Then $d(i, i)$, the distance between $t_{(r)}$ and $t_{(s)}$ at level i , is defined to be the absolute value of the difference between the level i elements of $t_{(r)}$ and $t_{(s)}$, that is, it is defined to be $|t_{(s)(i)} - t_{(r)(i)}|$, and is given by:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) Let $t_{(r)}$, $t_{(s)}$ be tuples consecutive at level i in T_A . Then $d(1, i)$, the distance between $t_{(r)}$ and $t_{(s)}$ at level 1, is defined to be the absolute value of the difference between the level 1 elements of $t_{(r)}$ and $t_{(s)}$, that is, it is defined to be $|t_{(s)(1)} - t_{(r)(1)}|$, and is given by:

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3}) \dots (2^{a_i})$$

Thus, in Fig. 1 in the section “Tuple-set” on page 7, the distance $d(3, 3)$ between $t_{8(3)} = 35$ and $t_{4(3)} = 17$ is $2 \cdot 3^{(3-1)} = 18$. The distance $d(1, 2)$ between $t_{12(1)} = 23$ and $t_{10(1)} = 19$ is $2 \cdot 2^1 = 4$.

Proof:

The proof is by induction.

Proof of Basis Step for Parts (a) and (b) of Lemma 1.0:

Let $t_{(r)}$ and $t_{(s)}$ be the first and second 2-level tuples, in the standard linear ordering of tuples based on their first elements, that are consecutive at level $i = 2$ in T_A . (See Fig. 2 (1).)

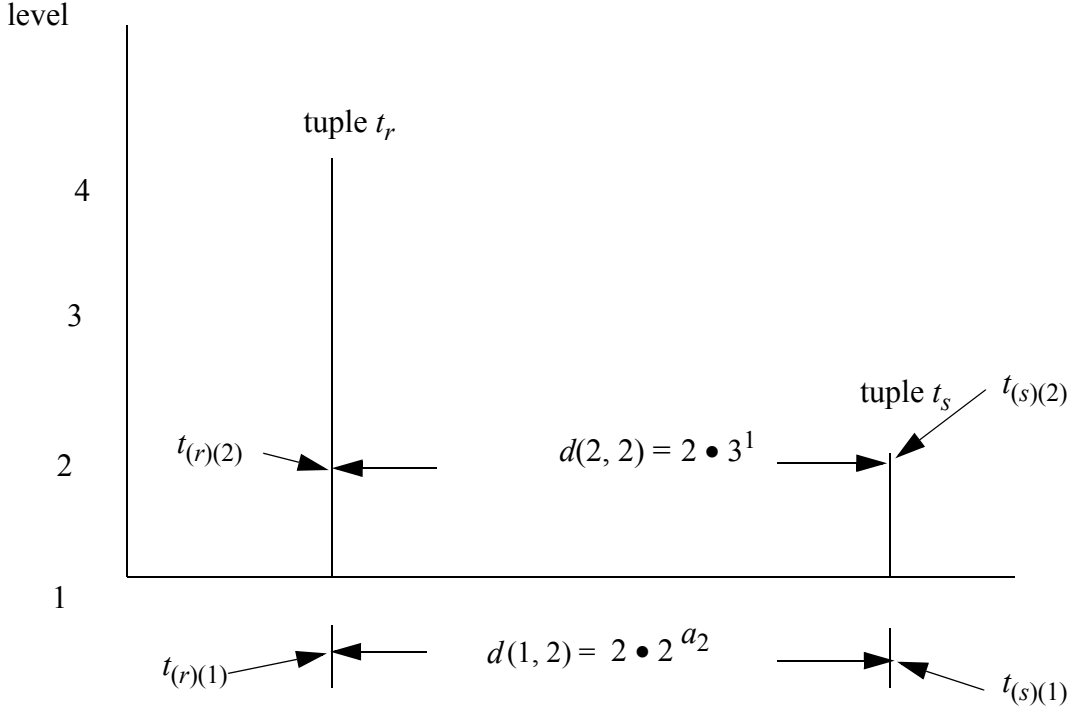


Fig. 2 (1). Illustration for proof of Basis Step of Lemma 1.0.

Then we have:

$$\frac{3t_{(r)(1)} + 1}{2^{a_2}} = t_{(r)(2)} \quad (1.1)$$

and since, by definition of $d(1, 2)$,

$$t_{(s)(1)} = t_{(r)(1)} + d(1, 2)$$

we have:

$$\frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}} = t_{(s)(2)} \quad (1.2)$$

Therefore, since, by definition of $d(i, i)$,

$$t_{(r)(2)} + d(2, 2) = t_{(s)(2)}$$

we can write, from (1.1) and (1.2):

$$\frac{3t_{(r)(1)} + 1}{2^{a_2}} + d(2, 2) = \frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}}$$

By elementary algebra, this yields:

$$2^{a_2}d(2, 2) = 3 \cdot d(1, 2)$$

Now $d(2, 2)$ must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level i , it must be the smallest such even number, whence it follows that $d(2, 2)$ must $= 3 \cdot 2$, and necessarily

$$d(1, 2) = 2 \cdot 2^{a_2}$$

A similar argument establishes that $d(2, 2)$ and $d(1, 2)$ have the above values for every other pair of tuples consecutive at level 2.

Thus we have our proof of the Basis Step for parts (a) and (b) of Lemma 1.0.

Proof of Induction Step for Parts (a) and (b) of Lemma 1.0

Assume the Lemma is true for all levels j , $2 \leq j \leq i$.

Let $t_{(r)}$ and $t_{(s)}$ be tuples consecutive at level i , and let $t_{(r)}$ and $t_{(f)}$ be tuples consecutive at level $i + 1$. (See Fig. 2 (2).)

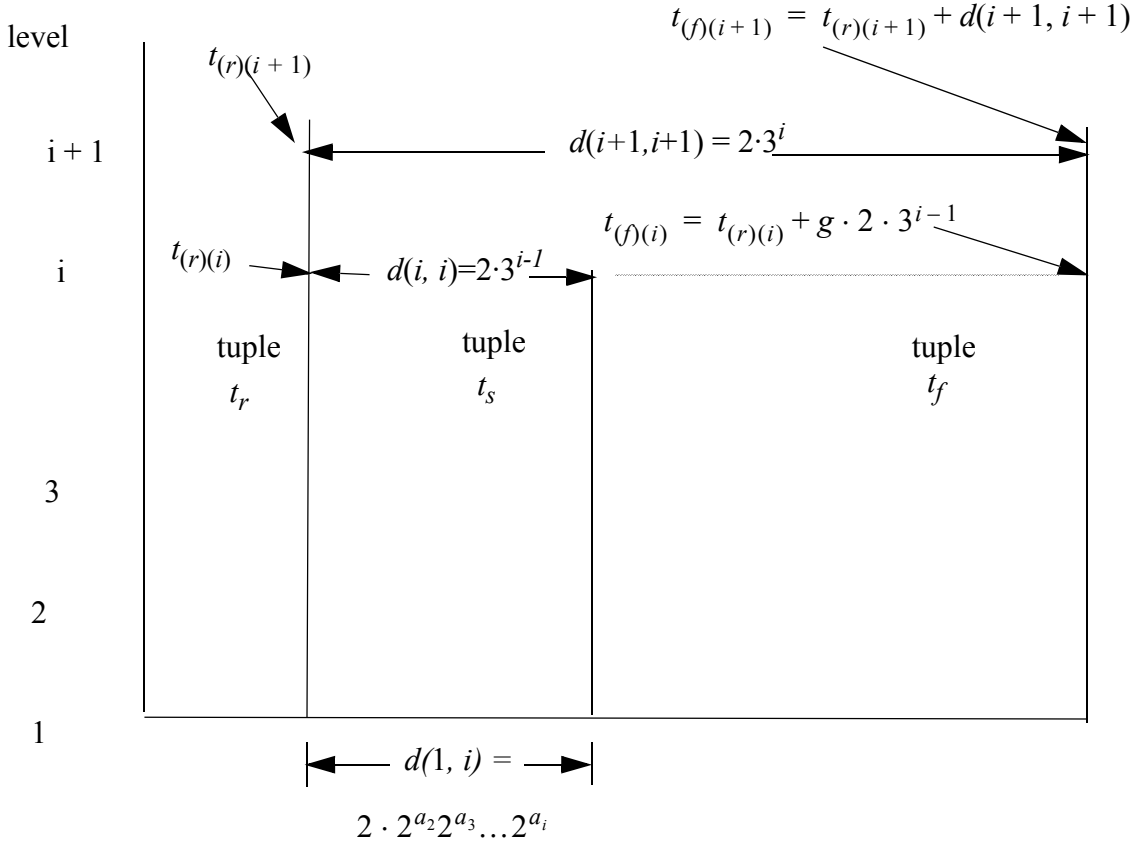


Fig. 2 (2). Illustration for proof of Induction Step of Lemma 1.0.

Then we have:

$$\frac{3t_{(r)(i)} + 1}{2^{a_{i+1}}} = t_{(r)(i+1)}$$

and since, by definition of $d(i, i)$,

$$t_{(f)(i)} = t_{(r)(i)} + g \cdot d(i, i)$$

for some $g \geq 1$, we have:

$$\frac{3(t_{(r)(i)} + g \cdot d(i, i)) + 1}{2^{a_{i+1}}} = t_{(f)(i+1)}$$

Thus, since

$$t_{(r)(i+1)} + d(i+1, i+1) = t_{(f)(i+1)}$$

we can write:

$$\frac{3t_{(r)(i)} + 1}{2^{a_{i+1}}} + d(i+1, i+1) = \frac{3(t_{(r)(i)} + gd(i, i)) + 1}{2^{a_{i+1}}}$$

This yields, by elementary algebra:

$$2^{a_{i+1}}d(i+1, i+1) = 3 \cdot gd(i, i)$$

As in the proof of the Basis Step, $d(i+1, i+1)$ must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level $i+1$, it must be the smallest such even number. Thus $d(i+1, i+1) = 3 \cdot d(i, i)$, and

$$g \cdot d(i, i) = 2^{a_{i+1}}d(i, i) \quad .$$

Hence

$$g = 2^{a_{i+1}}$$

Now g is the number of tuples consecutive at level i that must be “traversed” to get from $t_{(r)}$ to $t_{(f)}$. By inductive hypothesis, $d(1, i)$ for *each* pair of these tuples is:

$$d(1, i) = 2 \cdot 2^{a_2} \cdot 2^{a_3} \cdot \dots \cdot 2^{a_i}$$

hence, since

$$g = 2^{a_{i+1}}$$

we have

$$d(1, i+1) = d(1, i) \cdot 2^{a_{i+1}} \quad .$$

A similar argument establishes that $d(i+1, i+1)$ and $d(1, i+1)$ have the above values for every pair of tuples consecutive at level $i+1$.

Thus we have our proof of the Induction Step for parts (a) and (b) of Lemma 1.0. The proof of Lemma 1.0 is completed. \square

Lemma 2.0: Statement and Proof

For each exponent a_2 , a tuple-set T_A , where $A = \{a_2\}$, exists.

Proof:

By Lemma 12.0 (see “Lemma 12.0: Statement and Proof” on page 36) we know that each range element is mapped to by all exponents of one parity only. Then since 5 is mapped to by 3 via the exponent 1, we know that 5 is mapped to by all odd exponents. Since 1 is mapped to by 1 via the exponent 2, we know that 1 is mapped to by all even exponents. Both 1 and 5 are level-2 anchors, since each is less than $2 \cdot 3^{2-1} = 6$. Therefore each tuple $\langle x, 5 \rangle$, where x maps to 5 via the odd exponent a_2 is the anchor tuple of a tuple-set, and each tuple $\langle x', 1 \rangle$, where x maps to 1 via the even exponent a_2' , is the anchor tuple of a tuple-set. The result follows by Lemma 1.0 (a) and (b) (see “Lemma 1.0: Statement and Proof” on page 25), which assures us of an infinite number of tuples in each 2-level tuple-set. \square

Lemma 3.0: Statement and Proof

Each i -level tuple-set, where $i \geq 2$, can be extended by each even or odd exponent a_{i+1} .

Proof:

By Lemma 2.0 (see “Lemma 2.0: Statement and Proof” on page 30), for each exponent a_2 , a tuple-set $T_{A'}$, where $A' = \{a_2\}$, exists. So we show that for each exponent $a_2 = a_{i+1}$, the sequence of first elements of all tuples in $T_{A'}$ has at least one element in common with the sequence of i -level elements in T_A .

The sequence of i -level elements in the i -level tuple-set T_A is given by

$$2 \cdot 3^{i-1}k + y \tag{3.1}$$

where $k \geq 0$ and y is an i -level anchor, that is, y is an odd, positive integer that is less than or equal to, and relatively prime to, $2 \cdot 3^{(i-1)}$.

The sequence of 1-level elements of $T_{A'}$ is given by

$$\frac{2^{a_2}y' - 1}{3} + j2 \cdot 2^{a_2} \tag{3.2}$$

where $y' = 1$ or 5 is a 2-level anchor and $j \geq 0$ (see “Lemma 1.0: Statement and Proof” on page 25). Specifically, y' is 1 if $a_2 = a_{i+1}$ is even, and y' is 5 if $a_2 = a_{i+1}$ is odd. The left-hand term of (3.2) gives the value of the first element x of the level-1 sequence of $T_{A'}$, because

$$\frac{3x + 1}{2^{a_2}} = y'$$

and an anchor, namely, y' , is the smallest i -level element (in this case 2-level element) of an i -level tuple-set. The right-hand term of (3.2) is j times the difference between successive first elements of $T_{A'}$ (see “Lemma 1.0: Statement and Proof” on page 25).

Setting (3.1) equal to (3.2), we must prove that a solution j, k exists to the equation

$$2 \cdot 3^{i-1}k + y = \frac{2^{a_2}y' - 1}{3} + j2 \cdot 2^{a_2}$$

Multiplying through by 3, then dividing through by 2, which we can do since $3y + 1$ is even, we get

$$3^i k + \frac{3y+1}{2} = 2^{a_2-1} y' + 3j2^{a_2}$$

Rearranging terms, we have

$$3^i k - 3j2^{a_2} = -\frac{3y+1}{2} + 2^{a_2-1} y' \quad (3.3)$$

or

$$3(3^{i-1}k - j2^{a_2}) = -\frac{3y+1}{2} + 2^{a_2-1} y'$$

The right-hand side of the equation must be a multiple of 3, and so we can divide both sides by 3 and write:

$$3^{i-1}k - 2^{a_2}j = U$$

This is an equation of the form

$$au + bv = c$$

and a basic fact of Diophantine Equations states that such an equation has a solution u, v if and only if (a, b) divides c . In our case,

$$(3^{i-1}, 2^{a_2}) = 1$$

and so (3.3) has a solution j, k .

Lemma 1.0 (see “Lemma 1.0: Statement and Proof” on page 25) then assures us of an infinity of i -level elements in T_A that have extensions via the exponent $a_2 = a_{i+1}$, thus creating the tuple-set $T_{A''}$, where $A'' = \{a_2, a_3, \dots, a_i, a_{i+1}\}$. \square

Lemma 4.0: Statement and Proof

For each exponent sequence $A = \{a_2, a_3, \dots, a_i\}$, where $i \geq 2$, there exists a tuple-set T_A determined by A .

Proof:

The proof is by induction.

Basis Step:

By Lemma 2.0 (see “Lemma 2.0: Statement and Proof” on page 30) we know that there is a 2-level tuple-set for each exponent a_2 .

Inductive Step:

Assume the Lemma is true for all j -level exponent sequences $2 \leq j \leq i$. But then by Lemma 3.0 (see “Lemma 3.0: Statement and Proof” on page 30) it is true for all tuple-sets determined by $(i + 1)$ -level exponent sequences. \square

Lemma 5.0: Statement and Proof

Assume a counterexample exists. Then for all $i \geq 2$, each i -level tuple-set contains an infinity of i -level counterexample tuples and an infinity of i -level non-counterexample tuples.

Proof:

1. Assume counterexamples exist. Then:

There is a countable infinity of non-counterexample range elements.

Proof: Each non-counterexample maps to a range element, by definition of *range element*.

Each range element is mapped to by an infinity of elements

(“Lemma 12.0: Statement and Proof” on page 36). A countable infinity of these are range elements (proof of “Lemma 18.0: Statement and Proof” on page 42).

There is a countable infinity of counterexample range elements.

Proof: same as for non-counterexample case.

2. For each finite exponent sequence A , and for each range element y , non-counterexample or counterexample, there is an x that maps to y via A possibly followed by a buffer exponent (“Lemma 18.0: Statement and Proof” on page 42). The presence of the buffer exponent does not change the fact that x is the first element of a tuple associated with the exponent A . \square

Lemma 6.0: Statement and Proof

Let t be the i -level anchor tuple in an i -level tuple-set, where $i \geq 2$. Then the last element y of t , that is, the i -level element of t (which is the anchor), is a number less than $2 \cdot 3^{(i-1)}$.

Proof:

By definition of *i -level anchor tuple*, t is the first i -level tuple in an i -level tuple-set. Hence there are no i -level tuples to the left of t under our convention for ordering tuples from left to right in a tuple-set. By the distance function defined in Lemma 1.0 (a), the distance between the last elements of consecutive i -level tuples in an i -level tuple-set is $2 \cdot 3^{(i-1)}$. But if there is no i -level tuple to the left of t , it follows that the last element y of t must be less than $2 \cdot 3^{(i-1)}$. \square

Lemma 7.0: Statement and Proof

The set of i -level elements of all i -level tuples in an i -level tuple-set is a complete set of reduced residue classes modulo $2 \cdot 3^{(i-1)}$.

Proof:

Let T_A be an i -level tuple-set. Since the first i -level tuple t in T_A is an anchor tuple, the last element y of t is an anchor. By Lemma 6.0 (see “Lemma 6.0: Statement and Proof” on page 32), $y < 2 \cdot 3^{i-1}$. Since the range of the $3x + 1$ function C is the set of odd, positive integers not divisible by 3 (see “Lemma 9.0: Statement and Proof” on page 33), the result follows by part (a) of Lemma 1.0 (see “Lemma 1.0: Statement and Proof” on page 25), which states that the distance between i -level elements of successive i -level tuples in an i -level tuple-set is $2 \cdot 3^{i-1}$. \square

Lemma 8.0: Statement and Proof

For each odd, positive integer x there exists a minimum $i = i_0$ such that for each $i \geq i_0$, x is the first element of the first i -level tuple in some i -level tuple-set, that is, x is the first element of an i -level anchor tuple in some i -level tuple-set. In terms of infinite tuples, this lemma states: if x is an odd, positive integer, then in the infinite tuple $\bar{t} = \langle x, y, y', \dots \rangle$, there exists a minimum level i_0 such that:

- $\bar{t}(i_0)$ is the i_0 -level anchor tuple in an i_0 -level tuple-set;
- $\bar{t}(i_0 + 1)$ is the $(i_0 + 1)$ -level anchor tuple in an $(i_0 + 1)$ -level tuple-set;
- $\bar{t}(i_0 + 2)$ is the $(i_0 + 2)$ -level anchor tuple in an $(i_0 + 2)$ -level tuple-set;
- etc.

Proof:

The following proof is an edited version of a proof by Sanjai Gupta. Any errors it contains are entirely our own.

Let x be an odd, positive integer. Then x is the first element of an infinite tuple $\bar{t} = \langle x, y, \dots \rangle$. With each increment of i , $i \geq 2$, the element of \bar{t} at level i increases by at most a factor of 2, since for each exponent except 1, $C(y) < y$, and for exponent 1, $C(y) \leq 2y$. However, with each increment of i , $2 \cdot 3^{(i-1)}$ increases by a factor of 3. Therefore, a level $i = i_0$ must eventually be reached such that the element y' of \bar{t} at level i is less than $2 \cdot 3^{(i-1)}$. But then by definition y' is an anchor, and hence the prefix $\langle x, y, \dots, y' \rangle$ is an anchor tuple. By our rule, “once an anchor tuple, always an anchor tuple” (see under “Mark” on page 14), the final part of our result follows. \square

Lemma 9.0: Statement and Proof

No multiple of 3 is a range element.

Proof :

If

$$\frac{3x + 1}{2^a} = 3m$$

then $1 \equiv 0 \pmod{3}$, which is false. \square

Lemma 10.0: Statement and Proof

Each odd, positive integer (except a multiple of 3) is mapped to by a multiple of 3 in one iteration of the $3x + 1$ function.

Proof:

Since the domain of the $3x + 1$ function is the odd, positive integers, the only relevant generators are $3(2k + 1)$, $k \geq 0$. We show that, for each odd, positive integer y not a multiple of 3, there exists a k and an a such that

$$y = \frac{(3(3(2k + 1)) + 1)}{2^a} , \tag{11.1}$$

where a is necessarily the largest such a , since y is assumed odd.

Rewriting (11.1), we have:

$$y2^{a-1} - 5 = 9k . \tag{11.2}$$

Without loss of generality, we can let $y \equiv r \pmod{18}$, where r is one of 1, 5, 7, 11, 13, or 17 (since y is odd and not a multiple of 3, these values of r cover all possibilities mod 18). Or, in other words, for some q , r , $y = 18q + r$. Then, from (11.2) we can write:

$$18(2^{a-1})q + (2^{a-1})r - 5 = 9k . \tag{11.3}$$

Since the first term on the lefthand side is a multiple of 9, $(2^{a-1})r - 5$ must also be if the equation is to hold. We can thus construct the following table. (Certain larger a also serve equally well, but those given suffice for purposes of this proof.)

Table 2: Values of r , a , for Proof of Lemma

r	a	$(2^{a-1})r - 5$
1	6	27
5	1	0
7	2	9
11	5	171
13	4	99

Table 2: Values of r, a , for Proof of Lemma

r	a	$(2^a - 1)r - 5$
17	3	63

Given q and r (hence y), we can use r to look up a in the table, and then solve (11.3) for integral k , thus producing the multiple of 3 that maps to y in one iteration of the $3x + 1$ function. \square

Lemma 11.0: Statement and Proof

For each range element y there exists an infinity of x that map directly to y . Specifically, If

$$\frac{3x + 1}{2^a} = y$$

Then, for each $n \geq 1$,

$$\frac{3(x + (2^{a+2(0)} + 2^{a+2(1)} + \dots + 2^{a+2(n-1)})y) + 1}{2^{a+2(n)}} = y$$

Proof:

The proof is a matter of straightforward algebra.

From the antecedent, we have:

$$x = \frac{2^a y - 1}{3}$$

Substituting into the left-hand side of the consequent, multiplying the term in parentheses by 3, cancelling two 1's, and factoring out $(2^a)(y)$ yields:

$$\frac{2^a y (1 + 3(2^0 + 2^2 + 2^4 + \dots + 2^{2(n-1)}))}{2^{a+2(n)}}$$

The 2^a 's cancel, the term $(1 + 3(\dots))$ is easily shown to equal $2^{2(n)}$, the $2^{2(n)}$ in numerator and denominator cancel, and we are left with y , which gives us our result. \square

Remark

Lemma 11.0 and Lemma 10.0 (see “Lemma 10.0: Statement and Proof” on page 34) imply that if a counterexample exists, then there is an infinity of counterexamples.

Lemma 12.0: Statement and Proof

Each range element y is mapped to, in one iteration of the $3x + 1$ function, by every exponent of one parity only. Furthermore, for each of the two parities, there exists a range element that is mapped to by every exponent of that parity.

Proof:

Our proof is in three parts. The first two parts are slightly edited versions of proofs by Jonathan Kilgallin and Alex Godofsky. Any errors are entirely ours.

1. We first show that if y is mapped to by the exponent a , then y is mapped to by every exponent greater than a that is of the same parity as a .

Let y be a range element, and let x map to y via the exponent a . Then

$$y = \frac{3x + 1}{2^a}$$

We wish to show that there exists an x' such that x' maps to y via the exponent 2^{a+2} . That is, we wish to show that there exists an x' such that

$$y = \frac{3x' + 1}{2^{a+2}}$$

Rewriting, this gives

$$x' = \frac{2^{a+2}y - 1}{3}$$

Substituting for y yields

$$x' = \frac{2^{a+2} \left(\frac{3x + 1}{2^a} \right) - 1}{3}$$

Simplifying, this gives $x' = 4x + 1$. Since x is a positive, clearly x' is as well.

Thus, by induction, if y is mapped to via the exponent a , it is mapped to by every exponent greater than a of the same parity. \square

2. Next we show that if y is mapped to by the exponent a which is greater than 2, then it is mapped to by every exponent less than a that is of the same parity as a .

Let y be a range element, and let x map to y via the exponent a where $a > 2$. Then

$$y = \frac{3x + 1}{2^a}$$

We wish to show that there exists an x' such that x' maps to y via the exponent 2^{a-2} . That is, we wish to show that there exists an x' such that

$$y = \frac{3x' + 1}{2^{a-2}}$$

Rewriting, this gives

$$x' = \frac{2^{a-2}y - 1}{3}$$

Substituting for y yields

$$x' = \frac{2^{a-2} \left(\frac{3x + 1}{2^a} \right) - 1}{3}$$

Simplifying yields

$$x' = \frac{x - 1}{4}$$

We must now show that x' is a positive integer. Since, by the first line in this proof, a can be any exponent greater than 2, we can let $a = 4$. Therefore

$$y = \frac{3x + 1}{4}$$

where y is a positive integer. This implies that

$$3x \equiv -1 \pmod{4}$$

which in turn implies that

$$x \equiv 1 \pmod{4}$$

and therefore that

$$x' = \frac{x - 1}{4}$$

is a positive integer.

Thus, by induction, if y is mapped to via the exponent a , where $a > 2$, then it is mapped to by every exponent less than a of the same parity. \square

3. We now show that for each of the two parities there exists a range element that is mapped by every exponent of that parity.

Fix a range element y , and suppose that x maps to y via the exponent a . Now a is either even or odd, hence $a = 2n + h$, where h is either 0 or 1. Since $y = (3x + 1)/2^a$, it follows that $(2^a)y = 3x + 1$. Reduce the equation mod 3, and we get $(2^h)y \equiv 1 \pmod{3}$, by the following reasoning: $(2^a)y \equiv 1 \pmod{3}$ implies $(2^{2n+h})y \equiv 1 \pmod{3}$ implies $2^{2n} 2^h y \equiv 1 \pmod{3}$ implies $2^h y \equiv 1 \pmod{3}$ because $2^{2n} = 4^n \equiv 1 \pmod{3}$.

Since y is fixed, either $y \equiv 1$ or $y \equiv 2 \pmod{3}$. (We know that y , a range element, is not a multiple of 3 by “Lemma 9.0: Statement and Proof” on page 33). If $y \equiv 1 \pmod{3}$, then we have $2^h(1) \equiv 1 \pmod{3}$, which implies that h must be 0. If $y \equiv 2 \pmod{3}$, then we have $(2^h)(2) \equiv 1 \pmod{3}$, implying that h must be 1. \square

Lemma 13.0: Statement and Proof

There exists an explicit construction of the tuple-set whose exponent sequence is associated with a given tuple.

Proof:

A Solution to the $3x + 1$ Problem

Let x be the first element of a tuple and let $\{a_2, a_3, \dots, a_{n+1}\}$ be the sequence of exponents associated with the first n extensions of the tuple $\langle x \rangle$. The last element of the tuple is given by:

$$\frac{3^n x + r}{2^a}$$

where

$$a = \sum_{i=2}^n a_i$$

We find that r is most easily calculated by iterating from $x = 0$, then multiplying by the appropriate power of 2, as shown in the table at the end of this proof. We want the integral x that produce odd outputs:

$$\frac{3^n x + r}{2^a} = 2k + 1$$

which gives

$$3^n x - 2^{a+1} k = 2^a - r$$

This is a standard linear Diophantine equation. Since $(3^n, 2^{a+1}) = 1$, and 1 divides the right-hand side of the equation, the equation has a solution. One solution is:

$$x_0 = -(2^a - r) \left(\frac{2^{2 \cdot 3^{n-1} \cdot (a+1)} - 1}{3^n} \right)$$

$$k_0 = -(2^a - r) (2^{(2 \cdot 3^{n-1} - 1)(a+1)} - 1)$$

Note that the ratio in the expression for x_0 is an integer because

$$2^{2 \cdot 3^{n-1}} \equiv 1 \pmod{3^n}$$

The general solution is:

$$x = x_0 + t \cdot (-2^{a+1})$$

$$k = k_0 - t \cdot 3^n$$

where t ranges over the integers. Thus, the x 's are the inputs that iterate with the specified exponents and

$$2k + 1 = 2k_0 - t \cdot 2 \cdot 3^n + 1$$

are the outputs.

Table 3: Successive values of n , the x term, and r in proof of Lemma 14.0

n	x term	r	level of tuple element yielded, i.e., i in a_i
1	3^1x	1	2
2	3^2x	$3^1 + 2^{a_2}$	3
3	3^3x	$3^2 + 3^1 2^{a_2} + 2^{a_2} 2^{a_3}$	4
4	3^4x	$3^3 + 3^2 2^{a_2} + 3^1 2^{a_2} 2^{a_3} + 2^{a_2} 2^{a_3} 2^{a_4}$	5
...

□

Lemma 14.0: Statement and Proof

For each range element y , and for each finite sum a of exponents, a domain element x exists that maps to y via a sum a' that contains a .

Proof:

We are looking for an x such that the sequence of iterations represented by

$$\frac{3^n x + r}{2^a}$$

where n , a , and r are defined as in Lemma 13.0 (see “Lemma 13.0: Statement and Proof” on page 38), lead to a computation that ends with y . The numbers n , a , and r are determined by the exponent sequence we want. There also has to be an optional buffer iteration between the above and y , for example, to allow for parity constraints on the exponent leading to y (see “Lemma 11.0: Statement and Proof” on page 35). Thus, for example, if y is mapped to by even exponents, and our exponent sequence a ends with an odd exponent, then there must be a buffer exponent following the sequence a . So, we want

$$\frac{3\left(\frac{3^n x + r}{2^a}\right) + 1}{2^j} = y$$

or

$$\frac{3^{n+1}x + 3r + 2^a}{2^{a+j}} = y$$

which gives

$$3^{n+1}x = (2^a y)2^j - 3r - 2^a \tag{15.1}$$

or

$$(2^a y)2^j \equiv 3r + 2^a \pmod{3^{n+1}}$$

We are looking for x and j . Since y is a range element, it cannot be a multiple of 3 (see “Lemma 9.0: Statement and Proof” on page 33). Therefore $2^a y$ is relatively prime to 3^{n+1} , as is $3r + 2^a$. Since 2^j , where $j \geq 0$, is a member of a reduced residue class mod 3^{n+1} , the congruence is solvable. Hence we can find j , and then, from (15.1), x . \square

Remarks

The result would hold for an arbitrary finite number of buffer exponents following the exponent sum a , since they do not change the fact that a tuple generating each exponent sequence whose sum is a is guaranteed by the proof.

A recursive proof of the Lemma is possible because the set of odd, positive integers mapping to a given range element y in one iteration of the $3x + 1$ function C includes an infinite subset each element of which is mapped to by an infinity of even exponents, and an infinite subset each element of which is mapped to by an infinity of odd exponents.

Lemma 18.0: Statement and Proof

Let y be a range element of the $3x + 1$ function. Then for each finite exponent sequence A , there exists an x that maps to y via A possibly followed by a “buffer” exponent. (If y is mapped to by even exponents, and our exponent sequence A ends with an odd exponent, then there must be a “buffer” exponent following A , and similarly if y is mapped to by odd exponents and A ends with an even exponent.)

Proof:

1. Each range element y is mapped to by all exponents of one parity (“Lemma 12.0: Statement and Proof” on page 36).

2. Each range element y is mapped to by a multiple of 3 (“Lemma 10.0: Statement and Proof” on page 34).

Each range element is mapped to by an infinity of range elements (“Lemma 10.0: Statement and Proof” on page 34).

3. Let y be a range element and let $S = \{s_1, s_2, s_3, \dots\}$ be the set of all odd, positive integers that map to y in one iteration of the $3x + 1$ function. Furthermore, let the s_i be in increasing order of magnitude. It is easily shown that $s_{i+1} = 4s_i + 1$.

(In Fig. 18, $y = 13$, $S = \{17, 69, 277, 1109, \dots\}$)

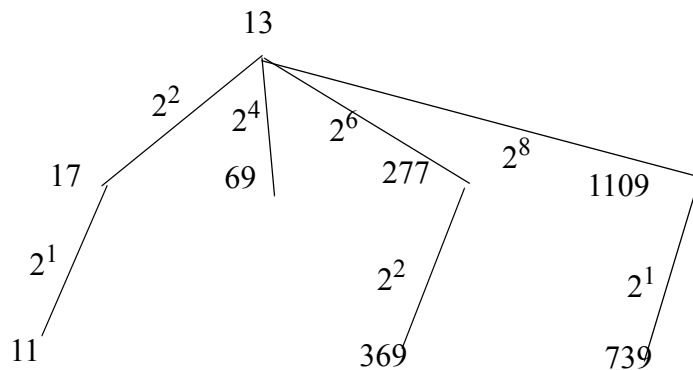


Fig. 18

4. If s_i is a multiple of 3, then $4s_i + 1$ is mapped to, in one iteration of the $3x + 1$ function, by all exponents of even parity.

To prove this, we need only show that x is an integer in the equation

$$4(3u) + 1 = \frac{3x + 1}{2^2}$$

Multiplying through by 2^2 and collecting terms we get

$$(48u) + 4 = 3x + 1$$

and clearly x is an integer.

5. If s_j is mapped to by all even exponents, then $4s_j + 1$ is mapped to, in one iteration of the $3x + 1$ function, by all exponents of odd parity.

(The proof is by an algebraic argument similar to that in step 4.)

6. If s_k is mapped to by all odd exponents, then $4s_k + 1$ is a multiple of 3.

(The proof is by an algebraic argument similar to that in step 4.)

7. The Lemma follows by an inductive argument that we now describe.

Let y be a range element. It is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. As a consequence of steps 1 through 6, we can represent an infinite sub-sequence of the sequence by

...3, 2, 1, 3, 2, 1, ...

where

“3” means “this odd, positive integer is a multiple of 3 and therefore is not mapped to by any odd, positive integer”;

“2” means “this odd, positive integer is mapped to by all even exponents”;

“1” means “this odd, positive integers is mapped to by all odd exponents”.

Each type “2” and type “1” odd, positive integer is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. We can represent an infinite sub-sequence of the sequence by

...3, 2, 1, 3, 2, 1, ...

where each integer has the same meaning as above.

Temporarily ignoring the case in which a buffer exponent is needed, it should now be clear that, for each range element y , and for each finite sequence of exponents B , we can find a finite path down through the infinitary tree we have just established, starting at the root y . The path will end in an odd, positive integer x . Let A denote the path B taken in reverse order. Then we have our result for the non-buffer-exponent case. The buffer-exponent case follows from the fact that the buffer exponent is one among an infinity of exponents of one parity. Thus y is mapped to by an infinite sequence of odd, positive integers. We then simply apply the above argument.. \square

Appendix B — $3x + C$ Functions

Motivation

Readers sometimes claim that our proofs of the $3x + 1$ Conjecture are invalid because they apply equally well to Conjectures involving functions similar to the $3x + 1$ function for which counterexamples are known to exist. In this Appendix, we attempt to show that these claims are not valid.

Definition of “ $3x + C$ Function” and the “ $3x + C$ Problem”

In the literature, a $3x + C$ function F_C is defined as

$$F_C(x) = \frac{3x + C}{2^{\text{ord}_2(3x + C)}}$$

where C and x are odd, positive integers.

For each C , the $3x + C$ Problem asks if for all x , repeated iterations of F_C , beginning with x , eventually terminate in 1. In some cases, for example, the $3x - 1$ and $3x + 5$ Problems, the answer is easily shown to be No. In other words, for these C , counterexamples to the $3x + C$ Conjecture exist.

In the case of the $3x - 1$ function, the smallest counterexample begins with 5, yielding the infinite cyclic tuple $\langle 5, 7, 5, \dots \rangle$. (In the $3x + 1$ function, 5 is the first element of the non-counterexample 2-level anchor tuple $\langle 5, 1 \rangle$.) Thus 5 and 7 are counterexamples to the $3x - 1$ Conjecture.

In the case of the $3x + 5$ function, the smallest counterexample begins with 19, yielding the infinite cyclic tuple $\langle 19, 31, 49, 19, \dots \rangle$. Thus 19, 31, 49, are counterexamples to the $3x + 5$ Conjecture. (In the $3x + 1$ function, 19 is the first element of the non-counterexample 4-level anchor tuple $\langle 19, 29, 11, 17 \rangle$.)

Definition of “ $3x + 1$ -like” Problem

Let F_C be a $3x + C$ function. Then we say that F_C gives rise to a $3x + 1$ -like Problem if $F_C(1) = 1$. It is by no means the case that all $3x + C$ functions give rise to $3x + 1$ -like Problems: For example, $F_7(1) = 5$.

We now establish all positive C that do, in fact, give rise to $3x + 1$ -like Problems.

All Positive C That Give Rise to $3x + 1$ -like Problems

Lemma 15.0

Let C define a $3x + C$ function F_C . Then F_C gives rise to a $3x + 1$ -like Problem iff $C = -1$ or $C = -1 + 2^1 + 2^2 + 2^3 + \dots + 2^k$.

Proof (if part):

Let $C = -1$. Then by direct calculation we confirm that $F_{-1}(1) = 1$.

Let $C = -1 + 2^1 + 2^2 + 2^3 + \dots + 2^k$. Then

(1)

$$\frac{3(1) - 1 + 2^1 + 2^2 + \dots + 2^k}{2^{k+1}} = \frac{1 + 2^0 + 2^1 + 2^2 + \dots + 2^k}{2^{k+1}} = \frac{2^{k+1}}{2^{k+1}} = 1$$

□

Proof (only if part):

If F_C gives rise to a $3x + 1$ -like Problem, then by definition there must exist a $k + 1$ such that

$$\frac{3(1) + C}{2^{k+1}} = 1$$

We find that solutions to this equation are $C = -1$ and $C = -1 + 2^1 + 2^2 + 2^3 + \dots + 2^k$. □

On the Claims That Our Proofs That There Are No Counterexamples to the $3x + 1$ Conjecture, Also Prove That There Are No Counterexamples to $3x + C$ Conjectures For Which Counterexamples Are Known to Exist

We have several concerns about claims that our proofs that there are no counterexamples to the $3x + 1$ conjecture, also prove that there are no counterexamples to $3x + C$ conjectures for which counterexamples are known to exist.

First of all, mathematics would come to a dead stop if the rule were instituted that each proof P had to be accompanied by a proof Q that P did not also apply to other cases known to be false. We believe that if a proof contains an error, then that error must be discoverable within the paper setting forth the proof.

We feel that it is perfectly legitimate to try to determine if one of our proofs also applies to a known false case, but we feel that the reader making the claim must then show us the error in our own paper.

Second, we are not sure what constitutes a legitimate refutation of the above type of claim. Suppose that one statement in our paper clearly does not apply to the $3x + C$ case. Is that sufficient for a refutation?

It would seem that for each such claim, the reader making the claim must implement all changes in our paper that are necessary to convert our paper to one about the $3x + C$ function. That includes changes pertaining to the domain and range of the function, if relevant. This is not a trivial task in itself. But, as shown in the section “All Positive C That Give Rise to $3x + 1$ -like Problems” on page 45, there is a countable infinity of $3x + 1$ -like functions. At present, we do not know which of these functions give rise to Conjectures for which counterexamples exist. It is possible that the number of such functions is infinite.

Suppose a reader is unable to show that our proof, when applied to a particular $3x + 1$ -like function, proves a falsity. It would seem that he cannot rest until he has done the same for all

other $3x + 1$ -like functions for which counterexamples are known to exist. And the number might be infinite.

One refutation of the above type of claim that we consider legitimate is the following. Suppose one of our proofs requires that the set of 2-, 3-, 4-, ..., i -level non-counterexample anchor tuples be complete, where *completeness* means that the set of all these j -level anchor tuples, where $2 \leq j \leq i$, is associated with the set of all j -level exponent sequences. Suppose we know that the set of j -level non-counterexample tuples in the $3x + C$ function is not complete, where $j \leq i$. Then we consider this a legitimate refutation of the claim that our proof applies to that $3x + C$ Conjecture.

“Second Proof” on page 20 specifically requires that all 2-level anchor tuples be non-counterexample. This holds for the $3x + 1$ function, but does not hold for the $3x - 1$ function.

An important fact about the $3x - 1$ case is the following. We begin by pointing out that, without loss of generality, the $3x - 1$ function can be regarded as the negative of the $3x + 1$ function on the negative integers. Thus, for example,

$$-\left(\frac{3(7)-1}{2^2}\right) = -5 = \left(\frac{3(-7)+1}{2^2}\right)$$

From here on, we will express the $3x - 1$ function in this negative form.

The known counterexamples to the $3x - 1$ Conjecture include -5 and -7 , because -5 gives rise to the infinite cyclic tuple $\langle -5, -7, -5, \dots \rangle$. This tuple is associated with the infinite exponent sequence $\{1, 2, 1, \dots\}$.

Now if the tuple-sets structure described in this paper is valid, and, in particular, if the distance functions described in “Lemma 1.0” on page 10 are valid (and no reader so far has denied their validity), then if the tuple-sets structure and the distance functions are extended into the odd, negative integers, the infinite cyclic tuple $\langle -5, -7, -5, \dots \rangle$ is simply a consequence of the distance functions! The same applies to all other infinite cyclic tuples that might exist in the odd, negative integers.

Thus to claim that one of our proofs of the $3x + 1$ Conjecture also applies to the (false) $3x - 1$ Conjecture, is (informally) to compare apples and oranges. It is a claim that a fact (infinite cyclic tuple) that arises in the odd, *negative* integer domain of the $3x + 1$ function, somehow invalidates logic whose domain is the odd, positive integers.

A Basic Property of the $3x + 1$ Function That Differs from That of All Other $3x + 1$ -like Functions But One

A basic property of the $3x + 1$ function is that each odd, positive integer not a multiple of 3 is mapped to by all powers of 2 of one and only one parity (“Lemma 12.0: Statement and Proof” on page 36). Thus, in particular, 1 is mapped to by all even powers of 2.

However, in all $3x + 1$ -like functions other than the $3x + 1$ and the $3x - 1$ functions, 1 is *not* mapped to by all powers of one and only parity, as we prove below in “Lemma 16.0” on page 48. Thus, as a result of this lemma, “Lemma 12.0: Statement and Proof” on page 36, which is used in

the proof of “Lemma 2.0” on page 11, and therefore our proof of the $3x + 1$ Conjecture, does not apply to these functions.

Putting the matter another way, if our proof of the $3x + 1$ Conjecture were offered as a proof of, say, the $3x + 5$ Conjecture to a mathematician who knew nothing about the counterexamples to the $3x + 5$ function, and the mathematician carefully checked the proof, he or she would find that it contained an error, namely, at least in Lemmas 12.0 and 2.0.

We have not yet investigated if, in $3x + 1$ -like functions other than the $3x - 1$ function, other odd, positive integers are also not mapped to by all powers of 2 of one and only one parity.

Lemma 16.0

Lemma 12.0 does not apply to any $3x + C$ function where $C = -1 + 2^1 + 2^2 + 2^3 + \dots + 2^k$, where $k \geq 2$.

Proof:

By (1) in the proof of the if part of Lemma 70.0 above, if F_C is a $3x + 1$ -like function such that $C > 1$, then 1 is mapped to by 2^{k+1} , where $k \geq 3$. If Lemma 12.0 is to apply to F_C , it is necessary, if parity is to be maintained, that there exists an x that maps to 1 via $2^{(k+1)-2} = 2^{k-1}$. That is, there must exist an x such that $(3x + C)/2^{k-1} = (3x - 1 + 2^1 + 2^2 + 2^3 + \dots + 2^k)/2^{k-1} = 1$ or $3x - 1 + 2^1 + 2^2 + 2^3 + \dots + 2^k = 2^{k-1}$ or $3x + (2^{k+1} - 2) = 2^{k-1}$. But there is no such x . \square

Appendix C — What You Can Do to Help Get This Paper Published

Although we believe that the solution to the $3x + 1$ Problem that is presented in this paper is valid, we have been faced with major difficulties in attempting to get the paper published. The main reason is that the author is not a professional mathematician¹, and *understandably*, journal editors, and their referees, are very skeptical that a solution to such a difficult problem could come from outside of the mathematical community. As a result, on the rare occasions when an editor does send the paper out for review, the report clearly indicates that the referee has felt that his task was to find the flaws in a paper that could not possibly be correct. We repeat: this is a perfectly understandable reaction, given the difficulty of the Problem and the fact that the author is not a member of the professional community.

On the other hand, the paper is not crackpot work. It has been read by as many qualified readers as we have been able to find, and continually improved as a result of readers' comments and our own ongoing attempts to make the underlying argument clearer and more convincing.

There are hundreds of mathematics and computer science journals. The speed of response to a submission varies greatly. In our experience, one journal failed to communicate with the author in any fashion for more than seven months, despite the author's repeated emails. Of course, this was an extreme case. But it is simply not practical for the author of a paper that will almost certainly be received with great skepticism, to continue to try to guess a journal that will give his paper the objective, open-minded reading that it requires.

On the other hand, it is entirely possible that the many readers of this paper will know of such journals. These readers are therefore encouraged to contact the author with their recommendations. If a reader wishes to remain anonymous, he or she can send his or her recommendations via surface-mail to the address on the title page.

1. He is a former researcher at Hewlett-Packard's main research laboratory — Hewlett-Packard Labs in Palo Alto, Calif. His degree is in computer science.

Appendix D — On Comparing “Counterexamples Exist” and “Counterexamples Do Not Exist”

Argument for the Validity of the Comparison

In our experience, readers who claim that it is illegitimate to compare the case that counterexamples exist with the case that counterexamples do not exist, assume that such a comparison implies that both cases exist at the same time, which, of course, would be absurd. But this assumption in essence declares that all pairs of sentences of the following form are illegitimate:

If counterexamples exist, then ...;
if counterexamples do not exist, then ...

We feel that readers who believe that such pairs of sentences are illegitimate must commit themselves to correcting all persons who utter pairs of sentences of this form, whether in speech or in writing. Furthermore, these readers must explain to such persons exactly where the illegitimacy lies.

Specific instances of such pairs of sentences are the following.

1. If counterexamples exist, then *not every* odd, positive integer maps to 1 under the $3x + 1$ function;
if counterexamples do not exist, then *every* odd, positive integer maps to 1 under the $3x + 1$ function.
2. If counterexamples exist, then the range of the $3x + 1$ function is a proper subset of the odd, positive integers;
if counterexamples do not exist, then the range of the $3x + 1$ function is the set of odd, positive integers.
3. If counterexamples exist, then each tuple-set consists of an infinity of counterexample tuples and an infinity of non-counterexample tuples (Lemma 5.0);
If counterexamples do not exist, then each tuple-set consists solely of an infinity of non-counterexample tuples.
4. Exactly one set J of odd, positive integers maps to 1, regardless whether counterexamples exist or not. In other words:

If counterexamples exist, then the set of odd, positive integers that map to 1 is J .
If counterexamples do not exist, then the set of odd, positive integers that map to 1 is J .

(This is Lemma 8.8 in our paper, “Are We Near a Solution to the $3x + 1$ Problem?”, on www.occampress.com. The recursive structure of the set J is described in the proof. Readers who believe that this statement — statement 4 — contradicts statement 1 should read “Motivation for Lemma 8.8” in “Are We Near a Solution to the $3x + 1$ Problem?”, where it is shown that the $3x + 1$ Conjecture is equivalent to the conjecture that the one set J consists of all odd, positive

integers. If the Conjecture is false, then the one set J consists of only a proper subset of the odd, positive integers. The $3x + 1$ Problem is thus a question about the constituents of a set.)

5. If counterexamples exist, then there is a first, odd, positive integer that is a counterexample; if counterexamples do not exist, then, trivially, this is not the case.

Argument that the Functions g , h are Well-defined

There are two classes of legitimate questions involving the functions g and h . One class is: *What is the value of the tuple-element ... if counterexamples exist?*, where ... specifies a specific tuple element in a specific tuple in a specific tuple-set. The other class is: *What is the value of the tuple-element ... if counterexamples do not exist?*, where ... likewise specifies a specific tuple element in a specific tuple in a specific tuple-set.

The correct answer to the first question is given by $g(\dots)$, and the correct answer to the second question is given by $h(\dots)$.

Sometimes a reader will attempt to argue that the functions are ill-defined by asking, *What is the value of $h(\dots)$ if counterexamples do not exist?* Our reply is that the question is illegitimate. It is an example of the kind of error that one can get into by introducing meta-levels into a question or discussion. The two legitimate classes of questions are questions about the values of tuple elements. The answers are given by two different functions, namely, g and h . The reader's question is a question about the values of a *function*. It is thus a question at a meta-level. Not all meta-level questions are meaningless, certainly, but this one is.