A Solution to the 3x + 1 Problem

by

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“Very often in mathematics the crucial problem is to recognize and to discover what are the relevant concepts; once this is accomplished the job may be more than half done.”¹

“One of the greatest contributions a mathematician can make is to spot something so simple and powerful that everybody else has missed it.”²

Note 1: The reader who is sorely pressed for time can gain a quick introduction to one of our proofs of the 3x + 1 Conjecture (a proof solves the 3x + 1 Problem) by reading:
“Statement of Problem” on page 4;
“Brief Description of Tuple-sets” on page 6;
“A Brief Summary of Our First Proof of the 3x + 1 Conjecture” on page 19.

A very short and, we believe, elegant possible proof of the Conjecture is given under “Possible Strategy for 1-Tree-Based Proof: Lemma 3.0 Approach” on page 40.

Note 2: We have discovered that the proof that was in Appendix F contained a fatal error, so we have deleted the Appendix. Our other two proofs of the 3x + 1 Conjecture remain correct as far as we know. These proofs are given under: “Theorem: The 3x + 1 Conjecture is True.” on page 21 and in “Appendix H — Second Proof of the 3x + 1 Conjecture” on page 54.

The first 11 pages of this paper provide sufficient background to understand either of these proofs.

Note 3: The reader can safely assume, initially, that all referenced lemmas in this paper are true, since their proofs have been checked and deemed correct by several mathematicians.

Note 4: We will offer shared-authorship to any mathematician who creates a proof of the 3x +1 Conjecture that differs from those in this paper, but that makes use of materials in this paper.

Note 5: The author is seeking a professional mathematician to help prepare this paper for publication. The author will pay any reasonable consulting fee, give generous credit in the Acknowledgments (but only with the mathematician’s prior written approval), and offer shared-authorship for significant contribution to content.

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Abstract

The 3x + 1 Problem asks if repeated iterations of the function $C(x) = (3x + 1)/(2^a)$ always terminate in 1. Here $x$ is an odd, positive integer, and $a = \text{ord}_2(3x + 1)$, the largest positive integer such that the denominator divides the numerator. The conjecture that the function always eventually terminates in 1 is the 3x + 1 Conjecture. An odd, positive integer that maps to 1 is called a non-counterexample; an odd, positive integer that doesn’t map to 1 is called a counterexample (to the Conjecture).

Our first proof (given under: “Theorem: The 3x + 1 Conjecture is True.” on page 21) is based on a structure called tuple-sets that represents the 3x + 1 function in the “forward” direction. In our proof, we show that the 35-level elements of all 35-level tuples in all 35-level tuple-sets are the same, regardless if counterexamples to the Conjecture exist or not. From this fact, a simple inductive argument allows us to conclude that all tuple-sets are the same, whether counterexamples exist or not, and hence that counterexamples do not exist.

Our second proof (given in “Appendix H — Second Proof of the 3x + 1 Conjecture” on page 54), like the first, is based on tuple-sets. In this proof, we define anchor, which is the $i$-level element of the first $i$-level tuple in an $i$-level tuple-set. We then show that there is one and only one set of anchors for all $i$, regardless if counterexamples exist or not. We then show that this implies that there is one and only one set of infinite tuples, regardless if counterexamples exist or not, and from this we deduce that, if counterexamples exist, then some infinite tuples must be both counterexample and non-counterexample tuples, which is absurd, hence counterexamples do not exist and the Conjecture is true.

As far as we have been able to determine, our approaches to a solution of the Problem are original.

1. A phrase of the form “$q$ regardless if $p$” is equivalent to “(if $p$ then $q$) and (if not-$p$ then $q$)”. It is meaningful and in fact true as long as $q$ is true, which it always is in this paper. Instances of the phrase occur in everyday speech, for example, “Fermat’s Last Theorem is true regardless if the Riemann Conjecture is true”,.
Introduction

Statement of Problem

For $x$ an odd, positive integer, set

$$C(x) = \frac{3x + 1}{2^{\text{ord}_2(3x + 1)}}$$

where $\text{ord}_2(3x + 1)$ is the largest exponent of 2 such that the denominator divides the numerator. Thus, for example, $C(17) = 13$ ($\text{ord}_2(3(17) + 1) = 2$), $C(13) = 5$ ($\text{ord}_2(3(13) + 1) = 3$), $C(5) = 1$ ($\text{ord}_2(3(5) + 1) = 4$). Each of these constitutes one iteration of the $3x + 1$ function. The $3x + 1$ Problem, also known as the $3n + 1$ Problem, the Syracuse Problem, Ulam’s Problem, the Collatz Conjecture, Kakutani’s Problem, and Hasse’s Algorithm, asks if repeated iterations of $C$ always terminate at 1. The conjecture that they do is hereafter called the $3x + 1$ Conjecture, or sometimes, in this paper, just the Conjecture. We call $C$ the $3x + 1$ function; note that $C(x)$ is by definition odd.

An odd, positive integer such that repeated iterations of $C$ terminate at 1, we call a non-counterexample. An odd, positive integer such that repeated iterations of $C$ never terminate at 1, we call a counterexample.

Other equivalent formulations of the $3x + 1$ Problem are given in the literature; we base our formulation on the $C$ function (following Crandall) because, as we shall see, it brings out certain structures that are not otherwise evident.

Summary of Research on the Problem

As stated in (Lagarias 1985), “The exact origin of the $3x + 1$ problem is obscure. It has circulated by word of mouth in the mathematical community for many years. The problem is traditionally credited to Lothar Collatz, at the University of Hamburg. In his student days in the 1930’s, stimulated by the lectures of Edmund Landau, Oskar Petron, and Issai Schur, he became interested in number-theoretic functions. His interest in graph theory led him to the idea of representing such number-theoretic functions as directed graphs, and questions about the structure of such graphs are tied to the behavior of iterates of such functions. In the last ten years [that is, 1975-1985] the problem has forsaken its underground existence by appearing in various forms as a problem in books and journals...”

Lagarias has performed an invaluable service to the $3x + 1$ research community by publishing several annotated bibliographies relating to the Problem. These are accessible on the Internet.

On the Structure of This Paper

To enhance readability, we have placed proofs of all lemmas in “Appendix A — Statement and Proof of Each Lemma” on page 24.
In Memoriam

Several of the most important lemmas in this paper were originally conjectured by the author and then proved by the late Michael O’Neill. He made a major contribution to this research, and is sorely missed.
Tuple-sets: The Structure of the $3x + 1$ Function in the “Forward” Direction

A Comment

The structure, tuple-sets, that we are about to describe, is one of two remarkably simple structures that we have discovered underlying the $3x + 1$ function, a function that is still referred to, at least informally, as “chaotic”. The reader can get an idea of the alternative structures that, at the time of this writing, are used throughout the $3x + 1$ literature, by browsing papers in Google that come up in response to the search string, “Collatz graphs”. For example, see:
https://www.fq.math.ca/Scanned/40-1/andaloro.pdf,
http://go.helms-net.de/math/collatz/aboutloop/collatzgraphs.htm

Brief Description of Tuple-sets

The following should be sufficient for the reader to understand our proofs of the $3x + 1$ Conjecture that are based on tuple-sets, namely those in “Theorem: The $3x + 1$ Conjecture is True.” on page 21 and “Appendix H — Second Proof of the $3x + 1$ Conjecture” on page 54.

1. We use the definition of the $3x + 1$ function in which all successive divisions by 2 are collapsed into a single exponent of 2 (see “Statement of Problem” on page 4). Thus, for example, the tuple $<9, 7, 11>$ represents the fact that
   
   $9$ maps to $7$ in one iteration of the function, via the exponent $2$, because $(3(9) + 1)/2^2 = 7$;
   $7$ maps to $11$ in one iteration of the function, via the exponent $1$, because $(3(7) + 1)/2^1 = 11$.

2. We see that the sequence of exponents associated with the tuple $<9, 7, 11>$ is $\{2, 1\}$.

3. A tuple-set $T_A$ is the set of all finite tuples that are associated with the exponent sequence $A$ (and “approximations” to $A$, but this is not important for our proofs of the $3x + 1$ Conjecture). In our example, $A = \{2, 1\}$. In addition to the tuple $<9, 7, 11>$, the tuple-set $T_A = T_{\{2, 1\}}$ contains the tuples $<25, 19, 29>$, $<41, 31, 47>$, and an infinity of others, each associated with the exponent sequence $\{2, 1\}$. (See “Lemma 1.0: the “Distance” Functions $d(i, i)$ and $d(1, i)$” on page 10.)

4. Facts about tuple-sets:

   An $i$-level tuple-set $T_A$, $i \geq 2$, contains (among other tuples, see previous step) all $(i + 1)$-element tuples that are associated with the exponent sequence $A$.

   There is an infinity of tuples in each tuple-set.

   The set of all tuple-sets contains tuples representing all finite iterations of the $3x + 1$ function.
A Solution to the 3x + 1 Problem

Full Description of Tuple-sets

Definitions

Iteration

An iteration takes an odd, positive integer, \( x \), to an odd, positive integer, \( y \), via one application of the \( 3x + 1 \) function, \( C \). Thus, in one iteration \( C \) takes 17 to 13 because \( C(17) = 13 \).

Tuple

A (finite) tuple is a finite sequence of zero or more successive iterations of \( C \), that is, \( < x, C(x), C^2(x), ..., C^k(x) > \), where \( k \geq 0 \).

A finite tuple is the prefix of an infinite tuple. If \( x \) is a non-counterexample, then \( x \) is the first element of an infinite tuple \( < x, y, ..., 1, 1, 1, ... > \). Of course, if \( x \) is a range element of \( C \), then \( x \) can be an element other than the first in another non-counterexample tuple.

In the literature, a tuple (finite or infinite) is usually called a trajectory or an orbit.

If \( x \) is a counterexample, then \( x \) is the first element of an infinite tuple \( < x, y, ... > \) which does not contain 1. Of course, if \( x \) is a range element of \( C \), then \( x \) can be an element other than the first in another counterexample tuple.

A counterexample tuple must be one of two types: either there is an infinitely-repeated finite cycle of elements (none of which is 1) in the infinite tuple having the counterexample \( x \) as first element, or else there is no such cycle, but there is no 1 in the infinite tuple having the counterexample \( x \) as first element — in other words, there is no upper bound to the elements of the infinite tuple.

Exponent, Exponent Sequence

If \( C(x) = y \), with \( y = (3x + 1)/2^a \), we say that \( a \) is the exponent associated with \( x \). In more formal language, this can be expressed as \( \text{ord}_2(3x + 1) = a \). Sometimes we simply write \( e(x) = a \).

The sequence \( A = \{ a_2, a_3, ..., a_i \} \), where \( a_2, a_3, ..., a_i \) are the exponents associated with \( x, C(x), ..., C^{(i-1)}(x) \) respectively, is called an exponent sequence. We number exponents beginning with \( a_2 \) in order that the subscript corresponds to a level number in the corresponding tuple-set. See “Levels in Tuples and Tuple-sets” on page 8. For all \( i \geq 2 \), there are always \( i - 1 \) exponents in the exponent sequence associated with an \( i \)-level tuple-set.

We say that \( x \) maps to \( y \) via \( a_i \) if \( C(x) = y \) and \( \text{ord}_2(3x + 1) = a_i \). By extension, we say that \( x \) maps to \( z \) if \( z \) is the result of a finite sequence of iterations of \( C \) beginning with \( x \), that is if the tuple \( < x, y, ..., z > \) exists.
A Solution to the 3x + 1 Problem

Tuple-set

Let \( A = \{a_2, a_3, \ldots, a_i\} \) be a finite sequence of exponents, where \( i \geq 2 \). The tuple-set \( T_A \) consists of all and only the tuples that are associated with all successive approximations to \( A \). Thus \( T_A \) consists of all and only the following tuples. (Note: First elements \( x \) in different tuples are different odd, positive integers. No two tuples in a tuple-set have the same first element.)

all tuples \(<x>\) such that \( x \) does not map to an odd, positive integer via \( a_2 \);

all tuples \(<x, y>\) such that \( x \) maps to \( y \) via \( a_2 \) but \( y \) does not map to an odd, positive integer via \( a_3 \);

all tuples \(<x, y, y'>\) such that \( x \) maps to \( y \) via \( a_2 \) and \( y \) maps to \( y' \) via \( a_3 \), but \( y' \) does not map to an odd, positive integer via \( a_4 \);

... all tuples \(<x, y, y', \ldots, y^{(i-3)}, y^{(i-2)}>\) such that \( x \) maps to \( y \) via \( a_2 \) and \( y \) maps to \( y' \) via \( a_3 \) and ... and \( y^{(i-3)} \) maps to \( y^{(i-2)} \) via the exponent \( a_i \). (The longest tuple in an \( i \)-level tuple-set has \( i \) elements.)

In other words, for each \( i \)-level exponent sequence \( A \):

there are tuples \(<x>\) whose associated exponent sequence is a prefix of \( A \) for no exponent of \( A \), and

there are other tuples \(<x, y>\) whose associated exponent sequence is a prefix of \( A \) for the first exponent of \( A \), and

there are other tuples \(<x, y, y'>\) whose associated exponent sequence is a prefix of \( A \) for the first two exponents of \( A \), and

... there are other tuples \(<x, y, z, \ldots, y^{(i-2)}>\) whose associated exponent sequence is a prefix of \( A \) for all \( i-1 \) exponents of \( A \).

Tuples are ordered in the natural way by their first elements.

The set of first elements of all tuples in a tuple-set is the set of odd, positive integers (see proof under “The Structure of Tuple-sets” on page 10). Thus, there is a countable infinity of tuples in each tuple-set.

For each \( i \geq 2 \), tuple-sets are a partition of the set of all \( i \)-level tuples.

Levels in Tuples and Tuple-sets

Let \( A \) be an \( i \)-level exponent sequence, \( \{a_2, a_3, \ldots, a_i\} \). The reason subscripts of exponents begin with 2, rather than with 0 or 1, is so that they correspond to levels in each tuple-set. (No

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1. The literature contains several results that establish properties of the \( 3x + 1 \) function that are equivalent to some of those for tuple-sets. However, the language is very different, and the definition of the \( 3x + 1 \) function that is used is not ours, but the original one, in which each division by 2 is a separate node in the tree graph representing the function.
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tuple-set has only one level, because that would mean it is associated with no exponent sequence.) Let \( T_A \) be the tuple-set determined by \( A \). Then, by definition of tuple-set, there exist \( j \)-level tuples in \( T_A \), where \( 1 \leq j \leq i \), that is, tuples \( t = <x, y, \ldots, z> \), where \( x \) is the 1-level element of \( t \), \( y \) is the 2-level element of \( t \), ..., and \( z \) is the \( j \)-level element of \( t \). We say that \( T_A \) is an \( i \)-level tuple-set, and we sometimes speak of the set of \( j \)-level tuple-elements in \( T_A \), where \( 1 \leq j \leq i \).

For \( 2 \leq j \leq i \), two tuples are said to be consecutive at level \( j \) if no \( j \)-level or higher-level tuple exists between them.

Example of Tuple-set

As an example of (part of) a tuple-set: in Fig. 1, where \( A = \{a_2, a_3, a_4\} = \{1, 1, 2\} \) and where we adopt the convention of orienting tuples vertically on the page, the tuple-set \( T_A \) includes:

- the tuple \(<1>\), because \( e(1) = 2 \neq (a_2 = 1) \);
- the tuple \(<3, 5>\), because \( e(3) = (a_2 = 1) \), but \( e(5) = 4 \neq (a_3 = 1) \);
- the tuple \(<5>\), because \( e(5) = 4 \neq (a_2 = 1) \);
- the tuple \(<7, 11, 17, 13>\) because \( e(7) = 1 (a_2 = 1) \) and \( e(11) = 1 (a_3 = 1) \) and \( e(17) = 2 (a_4 = 2) \);
- etc.

Fig. 1. Part of the tuple-set \( T_A \) associated with the sequence \( A = \{1, 1, 2\} \)

The number 18 between the arrows at level 3 and the number 4 between the arrows at level 1 are the values of the level 3 and level 1 distance functions, respectively, established by Lemma 1.0 (see “Lemma 1.0: the “Distance” Functions \( d(i, i) \) and \( d(1, i) \)” on page 10).

In each \( i \)-level tuple-set \( T_A \), where \( i \geq 2 \), for each odd, positive integer \( x \) there exists a tuple whose first element is \( x \). The tuple may be one-level \( (x>) \), or 2-level \( (x, y> \), or ... or \( i \)-level \( (x, y, y', \ldots, y^{(i-3)}, y^{(i-2)}> \). Thus each tuple-set is non-empty.
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Graphical Representation of the Set of All Tuple-sets
It is clear from the definition of tuple-set that the set of all tuple-sets can be represented by an infinitary tree in which each node is a tuple-set. We can imagine the tuple-set (which contains an infinity of tuples) extending into the page.

The Structure of Tuple-sets
It is important for the reader to understand that the structure of each tuple-set is unchanged by the presence or absence of counterexample tuples. Regardless if counterexample tuples exist or not, the set of first elements of all tuples in each tuple-set is always the same, namely, the set of odd, positive integers. Proof: Let $x$ be any odd, positive integer and let $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, be any exponent sequence. Then there are exactly two possibilities:

1. $x$ maps to a $y$ in a single iteration of the $3x + 1$ function, $C$, via the exponent $a_2$, or
2. $x$ does not map to a $y$ in a single iteration of $C$ via the exponent $a_2$.

But if (1) is true, then a tuple containing at least two elements, with $x$ as the first, is in $T_A$; if (2) is true, then the tuple $\langle x \rangle$ is in $T_A$. There is no third possibility. $\square$

For each tuple-set, the first element of the first tuple is 1, the first element of the second tuple is 3, the first element of the third tuple is 5, etc.

It can never be the case that, if counterexample tuples exist, then somehow there are “more” tuples in a tuple-set than if there are no counterexample tuples$^1$.

Furthermore, the distance functions defined in “Lemma 1.0: the “Distance” Functions $d(i, i)$ and $d(1, i)$” on page 10 are the same regardless if counterexample tuples exist or not.

Extensions of Tuple-sets
Since there is a tuple-set for each finite sequence $A$ of exponents, it follows that each tuple-set $T_A$ has an extension via the exponent 1, and an extension via the exponent 2, and an extension via the exponent 3, ... In other words, if $A = \{a_2, a_3, ..., a_i\}$, then there is a tuple-set $T_A$, where $A = \{a_2, a_3, ..., a_i\}$, and a tuple-set $T_A''$, where $A = \{a_2, a_3, ..., a_i, 2\}$, and a tuple-set $T_A'''$, where $A = \{a_2, a_3, ..., a_i, 3\}$, ...

All this is true whether or not the tuple-set $T_A$ and/or any of its extensions contains counterexample tuples or not.

For further details on extensions of tuple-sets, see “How Tuple-sets ‘Work’” and the proof that there exists an extension for each tuple-set (“Lemma 3.0 Statement and Proof”) in our paper, “Are We Near a Solution to the $3x + 1$ Problem?” on occampress.com.

Lemma 1.0: the “Distance” Functions $d(i, i)$ and $d(1, i)$
(a) Let $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, be a sequence of exponents, and let $t_{(r)}, t_{(s)}$ be tuples consecutive at level$^2 i$ in $T_A$. Then $d(i, i)$ is given by:

1. To make this statement more precise: in no tuple-set does there ever exist a first element of a tuple, regardless how large that first element is, such that there are more tuples in that tuple-set having smaller first elements if counterexamples exist, than if counterexamples do not exist.
Let \( t_{(r)} \), \( t_{(s)} \) be tuples consecutive at level \( i \) in \( T_A \). Then \( d(1, i) \) is given by:

\[
d(1, i) = 2 \cdot (2^{a_2})(2^{a_3})\ldots(2^{a_i})
\]

**Proof:** see “Lemma 1.0: Statement and Proof” on page 24

It follows from part (a) of the Lemma that the set of all \( i \)-level elements of all \( i \)-level first tuples in all \( i \)-level tuple-sets is \( \{ z \mid 1 \leq z < 2 \cdot 3^{i-1} \} \), where \( z \) is an odd, positive integer not divisible by 3.

**Remark:** Relationships similar to those described in parts (a) and (b) of the Lemma hold for successive \( j \)-level tuples, where \( 2 \leq j < i \). The following table shows these relationships for \((i-j)\)-level elements of tuples consecutive at level \((i-j)\) in an \(i\)-level tuple-set, where \(0 \leq j \leq (i-1)\). The distances are easily proved using Lemma 1.0.

**Table 1: Distances between elements of tuples consecutive at level \( i \)**

<table>
<thead>
<tr>
<th>Level</th>
<th>Distance between ((i-j))-level elements of tuples consecutive at level ((i-j)), where (0 \leq j \leq (i-1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i )</td>
<td>(2 \cdot 3^{i-1} )</td>
</tr>
<tr>
<td>( i-1 )</td>
<td>(2 \cdot 3^{i-2} \cdot 2^{a_i} )</td>
</tr>
<tr>
<td>( i-2 )</td>
<td>(2 \cdot 3^{i-3} \cdot 2^{a_{i-1}2^{a_i}} )</td>
</tr>
<tr>
<td>( i-3 )</td>
<td>(2 \cdot 3^{i-4} \cdot 2^{a_{i-2}2^{a_{i-1}2^{a_i}}} )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( 2 )</td>
<td>(2 \cdot 3 \cdot 2^{a_3} \ldots 2^{a_{i-1}2^{a_i}} )</td>
</tr>
<tr>
<td>( 1 )</td>
<td>(2 \cdot 2^{a_2}2^{a_3} \ldots 2^{a_{i-1}} 2^{a_i} )</td>
</tr>
</tbody>
</table>

Further details can be found in the section, “Remarks About the Distance Functions” in our paper, “Are We Near a Solution to the 3x + 1 Problem?”, on occampress.com.

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2. For \(2 \leq j \leq i\), two tuples are consecutive at level \( j \) if no \( j \)-level or higher-level tuple exists between them (see “Levels in Tuples and Tuple-sets” on page 8.)
Lemma 2.0

Assume a counterexample exists. Then for all $i \geq 2$, each $i$-level tuple-set contains an infinity of $i$-level counterexample tuples and an infinity of $i$-level non-counterexample tuples.

Proof: “Lemma 2.0: Statement and Proof” on page 29
Recursive “Spiral”s: The Structure of the 3x + 1 Function in the “Backward”, or Inverse, Direction

Definition of “Spiral”

We call the set of odd, positive integers that map to a range element \( y \) in one iteration of the \( 3x + 1 \) function, a recursive “spiral”, or just a “spiral” for short. (See “Fig. 4. Recursive “spirals” structure of odd, positive integers that map to 1.” on page 13.) Thus, for example, \( \{1, 5, 21, 85, 341, \ldots \} \) is a “spiral”. It maps to 1 in one iteration of the \( 3x + 1 \) function. It is easily shown that the elements of a “spiral” map to \( y \) either by all odd exponents, or by all even exponents (the latter is the case with \( \{1, 5, 21, 85, \ldots \} \)). It is likewise easily shown that if \( x \) is an element of a “spiral”, then the next larger “spiral” element is \( 4x + 1 \). (See our paper, “Are We Near a Solution to the 3x + 1 Problem?”, on occampress.com.)

Fig. 4. Recursive “spirals” structure of odd, positive integers that map to 1.

Bold-faced numbers are range elements (21 and 453 are multiples of 3, hence not range elements). Partial “spirals” surrounding the base elements 1 and 85 are shown. The line connecting 1813 to 85 is marked with a \( 2^6 \) because \( (3 \cdot 1813 + 1)/2^6 = 85 \). The line connecting 453 to 1813 is marked \( 85 \cdot 2^4 \) because \( 453 + 85 \cdot 2^4 = 1813 \). The quantity \( 85 \cdot 2^4 = 3 \cdot 453 + 1 \), and similarly for the difference between successive elements of a “spiral” in all “spiral”s. These facts follow from the fact that if \( x, y \) are consecutive elements of a “spiral”, with \( x < y \), then \( y = 4x + 1 \).

Not all elements of a “spiral” are mapped to by even exponents. For example, 5 is mapped to by all odd exponents, as is 341. It is easily shown that successive elements of a “spiral” have the pattern \( \ldots e, o, 3, e, o, 3, e, \ldots \), meaning that there is
an element mapped by all even exponents, then
an element mapped to by all odd exponents, then
a multiple-of-3, which is not mapped to by any odd, positive integer, then
an element mapped by all even exponents, then
an element mapped to by all odd exponents, then
a multiple-of-3, which is not mapped to by any odd, positive integer, then
an element mapped by all even exponents, then
...
(The first element of a “spiral” can have the e, o, or 3 property, but thereafter, the above pattern repeats for the entire “spiral”.)

Definition of y-Tree

Let y be a range element of the 3x + 1 function. As we have stated above, y is mapped to by a “spiral”. In each “spiral” there is a countable infinity of other range elements, each of which is in term mapped to by a “spiral”.

Thus each range element y is the root of a tree, which we call a y-tree. A y-tree, as is evident, is infinitary — each y in the tree is mapped to by an infinity of nodes (namely, those in a “spiral”) — and infinitely deep.

For each range element y, there is one and only one y-tree.

One y-tree is the 1-tree, which we describe below under “Definition of the 1-Tree” on page 15.

Can “Spiral”s and Tuple-sets Be Merged?

The answer is yes.

Consider the following sequence of tuple-sets:

\( T\{1\}, T\{3\}, T\{5\}, T\{7\}, \ldots \)

Consider the “spiral” \( \{x_1, x_2, x_3, \ldots\} \) and assume that each \( x_i \) maps to a range element y in one iteration of the 3x + 1 function via an odd exponent, which is one of two possibilities as described in “Definition of “Spiral”” on page 13. We can represent the “spiral” as the set \( \{<x_1, y>, <x_2, y>, <x_3, y>, \ldots\} \). Then

\(<x_1, y> \text{ is an element of } T\{1\},
<x_2, y> \text{ is an element of } T\{3\},
<x_3, y> \text{ is an element of } T\{5\},
\ldots \)

1. The tree is not strictly an infinitary tree, because if an element (node) of a “spiral” node is not a range element of the 3x + 1 function, that is, if an element is a multiple-of-3, then it has no descending nodes (no odd, positive integers map to a multiple-of-3). But as we stated above, a “spiral” nevertheless does contain an infinity of range elements.
Each “spiral” whose elements map to a range element via odd exponents, is a separate row in the sequence of tuple-sets \( T_{\{1\}}, T_{\{3\}}, T_{\{5\}}, T_{\{7\}}, \ldots \).
And similarly for the sequence of tuple-sets

\[ T_{\{2\}}, T_{\{4\}}, T_{\{6\}}, T_{\{8\}}, \ldots \]

So each “spiral” is a row of 2-level tuples in one of the sequences of tuple-sets.

**Definition of the 1-Tree**

We define a \( y \)-tree called the 1-tree as follows.

The set of all odd, positive integers that map to 1 in a single iteration\(^1\) of the \( 3x + 1 \) function is Level 1 in the tree; This set is \{1, 5, 21, 85, 341, ... \}. (Here and in all subsequent steps, we do not count iterations that take 1 to 1.)

The set of all odd, positive integers that map to 1 in two iterations of the \( 3x + 1 \) function is Level 2 in the tree;

The set of all odd, positive integers that map to 1 in three iterations of the \( 3x + 1 \) function is Level 3 in the tree;

...

Thus the 1-tree contains all and only the odd, positive integers that map to 1 in a finite number of iterations of the function.

Each node in the 1-tree is a “spiral” element.
Each “spiral” element \( y \) that is a range element of the \( 3x + 1 \) function is the root of a \( y \)-tree.
The definition of the \( y \)-tree is the same as that given above.
Each pair of \( y \)-trees at the same Level in the 1-tree, including those for which \( y \) is an element of \( S = \{1, 5, 21, 85, 341, ... \} \), the set of odd, positive integers that map to 1 in one iteration of the \( 3x + 1 \) function, is disjoint.

**Definition of Interval in a “Spiral”**

The integers between successive “spiral” elements we call an *interval* in the “spiral”. Thus, for example, in the “spiral” \( S = \{1, 5, 21, 85, 341, ... \} \), \( 2, 3, 4 \) are the elements in the first interval; \( 6, 7, 8, ..., 18, 19, 20 \) are the elements of the second interval, etc. Sometimes we will only be concerned with the odd elements in one or more intervals.

**Counterexamples and Intervals**

If a counterexample exists, then it is an element of an interval in some “spiral” in the 1-tree, but it is never a “spiral” element, because all the “spiral” elements map to 1. The tree or trees generated by counterexamples are of course separate (disjoint) from the 1-tree. There are “spiral”s, “spiral” elements, and intervals in counterexample trees just as there are in the 1-tree.

---

1. We remind the reader that throughout this paper we use, not the original definition of the \( 3x + 1 \) function, but Crandall’s equivalent definition, in which all successive divisions by 2 are collapsed into a single exponent of 2. See “Statement of Problem” on page 4.
The “Location” or “Address” of an Element in the 1-Tree
Suppose we ask, “What is the ‘location’ (the “address”) of an element \( x \) in the 1-tree?” A simple answer is: the tuple \(<x, \ldots, 1>\) (an element of a tuple-set), where \( x \) maps to the \( y \) of its “spiral” via exponent \( a_2, y \) maps to the \( z \) of its “spiral” via the exponent \( a_3, \ldots \) etc.

Properties of the 1-Tree
(Note: Some of the following properties are repeated from the above sections. Proofs of the existence of all the properties not given in this paper, can be found in our paper, “Are We Near a Solution to the 3x + 1 Problem?”, on occampress.com.)

There Is One and Only One 1-Tree, Whether or Not Counterexamples Exist\(^1\)
The reason there is one and only one 1-tree is that the 1-tree is a \( y \)-tree (with \( y = 1 \)) and all \( y \)-trees are monolithic. That is, they can never be changed. They are infinitary, infinitely deep trees formed under a strict rule (see “Definition of y-Tree” on page 14).

Distance Between Successive “Spiral” Elements
- If \( x \) is an element of a “spiral”, then \( 4x + 1 \) is the next larger element of the “spiral”.

Exponents That Map “Spiral” Elements to the “Spiral”’s \( y \)
- Each of the elements of a “spiral” map to the “spiral”’s \( y \) in one iteration of the \( 3x + 1 \) function either by all odd exponents, or by all even exponents.

Succession of Classes of Exponents That Map “Spiral” Elements to the “Spiral”’s \( y \)
- The successive elements of a “spiral” are mapped to in accordance with a rule that can be expressed as \( \ldots, 2, 1, 3, 2, 1, 3, \ldots \), where “2” means “is mapped to by all even exponents”, “1” means “is mapped to by all odd exponents”, and “3” means “is not mapped to because element is a multiple-of-3, hence not a range element”.

Thus, for example, in the “spiral” \( S = \{1, 5, 21, 85, \ldots\} \), 1 is mapped to by all even exponents, 5 is mapped to by all odd exponents, 21 is not mapped to because it is a multiple-of-3, 85 is mapped to by all even exponents, ...

The Interweaving of “Spiral”s
- The elements of each “spiral” other than \( S \), occupy an infinity of successive intervals in \( S \), one element per interval. Thus, consider the “spiral” \( \{3, 13, 53, 213, \ldots\} \). The same applies recursively down through the 1-tree, except that if a “spiral” \( s \) maps via even exponents to \( y \) in another “spiral” \( r \) in one iteration of the \( 3x + 1 \) function, then the elements of \( s \) skip one interval in \( r \) before occupying an infinity of successive intervals in \( r \), one element per interval.

“Spiral” Elements’ Relation to Tuples in Tuple-Set
- Each “spiral” element \( x \), and the range element \( y \) it maps to in one iteration of the \( 3x + 1 \) function, is a tuple \(<x, y>\) in a tuple-set. Since the set of elements of a “spiral” map to their range element either by all odd exponents or by all even exponents, it is easy to see that the set of all elements of all “spiral”s in the 1-tree is the set of all first elements of all non-counterexample 2-

---

1. “Lemma 3.0: Statement and Proof” on page 29
tuples in all 2-level tuple-sets.

The set of all (finite) upward paths (that is, paths in the direction of the root, 1, of the 1-tree, or away from 1 in the upward direction) is the set of all non-counterexample tuples in the set of all tuple-sets.

- The following table shows the first three 2-tuples in the 2-level tuple-sets $T_A$ for $A = \{1\}$, $\{3\}$, and $\{5\}$. (The reader might find it more natural to imagine the exponent 1 tuple-set as being foremost, with the exponent 3 tuple-set parallel and directly behind the exponent 1 tuple-set, and the exponent 5 tuple-set as parallel and directly behind the exponent 3 tuple-set, etc.)

  In brief, “spiral” elements run vertically upward, tuple-set elements run horizontally to the right.

  Observe the first three elements of the “spiral” that maps to 5 in one iteration of the $3x + 1$ function, namely, the elements 3, 13, 53 running vertically (and upward in the Table).

  Observe the first three elements of the “spiral” that maps to 11 in one iteration of the $3x + 1$ function, namely, the elements 7, 29, 117 running vertically (and upward).

  Observe the first three elements of the “spiral” that maps to 17 in one iteration of the $3x + 1$ function, namely, the elements 11, 45, 181 running vertically (and upward).

  In keeping with the rule governing successive elements of “spiral”s, namely, that if $x$ is a “spiral” element, then $4x + 1$ is the next element, we observe that $3(4) + 1 = 13$, etc.; $4(7) + 1 = 29$, etc., and $11(4) + 1 = 45$, etc.

  But observe also, in keeping with “Lemma 1.0: the “Distance” Functions $d(i, i)$ and $d(1, i)$” on page 10:

  the (horizontal) sequence of level-2 elements in each 2-level tuple-set is 5, 6, 11, ...

  The (horizontal) difference between level-1 elements in the tuple-set $T_A$ where $A = \{1\}$ is $2(2^1)$,

  The (horizontal) difference between level-1 elements in the tuple-set $T_A$ where $A = \{3\}$ is $2(2^3)$,

  The (horizontal) difference between level-1 elements in the tuple-set $T_A$ where $A = \{5\}$ is $2(2^5)$,

<table>
<thead>
<tr>
<th>Exponent</th>
<th>5</th>
<th>53</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5</td>
<td>11</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>17</td>
<td>117</td>
<td>29</td>
</tr>
<tr>
<td></td>
<td></td>
<td>181</td>
<td>45</td>
</tr>
</tbody>
</table>
|          |   |     |   |...
Similar 2-level tuple-sets and corresponding “spiral” elements exist for even exponents, that is, for 2-level tuple-sets $T_A$ where $A = \{2\}, \{4\}, \{6\}, \ldots$.

- It is possible to traverse from any point (“spiral” element) $x$ in the 1-tree to any other point (“spiral” element) $z$ in the 1-tree. Reason: the 1-tree is connected.

**Counterexamples and the 1-Tree**

- If counterexamples exist, it is not because a sub-tree of the 1-tree has somehow been “broken off” and become a counterexample tree. The structure of the 1-tree is the same, whether or not counterexamples exist.

- If counterexamples exist, then the intervals in each “spiral” — not just the “spiral” $\{1, 5, 21, 85, \ldots\}$ — contain an infinity of counterexamples (“Lemma 2.0” on page 12).

- It is not possible to tell, from a single “spiral” (ignoring the possibly colored contents of intervals), if counterexamples exist or not. For each range element $y$ of the $3x + 1$ function, there is exactly one “spiral”, whether or not counterexamples exist. The function is not itself a function of the existence or non-existence of counterexamples.

**Can the 1-Tree and Tuple-sets Be Merged?**

The better question is, “Can $y$-trees and tuple-sets be merged?” A legitimate response is, “It depends on what you mean by ‘merged’.” Certainly there are facts about the two structures that can readily be set forth (proofs are elementary). For example,

- A tuple is an upward path in a $y$-tree.

- If counterexamples do not exist, then for each $i \geq 2$, the set of all $i$-level tuple-sets contains every tuple in the 1-tree, broken down into $i$-level sub-tuples. Thus, for example, consider the tuple $<9, 7, 11, 17, 13, 5, 1>$, which is associated with an upward path in the 1-tree having exponent sequence $\{2, 1, 1, 2, 3, 4\}$. The 4-level tuple $<9, 7, 11, 17>$ is an element of the 4-level tuple-set $T_{^{1, 1}_2}$, and the 4-level tuple $<17, 13, 5, 1>$ is an element of the 4-level tuple-set $T_{^{1, 3, 4}_2}$. (Recall from “Exponent, Exponent Sequence” on page 7 that the number of exponents in the exponent sequence associated with an $i$-level tuple-set, is always $i – 1$.)

### Table 2: Partial View of Relationship Between 2-Level Tuple-sets and Odd-Exponent “Spiral”s

<table>
<thead>
<tr>
<th>Exponent</th>
<th>5</th>
<th>11</th>
<th>17</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Recall from “Exponent, Exponent Sequence” on page 7 that the number of exponents in the exponent sequence associated with an $i$-level tuple-set, is always $i – 1$.)
• A “spiral” and the \( y \) it maps to in one iteration of the \( 3x + 1 \) function, is equivalent to an infinity of 2-level tuples, \( \langle x_1, y \rangle, \langle x_2, y \rangle, \langle x_3, y \rangle, \ldots \), where \( x_i \) is an element of the “spiral” and \( x_{i+1} = 4x_i + 1 \). For more details, see “Can “Spiral”’s and Tuple-sets Be Merged?” on page 14.

• Each tuple \( \langle x_i, y \rangle \) is one of an infinity of tuples in the tuple-set \( T_{a_2} \), where \( a_2 \) is either an even integer, unique among all the other even \( a_2 \)s that map to \( y \), or an odd integer, unique among all the other odd \( a_2 \)s that map to \( y \).

Since each of these tuples occupies a separate tuple-set \( T_{a_2} \), we see that each range element is present in an infinity of 2-level tuple-sets. And this is true for tuple-sets of greater than two levels.

• For each \( i \geq 2 \), the set of all \( i \)-level elements of all \( i \)-level tuples in all \( i \)-level tuple-sets is the set of range elements of the \( 3x + 1 \) function.

• Each \( i \)-level element \( y \) in an \( i \)-level tuple in an \( i \)-level tuple-set is the root of a \( y \)-tree.

Thus as \( i \) increases, each of these \( y \)-trees merely extends downward by one tuple element (one level). There are no “new” \( y \)-trees (for example, counterexample \( y \)-trees), and no \( y \)-trees are lost. (Can these two facts be the basis for another proof of the \( 3x + 1 \) Conjecture similar to our first proof of the Conjecture? (See “Proof:” on page 21.))

A Brief Summary of Our First Proof of the \( 3x + 1 \) Conjecture

The \( 3x + 1 \) Problem asks if repeated iterations of the function \( C(x) = (3x + 1)/(2^a) \) always terminate in 1. Here \( x \) is an odd, positive integer, and \( a = ord_2(3x + 1) \), the largest positive integer such that the denominator divides the numerator. The conjecture that the function always eventually terminates in 1 is the \( 3x + 1 \) Conjecture.

(Note: the reader is asked to inform us of the first sentence that the reader believes contains an error, and what that error is.)

The following is a summary of our first solution to the Problem — that is, of our first proof of the \( 3x + 1 \) Conjecture. The full proof is given in “Proof:” on page 21.

1. We know, by computer test\(^1\), that for all \( i \), \( 2 \leq i \leq 35 \), the first \( i \)-level tuple in each \( i \)-level tuple-set is a non-counterexample tuple. Thus, the following argument does not apply to the \( 3x - 1 \) function, where already at \( i = 2 \), there is an \( i \)-level tuple that contains a counterexample. (The tuple is \( \langle 7, 5 \rangle \), which is the start of the infinite cycle \( \langle 7, 5, 7, 5, \ldots \rangle \), hence 5 is a counterexample (as is 7).)

\(^1\) See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than \( 20 \cdot 2^{58} \approx 5.76 \cdot 10^{18} \), which is greater than \( 3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35 - 1)} \), have been tested and found to be non-counterexamples. These include the set of all 35-level elements of all 35-level first tuples in all 35-level tuple-sets.
2. For each 35-level tuple-set, the set of 35-level tuples is a function of the first 35-level tuple (see the distance function (part (a) of “Lemma 1.0: the “Distance” Functions d(i, i) and d(1, i)” on page 10)) — a non-counterexample tuple, by step 1. Thus for each 35-level tuple-set, the set of 35-level tuples is the same whether or not counterexamples exist.

3. But since each \((i + 1)\)-level tuple-set is an extension of \(i\)-level tuples in an \(i\)-level tuple-set, we see that the value of the 36th-level element in the \(n\)th 36-level tuple in any 36-level tuple-set is the same, whether or not counterexamples exist. And so on for all \(i\)-level tuple-sets, where \(i > 36\).

5. We conclude that the set of all tuple-sets (structure and contents) if counterexamples exist is the same as the set of all tuple-sets if counterexamples do not exist, which implies that counterexamples are the same as non-counterexamples, which is absurd. We must conclude that counterexamples do not exist, and hence that the \(3x + 1\) Conjecture is true.
**Theorem: The 3x + 1 Conjecture is True.**

**Proof:**

(\textit{Note 1:} Another proof of the Conjecture is given in “Appendix H — Second Proof of the 3x + 1 Conjecture” on page 54. )

(\textit{Note 2:} we ask the reader to inform us of the first sentence that the reader believes contains an error, and what that error is.)

We show that there is no difference (in form or content) between tuple-sets if counterexamples do not exist, and tuple-sets if counterexamples exist. From that fact, we conclude that counterexamples do not exist.

1. It is easily shown that, for each $i \geq 2$, the set $E_i$ of $i$-level elements in \textit{first} $i$-level tuples in all $i$-level tuple-sets is the set of odd, positive integers less than \(2 \cdot 3^{(i-1)}\) that are not divisible by 3 (by part (a) of “Lemma 1.0: the “Distance” Functions $d(i, i)$ and $d(1, i)$” on page 10). Thus, for example, $E_2$ is the set of all the odd, positive integers less than \(2 \cdot 3^{(2-1)} = 6\), that are not divisible by 3, and these integers are 1 and 5.

2. By computer test, it is known that $E_2, E_3, E_4, \ldots$, up to at least $E_{35}$ each consists solely of non-counterexamples\(^1\).

3. 
   
   (1) For each 35-level tuple-set $T_A$, the sequence $S$ of 35-level elements in the sequence of 35-level tuples is given by $y + n(2 \cdot 3^{(35-1)})$, where $n \geq 0$ and $y$ is the 35-level element of the first 35-level tuple in the tuple-set $T_A$. The sequence $S$ is the sequence if counterexamples do not exist, and it is also the sequence if counterexamples exist. As we stated in step 2, $y$ is a non-counterexample element,

   \textit{Proof:} Follows from part (a) of “Lemma 1.0: the “Distance” Functions $d(i, i)$ and $d(1, i)$” on page 10. The Distance Functions are not themselves functions of the truth or falsity of the $3x + 1$ Conjecture. \(\square\)

   \textit{Note:} the fact that all elements of $E_{35}$ are non-counterexamples is emphatically not the case for the $3x – 1$ function, where one of the elements, 5, of $E_2$ is already a counterexample. Thus there exists a first 2-level tuple, namely <7, 5>, in a 2-level tuple-set that is a counterexample tuple. Each subsequent $E_i$ contains counterexamples, each of which is the $i$-level element of the first $i$-level tuple in an $i$-level tuple-set. Each of these tuples is therefore a counterexample tuple. \textit{So our proof cannot be used to prove the false 3x – 1 Conjecture}\(^2\).

4. Since each 35-level tuple-set— and indeed each $i$-level tuple-set, where $i \geq 2$ — has an

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1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$, which is greater than $3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35-1)}$, have been tested and found to be non-counterexamples. These include the set of all 35-level elements of all 35-level first tuples in all 35-level tuple-sets.
2. See “Appendix I — The 3x – 1 Test” on page 57.
extension via each possible exponent, namely, via 1, 2, 3, ..., we can use an inductive argument
beginning with (1) to arrive at the conclusion that the set of all non-counterexample tuples if
counterexamples do not exist, is the same as the set of all non-counterexample tuples if counter-
examples exist.

5. We must now ask if counterexamples can exist in $T_A$ in $j$-level tuples, where $j < 35$. The
answer is No, because each $j$-level tuple in $T_A$ extends to a 35-level tuple in some other 35-level
tuple-set, and $T_A$ is any 35-level tuple-set. From step 3 we know that there is one and only one
sequence $S$ of 35-level elements in the sequence of 35-level tuples in any tuple-set. Thus if any $j$-
level tuples are counterexample tuples, they always behave the same as non-counterexample
tuples.

6. So we must conclude from step 4 that counterexamples behave the same as non-counterex-
amples, which is absurd, or else that the set of counterexamples is the empty set. In either case,
our conclusion must be that the $3x + 1$ Conjecture is true. □

Remark 1

The reader might enjoy answering — or attempting to answer — question (I), below, which
arises from the following facts:

Let the tuple $t = <x, ..., 1, 1, ..., 1>$, which is clearly a non-counterexample tuple ($x$ is a non-
counterexample). Let $A$ be the $i$-level exponent sequence associated with $t$. Then $t$ is the first $i$-
level tuple in the tuple-set $T_A$. The 1-level (first) element of $t$ is $x$, the $i$-level element of $t$ is 1.

The 1-level (first) element of the $n$th $i$-level tuple in $T_A$ is given by

$$x + (n - 1)(2 \cdot (2^{a_2}) \cdot (2^{a_3}) \cdots (2^{a_i})),$$

and the $i$-level element of the $n$th $i$-level tuple in $T_A$ is given by

$$1 + (n - 1)(2 \cdot 3^{i-1}),$$

where $n \geq 1$ (by parts (b) and (a) of “Lemma 1.0: the “Distance” Functions
d(i, i) and d(1, i)” on page 10).

(I)

How does $T_A$ differ if (1) counterexamples exist, and (2) counterexamples do not exist?

Remark 2

The reader might enjoy reading at least some of the appendices below that give other possible
proofs of the $3x + 1$ Conjecture.

Remark 3

A wealth of additional results and possible strategies is available in our paper, “Are We Near a
Solution to the $3x + 1$ Problem?” on occampress.com.
References


Appendix A — Statement and Proof of Each Lemma

Lemma 1.0: Statement and Proof

Definition: let $T_A$ be an $i$-level tuple-set, where $i \geq 2$. Let $t(r)$, $t(s)$ denote tuples consecutive at level $i$, with $r < s$ in the natural ordering of tuples by first elements. Let $t(r)(h)$, $t(s)(h)$ denote the elements of $t(r)$, $t(s)$ at level $h$, where $1 \leq h \leq i$. Then we call $|t(s)(h) - t(r)(h)|$ the distance between $t(r)$ and $t(s)$ at level $h$. We denote this distance by $d(h, i)$ and call $d$ the distance functions (one function for each $h$).

Lemma 1.0

(a) Let $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, be a sequence of exponents, and let $t(r)$, $t(s)$ be tuples consecutive at level $i$ in $T_A$. Then $d(i, i)$ is given by:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) Let $t(r)$, $t(s)$ be tuples consecutive at level $i$ in $T_A$. Then $d(1, i)$ is given by:

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3})...(2^{a_i})$$

Thus, in “Fig. 1. Part of the tuple-set $T_A$ associated with the sequence $A = \{1, 1, 2\}$” on page 9, the distance $d(3, 3)$ between $t_{8(3)} = 35$ and $t_{4(3)} = 17$ is $2 \cdot 3^{(3-1)} = 18$. The distance $d(1, 2)$ between $t_{12(1)} = 23$ and $t_{10(1)} = 19$ is $2 \cdot 2^1 = 4$.

Proof:

The proof is by induction.

Proof of Basis Step for Parts (a) and (b) of Lemma 1.0:

Let $t(r)$ and $t(s)$ be the first and second 2-level tuples, in the standard linear ordering of tuples based on their first elements, that are consecutive at level $i = 2$ in the 2-level tuple-set $T_A$, where $A = \{a_2\}$. (See Fig. 2 (1).)
Then we have:

\[
\frac{3t_{(r)(1)} + 1}{2^{a_2}} = t_{(r)(2)}
\tag{1.1}
\]

and since, by definition of \(d(1, 2)\),

\[
t_{(s)(1)} = t_{(r)(1)} + d(1, 2)
\]

we have:

\[
\frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}} = t_{(s)(2)}
\tag{1.2}
\]

Therefore, since, by definition of \(d(i, i)\),

\[
t_{(r)(2)} + d(2, 2) = t_{(s)(2)}
\]
we can write, from (1.1) and (1.2):

\[
\frac{3t_{(r)(1)} + 1}{2^{a_2}} + d(2, 2) = \frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}}
\]

By elementary algebra, this yields:

\[
2^{a_2}d(2, 2) = 3 \cdot d(1, 2)
\]

Now \(d(2, 2)\) must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level \(i\), it must be the smallest such even number, whence it follows that \(d(2, 2) = 3 \cdot 2\), and necessarily

\[
d(1, 2) = 2 \cdot 2^{a_2}
\]

A similar argument establishes that \(d(2, 2)\) and \(d(1, 2)\) have the above values for every other pair of tuples consecutive at level 2.

Thus we have our proof of the Basis Step for parts (a) and (b) of Lemma 1.0.

**Proof of Induction Step for Parts (a) and (b) of Lemma 1.0**

Assume the Lemma is true for all levels \(j\), \(2 \leq j \leq i\) and that \(T_A\) is an \(i\)-level tuple-set, where \(A = \{a_2, a_3, ..., a_i\}\).

Let \(t_{(r)}\) and \(t_{(s)}\) be tuples consecutive at level \(i\), and let \(t_{(r)}\) and \(t_{(f)}\) be tuples consecutive at level \(i + 1\). (See Fig. 2 (2).)
Fig. 2 (2). Illustration for proof of Induction Step of Lemma 1.0.

Then we have:

$$\frac{3t_{(r)(i)} + 1}{2^{a_{i+1}}} = t_{(r)(i+1)}$$

and since, by definition of $d(i, i)$,

$$t_{(f)(i)} = t_{(r)(i)} + g \cdot d(i, i)$$

for some $g \geq 1$, we have:

$$\frac{3(t_{(r)(i)} + g \cdot d(i, i)) + 1}{2^{a_{i+1}}} = t_{(f)(i+1)}$$
Thus, since

\[ t_{(r)}(i + 1) + d(i + 1, i + 1) = t_{(f)}(i + 1) \]

we can write:

\[
\frac{3t_{(t(i + 1) + 1)} + 1}{2^{a_{i+1}}} + d(i + 1, i + 1) = \frac{3(t_{(f(i + 1))} + gd(i, i)) + 1}{2^{a_{i+1}}}
\]

This yields, by elementary algebra:

\[ 2^{a_{(i+1)}}d(i + 1, i + 1) = 3 \cdot gd(i, i) \]

As in the proof of the Basis Step, \(d(i+1, i+1)\) must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level \(i+1\), it must be the smallest such even number. Thus \(d(i+1, i+1) = 3 \cdot d(i, i)\), and

\[ g \cdot d(i, i) = 2^{a_{i+1}}d(i, i) \]

Hence

\[ g = 2^{a_{i+1}} \]

Now \(g\) is the number of tuples consecutive at level \(i\) that must be “traversed” to get from \(t_{(r)}\) to \(t_{(f)}\). By inductive hypothesis, \(d(1, i)\) for each pair of these tuples is:

\[ d(1, i) = 2 \cdot 2^{a_3} \cdot 2^{a_5} \cdot \ldots \cdot 2^{a_i} \]

hence, since

\[ g = 2^{a_{i+1}} \]

we have

\[ d(1, i + 1) = d(1, i) \cdot 2^{a_{i+1}} \]

A similar argument establishes that \(d(i+1, i+1)\) and \(d(1, i+1)\) have the above values for every pair of tuples consecutive at level \(i+1\).

Thus we have our proof of the Induction Step for parts (a) and (b) of Lemma 1.0. The proof of Lemma 1.0 is completed. \(\square\)
Lemma 2.0: Statement and Proof

Assume a counterexample exists. Then for all \( i \geq 2 \), each i-level tuple-set contains an infinity of i-level counterexample tuples and an infinity of i-level non-counterexample tuples.

Proof:
1. Assume a counterexample exists. Then:

There is a countable infinity of non-counterexample range elements.

Proof: Each non-counterexample maps to a range element, by definition of range element.

Each range element is mapped to by an infinity of elements

(“Lemma 6.0: Statement and Proof” on page 32). A countable infinity of these are range elements (proof of “Lemma 7.0: Statement and Proof” on page 35).

There is a countable infinity of counterexample range elements.

Proof: same as for non-counterexample case.

2. For each finite exponent sequence \( A \), and for each range element \( y \), non-counterexample or counterexample, there is an \( x \) that maps to \( y \) via \( A \) possibly followed by a buffer exponent (“Lemma 7.0: Statement and Proof” on page 35). The presence of the buffer exponent does not change the fact that \( x \) is the first element of a tuple associated with the exponent \( A \). □

Lemma 3.0: Statement and Proof

There is one and only one 1-tree, whether or not counterexamples exist.

Proof of “There is one and only one 1-tree...”

The 1-tree =

\[
\begin{align*}
\text{Level 1} &= \{ \text{odd, positive integers } y \mid y \text{ maps to 1 in one iteration of the } 3x + 1 \text{ function} \}^1 \\
\text{Level 2} &= \{ \text{odd, positive integers } y \mid y \text{ maps to an element of Level 1 in one iteration of the } 3x + 1 \text{ function} \} \\
\text{Level 3} &= \{ \text{odd, positive integers } y \mid y \text{ maps to an element of Level 2 in one iteration of the } 3x + 1 \text{ function} \} \\
& \quad \vdots \\
\end{align*}
\]

Since 1 is a range element of the 3x + 1 function, it is the root of a \( y \)-tree. Each \( y \)-tree has several basic, well-defined properties. (For full details, and references to the elementary proofs, see “Properties of the 1-Tree” on page 16 and “Recursive “Spiral”s: The Structure of the 3x + 1 Function in the “Backward”, or Inverse, Direction” on page 13):

Each \( y \) is mapped to by an infinity of odd, positive integers in one iteration of the 3x + 1 function. We call this infinity of odd, positive integers, a “spiral”.

---

1. This set is \( S = \{ 1, 5, 21, 85, 341, \ldots \} \).
If \( x \) is an element of a “spiral”, then \( 4x + 1 \) is the next larger element.

Each “spiral” contains an infinity of range elements, and an infinity of multiples of 3, which are not range elements because they are not mapped to by any odd, positive integer.

Each “spiral” element maps to \( y \) (in one iteration of the \( 3x + 1 \) function), by either all odd exponents, or by all even exponents.

The sequence of these types of “spiral” elements is given by a rule that can be expressed as ... \( 2, 1, 3, 2, 1, 3, ... \), where “2” means “is mapped to by all even exponents”, “1” means “is mapped to by all odd exponents”, and “3” means “is not mapped to because element is a multiple-of-3, hence not a range element”.

Because of the infinity of range elements in each “spiral”, it is clear that the structure of each \( y \)-tree is the result of an infinitely recursive process. Thus each \( y \)-tree is infinitely deep.

**Proof of “...whether or not counterexamples exist**

If an odd, positive integer \( x \) maps to 1 (that is, if \( x \) is a non-counterexample, hence an element of the 1-tree), then it maps to 1 regardless if counterexamples exist or not. Informally, we say, “Once a non-counterexample, always a non-counterexample.” Thus, for example,

- 13 maps to 1 today;
- if the \( 3x + 1 \) Conjecture is proved true tomorrow it will still map to 1;
- if the \( 3x + 1 \) Conjecture is proved false tomorrow it will still map to 1.

If it were not the case that “Once a non-counterexample, always a non-counterexample”, some odd, positive integers could map to two different odd, positive integers, contrary to the definition of the \( 3x + 1 \) function.

**Remark 1**

Some readers claim that the Lemma is trivial, “unnecessary”. But this claim is based on a false assumption. These readers assume (correctly) that 1 is mapped to by all exponents of only one parity, but they assume (incorrectly) that there are nevertheless two possibilities: (1) the 1-tree contains all odd, positive integers, or (2) the 1-tree contains only a proper subset of the odd, positive integers.

However, that implies that a given range element, although it is mapped to by one and only one exponent (in one iteration of the \( 3x + 1 \) function), nevertheless can be mapped to by two different odd, positive integers via that one exponent! But that is impossible, given the definition of the \( 3x + 1 \) function.

In actuality, we know that 1 is mapped to by all even exponents. There is no possibility that it might be mapped to by any odd exponents. Furthermore, for each range element \( y \) (and 1 is a range element) and each (even) exponent, \( y \) is mapped to by one and only odd, positive integer in one iteration of the \( 3x + 1 \) function.

Hence there is one and only one 1-tree, whether or not counterexamples exist.

**Remark 2**

The Lemma passes the \( 3x \) – 1 Test, that is, it doesn’t apply to the (false) \( 3x \) – 1 Conjecture. The reason is that at the time of this writing, no counterexample to the \( 3x + 1 \) Conjecture is known, even though all consecutive odd, positive integers between 1 and at least \( 10^{18} - 1 \) have been found, by computer test, to be non-counterexamples. But a counterexample to the \( 3x - 1 \)
Conjecture is known (the smallest is 5), and so it is emphatically not true that there is one and only 1-tree for the $3x - 1$ function, whether or not counterexamples exist. If no counterexamples to the $3x - 1$ Conjecture existed, the 1-tree for the $3x - 1$ function would certainly be different from the existing one.

**Remark 3**

The Lemma statement is, of course, very counter-intuitive. Even we who first stated it, and then proved it, found ourselves spending time trying to understand how it could be true. But it is true, as the reader can check by going over the proof.

At present, it seems to offer the possibility of a very short, elegant proof of the $3x + 1$ Conjecture. See “Possible Strategy for 1-Tree-Based Proof: Lemma 3.0 Approach” on page 40.

### Lemma 4.0: Statement and Proof

*No multiple of 3 is a range element.*

**Proof:**

If 

$$\frac{3x + 1}{2^a} = 3m$$

then $1 \equiv 0 \mod 3$, which is false. \(\square\)

### Lemma 5.0: Statement and Proof

*Each odd, positive integer (except a multiple of 3) is mapped to by a multiple of 3 in one iteration of the $3x + 1$ function.*

**Proof:**

Since the domain of the $3x + 1$ function is the odd, positive integers, the only relevant generators are $3(2k + 1), k \geq 0$. We show that, for each odd, positive integer $y$ not a multiple of 3, there exists a $k$ and an $a$ such that

$$y = \frac{(3(3(2k + 1)) + 1)}{2^a}, \quad (11.1)$$

where $a$ is necessarily the largest such $a$, since $y$ is assumed odd.

Rewriting (11.1), we have:

$$y2^{a-1} - 5 = 9k. \quad (11.2)$$

Without loss of generality, we can let $y \equiv r \mod 18$, where $r$ is one of 1, 5, 7, 11, 13, or 17 (since $y$ is odd and not a multiple of 3, these values of $r$ cover all possibilities mod 18). Or, in other words, for some $q, r, y = 18q + r$. Then, from (11.2) we can write:
18(2^{a-1})q + (2^{a-1})r - 5 = 9k . \hspace{1cm} (11.3)

Since the first term on the left-hand side is a multiple of 9, \((2^{a-1})r - 5\) must also be if the equation is to hold. We can thus construct the following table. (Certain larger \(a\) also serve equally well, but those given suffice for purposes of this proof.)

Table 3: Values of \(r, a\), for Proof of Lemma

<table>
<thead>
<tr>
<th>(r)</th>
<th>(a)</th>
<th>((2^{a-1})r - 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>27</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>171</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>99</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>63</td>
</tr>
</tbody>
</table>

Given \(q\) and \(r\) (hence \(y\)), we can use \(r\) to look up \(a\) in the table, and then solve (11.3) for integral \(k\), thus producing the multiple of 3 that maps to \(y\) in one iteration of the \(3x + 1\) function. \(\square\)

**Lemma 6.0: Statement and Proof**

(a) Each range element \(y\) is mapped to, in one iteration of the \(3x + 1\) function, by every exponent of one parity only. Furthermore,

(b) For each of the two parities, there exists a range element that is mapped to by every exponent of that parity.

**Proof of part (a):**

Steps 1 and 2 are slightly edited versions of proofs by Jonathan Kilgallin and Alex Godofsky. Any errors are entirely ours. Step 3 is a slightly edited version of a proof by Michael Klipper. Any errors are entirely ours.

1. We first show that if \(y\) is mapped to by the exponent \(a\), then \(y\) is mapped to by every exponent greater than \(a\) that is of the same parity as \(a\).

   Let \(y\) be a range element, and let \(x\) map to \(y\) via the exponent \(a\). Then
We wish to show that there exists an \(x'\) such that \(x'\) maps to \(y\) via the exponent \(2^{a+2}\). That is, we wish to show that there exists an \(x'\) such that

\[
y = \frac{3x + 1}{2^a}
\]

Rewriting, this gives

\[
x' = \frac{2^{a+2} y - 1}{3}
\]

Substituting for \(y\) yields

\[
x' = \frac{2^{a+2} \left(\frac{3x + 1}{2^a}\right) - 1}{3}
\]

Simplifying, this gives \(x' = 4x + 1\). Since \(x\) is an odd, positive integer, clearly \(x'\) is as well.

Thus, by induction, if \(y\) is mapped to via the exponent \(a\), it is mapped to by every exponent greater than \(a\) of the same parity. \(\square\)

2. Next we show that if \(y\) is mapped to by the exponent \(a\) which is greater than 2, then it is mapped to by every exponent less than \(a\) that is of the same parity as \(a\).

Let \(y\) be a range element, and let \(x\) map to \(y\) via the exponent \(a\) where \(a > 2\). Then

\[
y = \frac{3x + 1}{2^a}
\]

We wish to show that there exists an \(x'\) such that \(x'\) maps to \(y\) via the exponent \(2^{a-2}\). That is, we wish to show that there exists an \(x'\) such that
A Solution to the $3x + 1$ Problem

\[ y = \frac{3x' + 1}{2^{a-2}} \]

Rewriting, this gives

\[ x' = \frac{2^{a-2}y - 1}{3} \]

Substituting for $y$ yields

\[ x' = \frac{2^{a-2}(3x + 1) - 1}{2^a} \]

Simplifying yields

\[ x' = \frac{x - 1}{4} \]

3. We must now show that $x' = (x - 1)/4$ is an odd, positive integer. This means we must show that $(x - 1) = 4(2k + 1)$ for some $k \geq 0$, or that $(x - 1) = 8k + 4$, hence that $x = 8k + 5$. Thus, we must prove $x \equiv 5 \mod 8$.

We know that $x$ maps to $y$ via $a$, where $a \geq 3$. Thus, $y = (3x + 1)/2^a$, so $2^a y = 3x + 1$. Because $a \geq 3$, $2^a y$ is a multiple of 8. Thus, $3x + 1 \equiv 0 \mod 8$, and $3x \equiv 7 \mod 8$. This readily implies $x \equiv 5 \mod 8$.

4. Thus, by induction, if $y$ is mapped to via the exponent $a$, where $a > 2$, then it is mapped to by every exponent less than $a$ of the same parity. □

Proof of part (b):

We now show that for each of the two parities there exists a range element that is mapped to by every exponent of that parity.

1. Fix a range element $y$, and suppose that $x$ maps to $y$ via the exponent $a$. Now $a$ is either even or odd, hence $a = 2n + h$, where $h$ is either 0 or 1. Since $y = (3x + 1)/2^a$, it follows that $(2^a)y = 3x + 1$. Reduce the equation mod 3, and we get $(2^h)y \equiv 1 \mod 3$, by the following reasoning:
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$(2^n)y \equiv 1 \pmod{3}$ implies $(2^{2n} + h)y \equiv 1 \pmod{3}$ implies $2^{2n} 2^h y \equiv 1 \pmod{3}$ implies $2^h y \equiv 1 \pmod{3}$ because $2^{2n} = 4^n \equiv 1 \pmod{3}$.

2. Since $y$ is fixed, either $y \equiv 1$ or $y \equiv 2 \pmod{3}$. (We know that $y$, a range element, is not a multiple of 3 by “Lemma 4.0: Statement and Proof” on page 31). If $y \equiv 1 \pmod{3}$, then we have $2^h(1) \equiv 1 \pmod{3}$, which implies that $h$ must be 0. If $y \equiv 2 \pmod{3}$, then we have $(2^h)(2) \equiv 1 \pmod{3}$, implying that $h$ must be 1. □

Lemma 7.0: Statement and Proof

Let $y$ be a range element of the $3x + 1$ function. Then for each finite exponent sequence $A$, there exists an $x$ that maps to $y$ via $A$ possibly followed by a “buffer” exponent. (For example, if $y$ is mapped to by even exponents, and our exponent sequence $A$ ends with an odd exponent, then there must be an even “buffer” exponent following $A$, and similarly if $y$ is mapped to by odd exponents and $A$ ends with an even exponent. However, there are other cases in which a “buffer” exponent is required.)

Proof:
1. Each range element $y$ is mapped to by all exponents of one parity (“Lemma 6.0: Statement and Proof” on page 32).

2. Each range element $y$ is mapped to by a multiple of 3 (“Lemma 5.0: Statement and Proof” on page 31).

Each range element is mapped to by an infinity of range elements (“Lemma 5.0: Statement and Proof” on page 31).

3. Let $y$ be a range element and let $S = \{s_1, s_2, s_3, \ldots \}$ be the set of all odd, positive integers that map to $y$ in one iteration of the $3x + 1$ function. In other words, $S$ is the set of all elements in a “spiral”. Furthermore, let the $s_i$ be in increasing order of magnitude. It is easily shown that $s_{i+1} = 4s_i + 1$.
(In Fig. 18, $y = 13$, $S = \{17, 69, 277, 1109, \ldots \}$)
4. If \( s_i \) is a multiple of 3, then \( 4s_i + 1 \) is mapped to, in one iteration of the \( 3x + 1 \) function, by all exponents of even parity. To prove this, we need only show that \( x \) is an integer in the equation

\[
4(3u) + 1 = \frac{3x + 1}{2^2}
\]

Multiplying through by \( 2^2 \) and collecting terms we get

\[
(48u) + 4 = 3x + 1
\]

and clearly \( x \) is an integer.

5. If \( s_j \) is mapped to by all even exponents, then \( 4s_j + 1 \) is mapped to, in one iteration of the \( 3x + 1 \) function, by all exponents of odd parity. (The proof is by an algebraic argument similar to that in step 4.)

6. If \( s_k \) is mapped to by all odd exponents, then \( 4s_k + 1 \) is a multiple of 3.
(The proof is by an algebraic argument similar to that in step 4.)

7. The Lemma follows by an inductive argument that we now describe.

Let $y$ be a range element. It is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. As a consequence of steps 1 through 6, we can represent an infinite sub-sequence of the sequence by

$$...3, 2, 1, 3, 2, 1, ...$$

where

“3” means “this odd, positive integer is a multiple of 3 and therefore is not mapped to by any odd, positive integer”;

“2” means “this odd, positive integer is mapped to by all even exponents”;

“1” means “this odd, positive integer is mapped to by all odd exponents”.

Each type “2” and type “1” odd, positive integer is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. We can represent an infinite sub-sequence of the sequence by

$$...3, 2, 1, 3, 2, 1, ...$$

where each integer has the same meaning as above.

Temporarily ignoring the case in which a buffer exponent is needed, it should now be clear that, for each range element $y$, and for each finite sequence of exponents $B$, we can find a finite path down through the infinitary tree we have just established, starting at the root $y$. The path will end in an odd, positive integer $x$. Let $A$ denote the path $B$ taken in reverse order. Then we have our result for the non-buffer-exponent case. The buffer-exponent case follows from the fact that the buffer exponent is one among an infinity of exponents of one parity. Thus $y$ is mapped to by an infinite sequence of odd, positive integers. We then simply apply the above argument. □
Appendix B —Possible Strategies for Other Proofs of the 3x + 1 Conjecture Using Tuple-sets

**First Possible Strategy**

1. Each range element of the 3x + 1 function is mapped to by all finite exponent sequences, plus a possible final “buffer” exponent (“Lemma 7.0: Statement and Proof” on page 35).

2. *Definition:* an anchor tuple is the first \(i\)-level tuple in an \(i\)-level tuple-set, where \(i \geq 2\).
   
   If an odd, positive integer exists, then for some smallest \(i\), it must be an element of an anchor tuple. It is, furthermore, an element of all extensions of that anchor tuple, that is, it is an element of an anchor tuple for all greater \(i\). (Easy proofs."
   
   Thus, if counterexamples exist, each counterexample must eventually — for some \(i\) — be the first element of an anchor tuple.

3. For each finite exponent sequence \(A\), there is one and only one tuple-set \(T_A\) associated with \(A\).
   
   Thus for each finite exponent sequence \(A\), there is one and only one anchor tuple associated with \(A\).

4. Assume counterexamples exist. (By computer tests, we know that the minimum counterexample is greater than \(10^{18}\), hence our argument here does not also apply to the 3x – 1 function, where the smallest counterexample is known to be 5.)
   
   Then eventually, that is, for some \(i\), there must be anchor tuples that are counterexample tuples. However, since there is one and only one anchor tuple for each tuple-set, and one and only tuple-set associated with each exponent sequence, each of these counterexample anchor tuples takes the place of a non-counterexample anchor tuple.
   
   But then there are exponent sequences that do not map to 1, contradicting Lemma 7.0 (see step 1). Hence counterexamples do not exist, and the 3x + 1 Conjecture is true.

**Second Possible Strategy**

1. *Definition:* an anchor tuple is the first \(i\)-level tuple in an \(i\)-level tuple-set, where \(i \geq 2\).

   If an odd, positive integer exists, then for some smallest \(i\), it must be an element of an anchor tuple. It is, furthermore, an element of all extensions of that anchor tuple, that is, it is an element of an anchor tuple for all greater \(i\). (Easy proofs.)

2. If a counterexample exists, then there is a minimum counterexample, \(y_c\). It must be an element of an infinite tuple no element of which is less than \(y_c\) (otherwise, \(y_c\) would not be the minimum counterexample).

3. Call any tuple the last element of which is less than the first, a downward-slope tuple. Clearly, \(y_c\) can never be the first element of a downward slope tuple. We will say, \(y_c\) must always be the first element of a non-downward-slope tuple. In particular, if \(y_c\) is an element of an anchor tuple, then in all extensions of the anchor tuple, any sub-tuple of which \(y_c\) is the first element,
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must be a non-downward-slope sub-tuple.

4. But there is one and only one set of tuple-sets. Each finite exponent sequence is associated with one and only one tuple-set. So every downward-slope tuple is associated with a downward-slope exponent sequence; and similarly for every non-downward-slope tuple.

5. But this is true whether or not counterexamples exist. Thus, in particular, if counterexamples do not exist, there is nevertheless the same set of non-downward-slope exponent sequences as there is if counterexamples exist.

It appears, then, that there is no difference (in structure and content) between the set of all tuple-sets if counterexamples exist, and the set of all tuple-sets if counterexamples do not exist. We conclude that counterexamples must be the same as non-counterexamples, which is absurd. Therefore counterexamples do not exist.

A Failed Strategy

The following strategy has failed repeated attempts to make it yield a proof of the 3x + 1 Conjecture.

- Show that for all \( i \geq 2 \), each \( i \)-level counterexample tuple is always the second, or third, or fourth, or ..., but never the first \( i \)-level tuple in any tuple-set. That would mean that no counterexample tuple exists, for if an odd positive integer exists (for example, a counterexample), it must eventually, for some \( i \), and for all greater \( i \), be an element of a first \( i \)-level tuple in an \( i \)-level tuple-set. (Such tuples are called anchor tuples.)

The problem is that this assumes that non-counterexample tuples having the same \( i \)-level exponent sequence as counterexample tuples, are always anchor tuples, and therefore that counterexample tuples are always the second, or third, or fourth, or ... tuple in the tuple-set. This assumes the truth of what we are trying to prove, and hence is an invalid argument.
Appendix C — On the Possibility of a Proof of the $3x + 1$ Conjecture Based on the 1-Tree

Two Factors That Make Such a Proof Difficult

There are at least two factors that make a proof of the $3x + 1$ Conjecture based on the 1-tree, difficult to arrive at:

1. The 1-tree shows only part of the behavior of the $3x + 1$ function, whereas tuple-sets show all of the behavior.

2. At least one possible inductive proof is impossible because different $y$-trees at the same level below 1, are disjoint. Thus, for example, in Level 1, namely, in $S = \{1, 5, 21, 85, 341, \ldots \}$, the 5-tree and the 85-tree are disjoint.

We urge the reader to read “There Is One and Only One 1-Tree, Whether or Not Counterexamples Exist” on page 16 before reading any of the following possible strategies.

Note: Any of the following strategies that utilize “Lemma 3.0: Statement and Proof” on page 29 pass the $3x – 1$ Test. The reason is that the Lemma asserts that there is one and only one 1-tree, whether or not counterexamples exist. At the time of this writing, no counterexample to the $3x + 1$ Conjecture is known, even though all consecutive odd, positive integers between 1 and at least $10^{18} – 1$ have been found, by computer test$^1$, to be non-counterexamples. But a counterexample to the $3x – 1$ Conjecture is known (the smallest is 5), and so it is emphatically not true that there is one and only 1-tree for the $3x – 1$ function, whether or not counterexamples exist. If no counterexamples to the $3x – 1$ Conjecture existed, the 1-tree for the $3x – 1$ function would certainly be different than the existing one.

Possible Strategy for 1-Tree-Based Proof: Lemma 3.0 Approach

1. There is one and only one 1-tree, whether or not counterexamples exist (“Lemma 3.0: Statement and Proof” on page 29).

2. This implies that the existence of counterexamples has no effect on the 1-tree.

3. But this contradicts the fact that if counterexamples do not exist, the 1-tree contains all odd, positive integers; but if counterexamples exist, the 1-tree contains only a proper subset of the odd, positive integers.

4. Because of this contradiction, we conclude that counterexamples do not exist, and hence the $3x + 1$ Conjecture is true.

Possible Strategy for 1-Tree-Based Proof: Computer Tests Approach

Because of the importance of computer tests in our two proofs of the $3x + 1$ Conjecture (see proof of “Theorem: The $3x + 1$ Conjecture is True.” on page 21 and “Appendix H — Second
A Solution to the 3x + 1 Problem

Proof of the 3x + 1 Conjecture” on page 54) we believe that every effort must be made to see if computer tests can likewise be the source of a 1-tree-based proof of the Conjecture.

1. By computer tests we know that all consecutive odd, positive integers \( \geq 1 \) and less than \( 10^{18} \) are non-counterexamples.\(^1\) Let \( W \) denote the set of all consecutive odd, positive integers greater than \( 2 \) that, as of the present, are known, by computer test, to be non-counterexamples. (A 3x + 1 researcher has told us that the largest of these integers is now greater than \( 10^{20} \).)

2. Let \( F \) (for “First” part of 1-tree) denote the set of elements \( x \) of the 1-tree that are connected to elements of \( W \), where \( x \) is connected to an element of \( W \), that is, \( x \) is in \( F \), if \( x \) is in \( W \), and, if \( x \) is a range element, then all elements of the \( x \)-tree (apart from the root, which is already in \( F \)) are also in \( F \).

3. \( F \) is thus a part of the 1-tree (excluding 1) if counterexamples do not exist. (As a conceptual aid, we can imagine all and only the elements of \( F \) to be colored green.)

4. Let \( R \) denote the set of odd, positive integers in the 1-tree (excluding 1) that are not in \( F \). There are now two possibilities: (1) \( R \) contains all odd, positive integers that are not in \( F \); and (2) \( R \) contains only a proper subset of odd, positive integers that are not in \( F \).

5. But since there is one and only one 1-tree ( “Lemma 3.0: Statement and Proof” on page 29) and the 1-tree, like all \( y \)-trees, is monolithic and rigorously defined (see “Properties of the 1-Tree” on page 16), \( R \) is the same in both cases. Putting this in formal logical terms, let:

\[ c \text{ denote “counterexamples exist”;} \]
\[ R \text{ denote the set of odd, positive integers in the 1-tree (excluding 1) that are not in } F; \]
\[ r \text{ denote “} R \text{ is the same whether or not counterexamples exist”;} \]
\[ not-r \text{ denote “} R \text{ is not the same if counterexamples exist”}. \]

Then consider the statement:

\[ (c \text{ implies not-r) and r}) \text{ implies not-c} \]

We know that \( r \) is true, and that, therefore \( not-r \) is false.
To prevent the innermost implication from being false, \( c \) must be false.
Hence \( not-c \) is true, and the implication as a whole is true.

Thus counterexamples do not exist, and the 3x + 1 Conjecture is true.

### Possible Strategy for 1-Tree-Based Proof: First Inductive Approach

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1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than \( 20 \cdot 2^{58} \approx 5.76 \cdot 10^{18} \), which is greater than \( 3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35-1)} \), have been tested and found to be non-counterexamples.
2. If we include 1, the set \( F \) we are trying to define will trivially contain all elements of the 1-tree.
Note: We will offer shared-authorship to the first mathematician who correctly proves the Interval Conjecture, which is described below. We cannot believe that there are not number theorists in the world who could prove this Conjecture.

1. Let $S$ denote the set $\{1, 5, 21, 85, 341, \ldots \}$. This is the set of odd, positive integers that map to 1 in one iteration of the $3x + 1$ function.

2. Definition: an interval in the above set $S$ is the set of odd integers between successive elements of the set. Thus, $\{3\}$ is the first interval, $\{7, 9, 11, 13, 15, 17, 19\}$ is the second interval, etc.

The Interval Conjecture

Let $k$ be the largest $k$ such that the first $k$ intervals of $S$ are known to be filled by non-counterexamples.

Then the next interval (the $(k + 1)$th) interval is also filled by non-counterexamples.

Remark 1

At the time of this writing, $k \geq 29$.

Proof:

1. The number of elements in the first few intervals of $S$ are:

   $(1 + 2^2) = 5,$
   $(1 + 2^2 + 2^4) = 21,$
   $(1 + 2^2 + 2^4 + 2^6) = 85,$
   ...

   We assume that the element $h$ of $S$ is the sum of the number of elements in all previous intervals of $S$. We further assume that the highest exponent of 2 in the sequence of powers of 2 that yield the next element $h$, is twice the number of intervals − 1 that precede $h$. Thus in the sum for 85 above, $6/2 = 3$, minus 1, = 2, and there are two intervals prior to 85.

2. By computer test\(^1\), we know that all odd, positive integers less than $10^{18}$ are non-counterexamples.

   Now $10^{18} > 2^{58}$. If $2^{58}$ is the highest exponent of 2 in the sequence of powers of 2 that yield the element $h$, and if the sum of the lesser powers is less than the highest power, then we can con-

\(^1\) See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$, which is greater than $3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35 - 1)}$, have been tested and found to be non-counterexamples.
clude that $k$ in the statement of our Interval Conjecture $\geq 29$. □

**Remark 2**

It must be emphasized that the intervals in the set $S$ are solely a means of keeping a *tally* of odd, positive integers in the 1-tree. In no sense are they somehow the location — the node — where these integers occur. Thus, for example, 53 is an element of the “spiral” $\{3, 13, 53, \ldots\}$, that maps to 5. That is the sole location (node) of 53 in the 1-tree.

**Remark 3**

We remind the skeptic that successive elements of each “spiral” occupy (in the tally sense — see Remark 2) successive intervals in $S$. Thus the fact that at least the first 29 intervals in $S$ are occupied solely by elements of the 1-tree, means that elements of the “spiral” of each of these elements that are range elements, occupy an infinity of successive intervals in $S$. This fact, plus the fact that each $y$-tree such that $y$ is an element of $S$, also produces infinities of “spiral” elements to occupy successive intervals in $S$, means there are fewer “empty” interval elements in $S$ for counterexamples. And the reader must keep in mind that, if one counterexample exists, then an infinity of counterexamples exists (“Lemma 2.0” on page 12). So counterexamples might simply be “squeezed out” by at least the first 20 intervals in $S$ being solely occupied by elements in the 1-tree.

**Remark 4**

It will follow trivially from the truth of the Interval Conjecture that each odd, positive integer maps to 1. Another way of showing this would be by a straightforward inductive argument: let $x$ denote the largest odd, positive integer that has been found, by computer test, to be a non-counterexample, and then show that $x + 2$ is likewise a non-counterexample. Finally, show that this argument is valid for $x + 2$, etc.

However, there does not seem to be a systematic way to find $x + 2$. Thus, for example, 5 maps to 1. However, to find 7, we must find, by trial and error, that it is an element of the tuple $<7, 11, 13, 5>$. It is true that 9 is an element of the tuple $<9, 7, 11, 13, 5>$, as are 11 and 13. But 15 is an element of the tuple $<15, 23, 35, 53, 5>$, which we must find by trial and error.

However, we must not fail to point out that the first 12 consecutive odd, positive integers 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23 occur at, or near, the first three elements of $S = \{1, 5, 21, 85, \ldots\}$, the set of odd, positive integers that map to 1 in one iteration of the $3x + 1$ function.

Understanding that a finite upward path in the 1-tree is a tuple (as in a tuple-set), we observe:

$<3, 5>$;
$<9, 7, 11, 17, 13, 5>$;
$<15, 23, 35, 53, 5>$;
$<19, 29, 11, 17, 13, 5>$;
$<21>$;

This phenomenon does not occur in the case of the $3x – 1$ function, since 5 and 7 are counterexamples to the $3x – 1$ Conjecture.
If we can show that this phenomenon of consecutive odd, positive integers clustering around elements of $S$, persists for all elements of $S$, we will have a proof of the Interval Conjecture.

**Possible Strategy for 1-Tree-Based Proof: “Squeezing Out the Counterexamples”**

1. At least the first $10^{18}$ positive integers are known by computer test to be non-counterexamples. (The testers used the original definition of the $3x + 1$ function, which means they tested even integers as well. We only use odd integers, in accordance with Crandall's equivalent definition of the function.)

2. About $1/3$ of odd integers are multiples of 3, and therefore are not range elements of the function. But $2/3$ are.

3. That means that the first $(1/2)(2/3)(10^{18})$, or about $1.67 \times 10^{17}$, odd, positive integers are known, by computer test, to be non-counterexamples that are range elements of the function.

4. Each of these range elements is the root of a $y$-tree. A $y$-tree contains an infinity of “spiral”s. Thus, e.g., the range element 5 is mapped to, in one iteration of the function, by the “spiral”\{ 3, 13, 53, 213, \ldots \}  
   (We will, in the following, limit ourselves to the single “spiral” below $y$. Taking into account the infinity of lower “spiral”s only reinforces our argument.)

5. It is easily shown that the successive elements of a “spiral” occupy an infinity of successive intervals in the set $S = \{ 1, 5, 21, 85, 341, \ldots \}$, which is the set of odd, positive integers that map to 1 in one iteration of the function. (The first interval in $S$ is $\{2, 3, 4\}$; the second interval is $\{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$; etc.)
   The reader can observe that 3, 13, 53, 213 occupy the first four successive intervals in $S$.

6. So each of our approximately $1.67 \times 10^{17}$ range elements $y$ places a non-counterexample in each of an infinity of successive intervals in $S$.

7. Each of these non-counterexamples in each successive interval cannot, of course, be a counterexample.

8. Now since, if counterexamples exist, an infinity of counterexamples exists (“Lemma 2.0: 

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2. It must be emphasized that the intervals in the set $S$ are solely a means of keeping a tally of odd, positive integers in the 1-tree. In no sense are they somehow the location — the node — where these integers occur. Thus, for example, 29 is an element of the “spiral” $\{7, 29, 117, \ldots \}$, which maps to 11 in one iteration of the $3x + 1$ function. That is the sole location (node) of 29 in the 1-tree.
3. A proof that the elements of a “spiral” occupy an infinity of successive intervals in the “spiral” $S = \{ 1, 5, 21, 85, 341, \ldots \}$ will be found in our paper, “Are We Near a Solution to the $3x + 1$ Problem?”, on occampress.com, in the sub-section, “ ‘Spiral’s, Intervals and Levels” of the section, “Strategy of Filling-in of Intervals in the Base Sequence Relative to 1”.

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Statement and Proof” on page 29), we conjecture that there is simply not enough “room” for counterexamples in the intervals of \( S \). (Counterargument to this conjecture: but there is room for non-counterexamples if counterexamples do not exist.)

9. This may be why the first counterexamples in the case of the \( 3x - 1 \) function occur so early (namely, at 5 and 7): informally the counterexamples had to get their “spirals” into the intervals of the \( 3x - 1 \) equivalent of the set \( S \), while at the same time preventing some non-counterexample “spirals” from occupying some infinite successive intervals.

**Remark 1**

Let \( y_m \) be the largest element of the set \( S \) such that all intervals prior to \( y_m \) are filled with odd, positive integers known, by computer test, to be non-counterexamples. Call this set of odd, positive integers, \( W \).

All range elements \( y \) of \( S \) that are greater than \( y_m \) are roots of \( y \)-trees. There is one and only one set of these \( y \)-trees. We assert that all these \( y \) — let \( V \) denote the set of them — would be present if counterexamples did not exist. (Otherwise, level 1 would vary if counterexamples existed, contrary to “Lemma 3.0: Statement and Proof” on page 29.)

Let \( G \) denote all the \( y \)-trees such that \( y \) is an element of \( W \) (\( y \) is necessarily a range element).

(We exclude 1 from these \( y \), to avoid unnecessary redundancy.)

Then it appears that \( G \cup V \) is the set of odd, positive integers, and so counterexamples do not exist.  
*Note: this argument is not valid."

**Remark 2**

It might be of interest to the reader to view the first few intervals of the equivalent of the set \( S \) for the \( 3x - 1 \) function. We will call the equivalent of \( S, S' \).

We boldface the elements of \( S \) and \( S' \) to distinguish them from (odd) elements of the intervals. A non-counterexample in an interval is in normal typeface; a counterexample is in underlined italics.

\[
S = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, \ldots, 81, 83, 85, \ldots \}
\]

\[
S' = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, \ldots, 41, 43, 45, 47, 49, 51, 53, \ldots, 75, \ldots, 299, \ldots \}
\]

**Possible Strategy for 1-Tree-Based Proof: Second Inductive Approach**

1. Let \( S \) denote the set \( \{1, 5, 21, 85, 341, \ldots \} \). This is the set of odd, positive integers that map to 1 in one iteration of the \( 3x + 1 \) function.

2. Call *First Part of 1-Tree* the set of all \( y \)-trees such that \( y \) is a range element \( 1 \leq y \leq w \), where \( w \) is the largest consecutive range element, starting at 1, that has been confirmed by computer test to be a non-counterexample. Computer tests as of this writing put \( w \) considerably larger than
Informally, we say that *First Part of 1-Tree* is what this part of the 1-tree would be if counterexamples did not exist.

3. The set of *y*-trees, where *y* is an element of *S* greater than *w*, is the same whether or not counterexamples exist. Call this set *W*.

We ask how, and at one point, the minimum counterexample would come into being as we moved up through the *y*-trees of *W*.

We point out that every possible finite “downward” path through the 1-tree, beginning at 1, exists. Hence every possible finite “upward” path terminating at 1, exists. Does this allow us to conclude that the 1-tree contains all odd, positive integers?

We also point out that the 1-tree is what it is, regardless if counterexamples exist. (No downward path from 1 is changed if counterexamples exist, or if they do not exist.) Must we conclude that this could imply that counterexamples are the same as non-counterexamples, which, of course is absurd, and, if true, implies that counterexamples do not exist?

**Possible Strategy for 1-Tree-Based Proof: Show That Counterexamples Eliminate Exponent Sequences That Cannot Be Eliminated**

1. We must remember that there is exactly one 1-tree, whether or not counterexamples exist (see “Lemma 3.0: Statement and Proof” on page 29). If the set of all “spiral” elements in the 1-tree is the set of all odd, positive integers, then counterexamples do not exist. Otherwise, counterexamples exist.

2. If a counterexample exists, it is the first element of a 2-tuple in a 2-level tuple-set. This means that the counterexample is an element of a counterexample “spiral”. The “spiral” maps to a counterexample range-element *y* in a single iteration of the 3x + 1 function, and therefore *y* is the root of a *y*-tree. The *y*-tree contains an infinity of counterexample “spiral”s.

3. There are no counterexample tuples in the set of all tuple-sets if the set contains only non-counterexample tuples.

It is easily shown that in order for there to be an infinity of tuples associated with each finite exponent sequence there must be an infinity of non-counterexample tuples in each tuple-set that are extended by the exponent 1, and an infinity that are extended by the exponent 2, etc. (See “Extensions of Tuple-sets” on page 10.) However, this property cannot survive the omission of a countable infinity of “spiral”s.

It seems, therefore, that we must conclude that the existence of a counterexample results in the failure of non-counterexample tuples to be associated with each finite exponent sequence, contradicting “Lemma 2.0” on page 12.

If this reasoning can be made precise and valid, we have a proof of the 3x + 1 Conjecture.

(Note: we must reconcile the omission of non-counterexample “spiral”s from tuple-sets, with the fact that the 1-tree is the same, whether or not counterexamples exist.

We must also explain how it is that there are nevertheless counterexamples to the 3x – 1 Conjecture.)
Possible Strategy for a 1-Tree-Based Proof: Color Green All Elements of Tree Derived From Computer Tests (First Attempt)

1. If someone were to ask us, informally, “What does the 1-tree look like if counterexamples do not exist?”, our answer would be: “It looks exactly like it does at present!” The reason for this answer is “Lemma 3.0: Statement and Proof” on page 29, which states that there is one and only one 1-tree, whether or not counterexamples exist.

The real question is, “Is every odd, positive integer in the 1-tree we see at present?” If the answer is yes, then counterexamples do not exist. If no, then they do exist.

2. It is known by computer test\(^1\) that all successive odd, positive integers less than approximately \(5.76 \times 10^{18}\) are non-counterexamples. Let \(W\) denote the set of all odd, positive integers less than \(10^{18}\). Recall the “spiral” \(S = \{1, 5, 21, 85, \ldots\}\), all of whose elements map to 1 in one iteration of the \(3x + 1\) function, and proceed as follows.

3. Definition: If \(y\) is a range element (that is, not a multiple-of-3), then \(y\) is the root of a \(y\)-tree, which has the same structure and properties as the 1-tree, except, of course, that the root is \(y\), and not 1. Thus a \(y\)-tree, like the 1-tree, is an infinitary, infinitely-deep tree.

4. We can color green the odd, positive integer associated with each of the nodes of all \(y\)-trees such that \(y\) is a range element that is an element of \(W\).

The green odd, positive integers are all those that map to 1 as a result of the computer tests having shown that at least all odd, positive integers less than \(10^{18}\) map to 1. There is an infinity of green integers because they are elements of \(y\)-trees, and each \(y\)-tree is infinitary and infinitely deep.

5. Let \(G\) denote the set of all green odd, positive integers. Let \(B\) (for blue) denote the set of all other odd, positive integers that map to 1. Thus \(G \cup B\) is the set of all odd, positive integers that map to 1. This set is fixed, by “Lemma 3.0: Statement and Proof” on page 29. It is one and the same, whether or not counterexamples exist.

If a counterexample exists, then it must be in a \(y\)-tree containing counterexamples only (which we will call a red \(y\)-tree), since if it were in a \(y\)-tree containing non-counterexamples, that would mean that it also mapped to 1, which is a contradiction.

And yet, by “Lemma 3.0: Statement and Proof” on page 29, the set \(G \cup B\) remains the same.

If counterexamples do not exist, there are no red \(y\)-trees. All \(y\)-trees are green or blue — they are all in the set \(G \cup B\).

It would seem that the assumption of the existence of counterexamples introduces a contradiction, namely, that \(G \cup B\) is not fixed. If our reasoning is correct, or can be made correct, we have a proof of the \(3x + 1\) Conjecture.

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1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than \(20 \times 2^{58} \approx 5.76 \times 10^{18}\), which is greater than \(3.33 \times 10^{16} \approx 2 \times 3^{(35 - 1)}\), have been tested and found to be non-counterexamples.
**Note 1:** One fact that might be promising for a proof of the $3x + 1$ Conjecture is that each successive element of each ‘spiral’ is an element of an infinity of successive intervals in the “spiral” $S = \{1, 5, 21, 85, 341, \ldots \}$. For example, consider the elements of the “spiral” $\{3, 13, 53, 213, \ldots \}$.

A proof that the elements of a “spiral” occupy an infinity of successive intervals in the “spiral” $S = \{1, 5, 21, 85, 341, \ldots \}$ will be found in our paper, “Are We Near a Solution to the $3x + 1$ Problem?”, on occampress.com, in the sub-section, “‘Spiral’s, Intervals and Levels” of the section, “Strategy of ‘Filling-in’ of Intervals in the Base Sequence Relative to 1”.

**Note 2:** In the case of the 1-tree for the $3x – 1$ function, the first counterexample, 5, occurs in the second interval in the “spiral” $\{1, 3, 11, 43, \ldots \}$ that maps to 1 in one iteration of the $3x – 1$ function — namely, in the interval whose elements are 5, 7, 9. (The element 7 is also a counterexample, because $<5, 7, 5, \ldots >$ is an infinite cycle that never yields 1.)

**Possible Strategy for a 1-Tree-Based Proof: Color Green All Elements of Tree Derived From Computer Tests (Second Attempt)**

1. Let $S$ denote the “spiral” $\{1, 5, 21, 85, 341, \ldots \}$ in the 1-tree. (For background on “spiral”s see “Recursive “Spiral”s: The Structure of the $3x + 1$ Function in the “Backward”, or Inverse, Direction” on page 13.)

This “spiral” is the set of odd, positive integers that map to 1 in one iteration of the $3x + 1$ function. The set of odd, positive integers lying between “spiral” elements is called an interval. Thus, $\{3\}$ is the first interval in $S$, $\{7, 9, 11, 13, 15, 17, 19\}$ is the second interval, etc.

2. By computer test$^1$, we know that all odd, positive integers less than $5 \cdot 10^{18}$ are non-counterexamples. These integers occupy more than the first 30 successive intervals in $S$.

   Call the largest odd, positive integer less than $5 \cdot 10^{18}$, $w$.

3. A basic fact governing “spiral”s is that the elements of each “spiral” other than $S$, occupy an infinity of successive intervals in $S$. Thus, consider the “spiral” $\{3, 13, 53, 213, \ldots \}$ and the intervals in the “spiral” $S$. (The proof of this basic fact will be found in our paper, “Are We Near a Solution to the $3x + 1$ Problem?”, on occampress.com, in the sub-section, “‘Spiral’s, Intervals and Levels” of the section, “Strategy of ‘Filling-in’ of Intervals in the Base Sequence Relative to 1”.)

   We remark in passing that each “spiral” element $x$, and the range element $y$ it maps to in one iteration of the $3x + 1$ function, is a tuple $<x, y>$ in a tuple-set. Since the set of elements of a “spiral” map to their range element either by all odd exponents or by all even exponents, it is easy to see that the set of all elements of all “spirals” in the 1-tree is the set of all first elements of all non-counterexample 2-tuples in all 2-level tuple-sets.

   The set of all (finite) upward paths (that is, paths in the direction of the root, 1, of the 1-tree) is

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$^1$ See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$, which is greater than $3.33 \cdot 10^{16} \approx 2 \cdot 3^{35 - 1}$, have been tested and found to be non-counterexamples.
the set of all non-counterexample tuples in the set of all tuple-sets.

4. Therefore we can go through 1, 3, 5, ..., w. Call the set of these odd, positive integers, W. We can color green each interval element y in W that is a range element of the 3x + 1 function, and then color green each element of the y-tree (same structure as the 1-tree, except with y as its root — see the above-mentioned section on “spiral”). We can then color green the elements of all extensions of each y in W, stopping at 1.

We can then color green each interval element v in W that is not a range-element (that is, which is a multiple-of-3), and all extensions of each such v, stopping at 1.

We can then color green each interval element in S that equals an element that we have already colored green. Step 3 assures us that infinities of interval elements in S greater than w will be colored green.

5. Counterexamples occupy intervals only. They are never “spiral” elements.

If counterexamples exist, they can never themselves fill an interval in S as non-counterexamples less than w have done. The reason is that this would prevent non-counterexample “spiral” elements from having “room” in the interval, as these elements must, by step 3.

But after the first counterexample, non-counterexamples can never themselves fill an interval in S as non-counterexamples have done in smaller intervals. The reason is that this would prevent counterexample elements from having room in the interval, as these elements must, by step 3.

So after the first counterexample, all subsequent intervals in S must be shared between non-counterexamples and counterexamples. There can never again be an interval containing solely non-counterexamples, and there can never be one containing solely counterexamples.

It seems hard to believe that the behavior of the 3x + 1 function would change so radically once the first counterexample has appeared — especially since a counterexample is simply another odd, positive integer that, if it and all other counterexamples were non-counterexamples, would result in an infinite succession of intervals containing solely non-counterexamples.

We might ask, informally, “How does the function ‘know’ what to do when it is given an odd, positive integer that may or may not be a counterexample?” Keeping in mind that, as far as the function is concerned, a counterexample is simply an odd, positive integer that is not in the 1-tree, one answer (invalid!) is the following.

For each odd, positive integer the function is given, it asks:
1. Is this integer in the 1-tree?

   If the answer is yes, then the function asks,
   Have I been given an integer not in the 1-tree before?

      If the answer is yes, then the function simply proceeds with its mixed-intervals-only rule (each interval contains both non-counterexamples and counterexamples);

      If the answer is no, then the function continues its practice of filling successive intervals entirely with non-counterexamples.

2. If the answer to 1. is no, then the function asks,
   Have I been given an integer not in the 1-tree before?

      If the answer is yes, then the function simply proceeds with its mixed-intervals-only rule;
If the answer is no, then the function begins the permanent application of its mixed-intervals-only rule.

But of course there is nothing like this question/answer sequence in the definition of the 3x + 1 function.

Whether all this can lead to a proof of the 3x + 1 Conjecture, remains to be seen.

Possible Strategy for a 1-Tree-Based Proof: Color Green All Elements of Tree Derived From Computer Tests (Third Attempt)

1. If a counterexample exists, then an infinity of counterexamples exists (“Lemma 2.0: Statement and Proof” on page 29). So if a counterexample exists, an infinity of odd, positive integers is not in the 1-tree.

2. 1. By computer test\(^1\), we know that all odd, positive integers less than \(10^{18}\) are non-counterexamples. Let \(w\) denote the largest of these odd, positive integers.

3. We can color green all “spiral” elements that must be in the 1-tree as a result of all consecutive odd, positive integers in the range 1 through \(w\) being non-counterexamples.

   If counterexamples do not exist, then all these “spiral” elements are part of the 1-tree.

4. Let \(S\) denote the “spiral” \(\{1, 5, 21, 85, 341, \ldots\}\) in the 1-tree. This is the set of all odd, positive integers that map to 1 in one iteration of the 3x + 1 function.

   If counterexamples do not exist, then all the “spiral” elements of each \(y\)-tree, where \(y\) is a range element in \(S\) that is greater than \(10^{18}\), are part of the 1-tree.

4. We conclude that counterexamples do not exist, by the following reasoning:

   Let \(p\) denote “counterexamples do not exist”.
   Let \(q\) denote our conclusion in step 3.
   Let \(q'\) denote our conclusion in step 4.

   Then the statement

   \[(1)\]

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1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than \(20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}\), which is greater than \(3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35 - 1)}\), have been tested and found to be non-counterexamples.
((p implies q) and (p implies q′) and q and q′) implies p

if q and q′ are both true, can only be valid (true) if p is true.

(Proof:

If p is false, and q and q′ are true, then we have, from (1):

((false implies true) and (false implies true) and true and true) implies false, that is,

((true) and (true) and true and true) implies false, that is

(true) implies false,

which is false.)
Appendix D — For Professional Mathematicians Only

Understandable Reluctance of Mathematicians to Read This Paper

There has been an understandable reluctance on the part of professional mathematicians to give serious attention to this paper, or its predecessors. It seems clear to us that the main reason is mathematicians’ difficulty in believing that such an extraordinarily difficult problem can have been solved by a non-mathematician (our degree is in computer science, and we have spent most of our working life doing research in the computer industry). This skepticism is reinforced by the fact that there have been many false claims of solutions to the 3x + 1 Problem, the overwhelming majority of which having been made by non-mathematicians.

But we must point out that the occasionally-heard remark, “Nothing of importance in mathematics has ever come from outside the university”, is, in fact, false, considering that some of the best of the best worked entirely outside the university — Descartes, Pascal, Fermat, Leibniz, and Galois, to name only the most famous.

We must also not fail to mention another reason for mathematicians’ reluctance to read this paper, and that is the online presence of obsolete criticisms of the paper. For example, Stack Overflow has a website containing criticisms of a proof in a 2015 version of the paper. Not only were the criticisms false, but the proof that was criticized has long since been removed from the paper. This website appears next to the website containing this paper, and thus unquestionably discourages potential readers — especially mathematicians — from reading this paper. Yet despite many pleas on our part, the managers of the website have refused to delete the criticisms or to add a note to the website stating that the criticisms do not apply to the current version of the paper. Nor have they explained to us the reason for their refusals. Apparently we have no recourse in this matter, except to encourage others to boycott the Stack Overflow websites, and to write to the organization explaining the reason for the boycott. The email address is team@stackoverflow.com, the item no. is 201708202111462820.

It appears that the managers of the above website have no experience of actually doing research. They believe that if a paper is published online and contains an error, that means that the author is incapable of correcting the error, and that his underlying ideas do not deserve any attention. But errors are almost inevitable in the course of attempting to solve very difficult problems. We remind the reader that Wiles’ first proposed proof of the Taniyama–Shimura–Weil Conjecture in the early 90s, which implied a proof of Fermat’s Last Theorem, contained an error that took Wiles, with the help of the mathematician Richard Taylor, more than a year to repair. The important question obviously was, Do the underlying ideas in this paper offer hope for correcting the error? And the answer was yes.

We have been struck by the eagerness with which readers of this paper look for anything they can regard as an error, and the indifference they display to understanding, and thinking about, the underlying ideas.

If You Do Not Accept Our Proofs of the 3x + 1 Conjecture...

If you do not accept our proofs of the Conjecture, or any of the possible strategies for a proof that are set forth in the above appendices, we urge you to at least peruse our paper, “Are We Near a Solution to the 3x + 1 Conjecture?” on occampress.com. This paper contains a wealth of results, insights, possible strategies for a proof, plus a section on what we have called “3x + 1-like functions”. We will welcome comments.
We are confident that at least two publishable, significant papers can be produced from the material in our $3x + 1$ papers, and that this is true even if the proof of the Conjecture in the present paper and all the possible strategies in the appendices, are faulty and cannot be repaired.

We feel that the two structures we have discovered that underlie the $3x + 1$ function, namely, tuple-sets and recursive “spiral”s, are of fundamental importance, and should be brought to the attention of the entire $3x + 1$ research community.

**Difficulty, So Far, In Getting This Paper Published**

Not surprisingly, so far, no journal that we know of is willing to even consider this paper for possible publication. The reason seems to be that editors cannot believe that such a difficult problem might have been solved by a non-mathematician.

**Incentives for Mathematicians to Take This Paper Seriously and to Spread the Word About It**

Unquestionably, if this paper contains a solution to the $3x + 1$ Problem, or can easily be modified to contain a solution, considerable prestige will be gained by the first mathematicians who promote the paper. Of course, mathematicians who believe that the proof of the Conjecture is correct, and/or that at least one of the possible strategies in the appendices look promising, but do not want to risk their reputations by saying so, especially given that the author of the paper is not an academic mathematician, will not be recipients of that prestige. We are offering three incentives:

1. Any reasonable consulting fee;
2. Generous mention in the Acknowledgments when the paper is published. (But no name will be mentioned without the prior written approval of the mathematician concerned.)
3. An offer of shared authorship to the first mathematician who makes a significant contribution to the paper prior to publication.

In any case, all communications we receive about this paper will be kept strictly confidential.
Appendix H — Second Proof of the $3x + 1$ Conjecture

The following proof of the $3x + 1$ Conjecture is based on the idea underlying the proof of (“Theorem: The $3x + 1$ Conjecture is True.” on page 21). Like that one, it shows that the set of all tuple-sets (structure and contents) is the same, whether or not counterexamples exist. This implies that, if counterexamples exist, there is a contradiction, hence counterexamples do not exist.

(Note: we ask the reader to inform us of the first sentence that the reader believes contains an error, and what that error is.)

1. Definitions: an anchor tuple is the first $i$-level tuple in an $i$-level tuple-set, where $i \geq 2$. An anchor is the $i$-level element of the anchor tuple in an $i$-level tuple-set.

(a) If $x$ is a range element of the $3x + 1$ function, then $x$ is eventually — for some $i \geq 2$ — an anchor.

Proof: If $x$ exists, then for some $i \geq 2$, $x < 2 \cdot 3^{(i-1)}$. Therefore, $x$ is an anchor (part (a) of “Lemma 1.0: the “Distance” Functions d(i, i) and d(1, i)” on page 10). □

It follows trivially that $x$ is also an anchor for all greater $i$. (“Once an anchor, always an anchor.”)

(b) If $x$ is a non-counterexample anchor, then it is a non-counterexample anchor whether or not counterexamples exist.

Proof: The arithmetic defining the $3x + 1$ function is not itself a function of the truth or falsity of the $3x + 1$ Conjecture. □

Thus, for example, 13 is a non-counterexample (maps to 1) today, and if the Conjecture is proved true tomorrow, it will be a non-counterexample tomorrow, and if the Conjecture is proved false tomorrow it will still be a non-counterexample.

(Actually, the statement (b) holds for non-counterexamples in general, not just non-counterexample anchors.)

2. At this point, it is reasonable to assume that there are two possible sets of anchors: one containing counterexamples if counterexamples exist, and one not containing counterexamples, if counterexamples do not exist.

However this assumption is false.

(1) There is one and only one set of anchors, regardless if counterexamples exist or not.

Proof:

(a) The “distance” between consecutive $i$-level elements of an $i$-level tuple-set is $2 \cdot 3^{(i-1)}$ (follows from part (a) in “Lemma 1.0: the “Distance” Functions d(i, i) and d(1, i)” on page 10).

Thus, for example, the distance between the first and second 2-level elements of any 2-level tuple-set having 1 as first element, namely, between the elements 1 and 7, is $2 \cdot 3^{(2-1)} = 2 \cdot 3^1 = 6$. The distance between the second and third elements, that is, between the elements 7 and 13, is
likewise 6.

(b) Each $i$-level anchor is less than $2 \cdot 3^{(i-1)}$ (follows from part (a) in “Lemma 1.0: the “Distance” Functions $d(i, i)$ and $d(1, i)$” on page 10). Of course, each anchor is greater than 0, by definition of the domain of the $3x + 1$ function.

(c) For each $i \geq 2$, the number of anchors in all $i$-level tuple-sets is $2 \cdot 3^{((i-1)-1)}$ (follows from part (a) in “Lemma 1.0: the “Distance” Functions $d(i, i)$ and $d(1, i)$” on page 10).

Thus, for example, the number of anchors in all 2-level tuple-sets is $2 \cdot 3^{((2-1)-1))} = 2 \cdot 3^0 = 2$. These anchors are 1 and 5. It is easy to show that 1 is mapped to by all even exponents, and 5 is mapped to by all odd exponents. Those are the only two possibilities for the anchors of 2-level tuple-sets.

The number of anchors in all 3-level tuple-sets is $2 \cdot 3^{((3-1)-1))} = 2 \cdot 3^1 = 6$. These anchors are 1, 5, 7, 11, 13, 17.

(d) The set of $(i+1)$-level anchors comes into being as follows:

If $a$ is an $i$-level anchor then $a$ is an $(i+1)$-level anchor, because if $a$ is less than $2 \cdot 3^{(i-1)}$, as it must be if $a$ is an $i$-level anchor, then $a$ is certainly less than $2 \cdot 3^{((i+1)-1)}$.

Since the $i$-level tuple-set element $a + 1 \cdot (2 \cdot 3^{(i-1)})$ is less than $2 \cdot 3^{((i+1)-1)}$, the element is an $(i+1)$-level anchor.

Since the $i$-level tuple-set element $a + 2 \cdot (2 \cdot 3^{(i-1)})$ is less than $2 \cdot 3^{((i+1)-1)}$, the element is an $(i+1)$-level anchor.

No other element of an $i$-level tuple-set is less than $2 \cdot 3^{((i+1)-1)}$, and therefore no other element of an $i$-level tuple-set is an $(i+1)$-level anchor.

The reader can see an example of this increase in anchors from level 2 to level 3 in step 2 (c).

(e) The process we have described is unique. It yields all $i$-level anchors for all $i \geq 2$. There is thus one and only one set of anchors. In other words:

If counterexamples do not exist, then the set of all anchors is exactly the set that results from the process we have described. Call that set $S$.

If counterexamples exist, then the set of all anchors is exactly the set that results from the process we have described. In other words, if counterexamples exist, then the set of all anchors is the same set $S$. □

3. Computer tests\(^1\) have shown the Conjecture to be valid for all consecutive odd positive integers up to at least $10^{18} + 1$, which includes all the anchors (each of which is a non-counterexample anchor) from level 2 through level 35.

\(^1\) See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$, which is greater than $3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35 - 1)}$, have been tested and found to be non-counterexamples. These include the set of all 35-level anchors.
A Solution to the $3x + 1$ Problem

(This is not true in the case of the $3x - 1$ function since the first counterexample in the case of that function is 5.)

Furthermore, there is an infinity of non-counterexample anchors at levels greater than 35. \textit{Proof:} each range element, hence each non-counterexample anchor at levels 2 through 35 is mapped to by an infinity of odd, positive integers (see Lemma 13.0 in our paper, “Are We Near a Solution to the $3x + 1$ Problem?”, on occampress.com), and range element there is an infinity of them in that infinity of odd, positive integers is also mapped to by an infinity of odd, positive integers, etc.. The fact that there is an infinity of these range elements in each case means that an infinity of them are greater than the largest anchor at level 35. $\square$

The unique process for generating anchors (step 2) then continues to generate anchors for all levels beyond 35. The set of anchors so generated for each level is the same whether or not counterexamples exist (the process is unique). So, in particular, we can regard the process as generating the set of all non-counterexample anchors.

If counterexamples exist, the set of anchors so generated is the same as the set of anchors if counterexamples do not exist. Each anchor is an element of an infinite tuple. Non-counterexample infinite tuples are, by definition, of the form $<x, ..., 1, 1, 1, ...>$, whereas counterexample infinite tuples are of the form $<y, ...>$, with no element equal to 1.

And so if counterexamples exist, then some counterexample anchors are the same as non-counterexample anchors, which is absurd. Therefore the $3x + 1$ Conjecture is true.

Another way of stating our argument here is:

The set of all tuple-sets (structure and contents) is the same, whether or not counterexamples exist. Therefore there is no difference between the set of all counterexamples and the set of all non-counterexamples. Therefore, counterexample tuples behave exactly the same as non-counterexample tuples, which is absurd. Therefore counterexamples do not exist, and the Conjecture is true. $\square$

\textbf{Remark}

Suppose that the anchors were all and only those odd, positive integers that map to 1. Suppose, further, that if counterexamples exist, they never become anchors. Then there would be no difficulty: the set of anchors would be fixed, whether or not counterexamples existed, and they would all map to 1.

However, the simple argument in step 1 (a) shows that if counterexamples exist, they must eventually be anchors. And so there is, in reality, a difficulty: how to reconcile this fact with the fact that the set of anchors is fixed whether or not counterexamples exist. Our proof, above, shows one way to reconcile this fact.
Appendix I — The 3x – 1 Test

There is no question but that what we have called the “3x – 1 Test” has helped us to recognize errors in our proposed proofs of the Conjecture. (In this Test, we ask if our proof also proves the 3x – 1 Conjecture. If it does, then it is claimed that our proof is in error, because there are known counterexamples to the 3x – 1 Conjecture, the smallest of which is 5.)

However, we make the following counterargument to this claim, and hence to the validity of the 3x – 1 Test:

The question we ask those who use the 3x – 1 Test to claim a proof of ours has an error, is: “Suppose you didn’t know about the 3x – 1 function, and hence about the 3x – 1 Test. What would your criticism be then?”

A proof must stand or fall on its own terms. Mathematical logic, and, in particular, computer-ized proof-checking, would face an insurmountable obstacle if each proof of a conjecture required that the author, or the proof-checker, find all and only the related conjectures (whatever “related” means) that are known to be false, and then prove that the proof in question did not also prove one of those conjectures.

Furthermore, even if our proof of the Conjecture proves none of the related conjectures, each of which is known to be false, that in no way confirms the validity of our proof! There may be other infinite cycles in one or more of the 3x + 1-like functions, and/or there may be computations that never yield 1, although they are not infinite cycles.

Our proofs in this paper apply to a function having the property that all consecutive odd, positive integers between 1 and 10^{18} – 1 are known, by computer test, to be non-counterexamples. No counterexample is known.

The 3x – 1 function, on the other hand, has the property that only the first two consecutive odd, positive integers, starting at 1, are known, by computer test, to be non-counterexamples. The first counterexample is 5. (In passing, we remark that it might be significant that the first two consecutive odd, positive integers that are non-counterexamples, namely, 1 and 3, are both integers that map directly to 1: (3(1) – 1)/2^1 = 1, and (3(3) – 1)/2^3 = 1.)

So, it seems to us entirely possible that a proof can be valid when applied to the 3x + 1 function, and invalid when applied to the 3x – 1 function.