A Solution to the 3*x* + 1 Problem

by

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"Very often in mathematics the crucial problem is to recognize and to discover what are the relevant concepts; once this is accomplished the job may be more than half done."[1]

"One of the greatest contributions a mathematician can make is to spot something so simple and powerful that everybody else has missed it."[2]

***Note* 1:** The reader who is sorely pressed for time is encourage to begin with "Statement of Problem" on page 3, then go to "Brief Description of Tuple-sets" on page 4, and then go to "Appendix H — Third Proof of the 3x + 1 Conjecture" on page 44. Any competent mathematician should be able to read and understand all this material in less than 20 min.

Note* 2*: The reader can safely assume, *initially*, that all referenced lemmas in this paper are true, since their proofs have been checked and deemed correct by several mathematicians. Thus it is only necessary to read pages 1 - 9 in order to understand in detail our first proof of the $3x + 1$ Conjecture.

***Note* 3:** The author is seeking a professional mathematician to help prepare this paper for publication. The author will pay any reasonable consulting fee, and, of course, give generous credit in the Acknowledgments (but only with the mathematician's prior written approval).

Key words: $3x + 1$ Problem, $3n + 1$ Problem, Syracuse Problem, Ulam's Problem, Collatz Conjecture, computational number theory, proof of termination of programs, recursive function theory

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1. Herstein, I. N., *Topics in Algebra*, John Wiley & Sons, N.Y., 1975, p. 50.
2. Stewart, Ian, *The Problems of Mathematics*, Oxford University Press, N.Y., 1992, pp. 279-280.

Abstract

The $3x + 1$ Problem asks if repeated iterations of the function $C(x) = (3x + 1)/(2^a)$ always terminate in 1 Here x is an odd, positive integer, and $a = ord_2(3x + 1)$, the largest positive integer such that the denominator divides the numerator. The conjecture that the function always eventually terminates in 1 is the *3x + 1 Conjecture*. An odd, positive integer that maps to 1 is called a *non-counterexample*; an odd, positive integer that doesn't map to 1 is called a *counterexample* (to the Conjecture).

Our first proof is based on a structure called *tuple-sets* that represents the $3x + 1$ function in the "forward" direction. In our proof, we show that the 35-level elements of all 35-level tuples in all 35-level tuple-sets are the same, regardless if counterexamples to the Conjecture exist or not[1]. From this fact, a simple inductive argument allows us to conclude that all tuple-sets are the same, whether counterexamples exist or not, and hence that counterexamples do not exist.

Our second proof is based on a structure called *the 1-tree*, which is at least part of the $3x + 1$ function in the "inverse" direction. It represents all odd, positive integers that map to 1. We show that, counter-intuitively, there is only one 1-tree, and from that fact we derive our proof.

Our third proof, like the first, is based on tuple-sets. In this proof, we define *anchor*, which is the i-level element of the first i-level tuple in an i-level tuple-set. We then show that there is one and only one set of anchors for all i, regardless if counterexamples exist or not. We then show that this implies that there is one and only one set of infinite tuples, regardless if counterexamples exist or not, and from this we deduce that, if counterexamples exist, then some infinite tuples must be both counterexample and non-counterexample tuples, which is absurd, hence counterexamples do not exist and the Conjecture is true.

As far as we have been able to determine, our approaches to a solution of the Problem is original.

1. A phrase of the form "q regardless if p" is equivalent to "(if p then q) and (if *not-p* then q)". It is meaningful and in fact true as long as q is true, which it always is in this paper. Instances of the phrase occur in everyday speech, for example, "Fermat's Last Theorem is true regardless if the Riemann Conjecture is true".,

Introduction

Statement of Problem

For x an odd, positive integer, set

$$C(x) = \frac{3x+1}{2^{ord_2(3x+1)}}$$

where $ord_2(3x + 1)$ is the largest exponent of 2 such that the denominator divides the numerator. Thus, for example, C(17) = 13, C(13) = 5, C(5) = 1. The $3x + 1$ Problem, also known as the $3n$ + 1 Problem, the Syracuse Problem, Ulam's Problem, the Collatz Conjecture, Kakutani's Problem, and Hasse's Algorithm, asks if repeated iterations of C always terminate at 1. The conjecture that they do is hereafter called the *3x + 1 Conjecture*, or sometimes, in this paper, just *the Conjecture*. We call C the $3x + 1$ function; note that $C(x)$ is by definition odd.

An odd, positive integer such that repeated iterations of C terminate at 1, we call a *non-counterexample*. An odd, positive integer such that repeated iterations of C never terminate at 1, we call a *counterexample*.

Other equivalent formulations of the $3x + 1$ Problem are given in the literature; we base our formulation on the C function (following Crandall) because, as we shall see, it brings out certain structures that are not otherwise evident.

Summary of Research on the Problem

As stated in (Lagarias 1985), "The exact origin of the $3x + 1$ problem is obscure. It has circulated by word of mouth in the mathematical community for many years. The problem is traditionally credited to Lothar Collatz, at the University of Hamburg. In his student days in the 1930's, stimulated by the lectures of Edmund Landau, Oskar Petron, and Issai Schur, he became interested in number-theoretic functions. His interest in graph theory led him to the idea of representing such number-theoretic functions as directed graphs, and questions about the structure of such graphs are tied to the behavior of iterates of such functions. In the last ten years [that is, 1975-1985] the problem has forsaken its underground existence by appearing in various forms as a problem in books and journals..."

Lagarias has performed an invaluable service to the $3x + 1$ research community by publishing several annotated bibliographies relating to the Problem. These are accessible on the Internet.

On the Structure of This Paper

To enhance readability, we have placed proofs of all lemmas in "Appendix A — Statement and Proof of Each Lemma" on page 17.

In Memoriam

Several of the most important lemmas in this paper (though not Lemma 1.0) were originally conjectured by the author and then proved by the late Michael O'Neill. He made a major contribution to this research, and is sorely missed.

Tuple-sets: The Structure of the 3*x* + 1 Function in the "Forward" Direction

Brief Description of Tuple-sets

The following should be sufficient for the reader to understand at least our latest, and, we believe, our best, proof of the $3x + 1$ Conjecture (see "Appendix H — Third Proof of the 3x + 1 Conjecture" on page 44).

1. We use the definition of the $3x + 1$ function in which all successive divisions by 2 are collapsed into a single exponent of 2 (see "Statement of Problem" on page 3). Thus, for example, the tuple <9, 7, 11> represents the fact that

9 maps to 7 in one iteration of the function, via the exponent 2, because $(3(9) + 1)/2^2 = 7$;
7 maps to 11 in one iteration of the function, via the exponent 1, because $(3(7) + 1)/2^1 = 11$.

2. We see that the sequence of exponents associated with the tuple <9, 7, 11> is {2, 1}.

3. A tuple-set T_A is the set of all finite tuples that are associated with the exponent sequence A (and "approximations" to A, but this is not important for our proofs of the 3x + 1 Conjecture). In our case, $A = \{2, 1\}$.

In addition to the tuple <9, 7, 11> , the tuple-set $T_A = T_{\{2,\ 1\}}$ contains the tuples <25, 19, 17> , <41, 31, 47>, and an infinity of others, each associated with the exponent sequence {2, 1}. (See "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 8.)

4. Facts about tuple-sets:

An i-level tuple-set T_A, $i \geq 2$, contains (among other tuples, see previous step) all $(i + 1)$-element tuples that are associated with the exponent sequence A.

There is an infinity of tuples in each tuple-set.

The set of all tuple-sets contains tuples representing all finite iterations of the $3x + 1$ function.

Full Description of Tuple-sets

Definitions

Iteration

An *iteration* takes an odd, positive integer, x, to an odd, positive integer, y, via one application of the $3x + 1$ function, C. Thus, in one iteration C takes 17 to 13 because $C(17) = 13$.

Tuple

A (finite) *tuple* is a finite sequence of zero or more successive iterations of C, that is, $\langle x, C(x), C^2(x), ..., C^k(x)\rangle$, where $k \geq 0$.

A finite tuple is the prefix of an infinite tuple. If x is a non-counterexample, then x is the first element of an infinite tuple $\langle x, y, ..., 1, 1, 1, ... \rangle$. Of course, if x is a range element of C, then x can be an element other than the first in another non-counterexample tuple.

In the literature, a tuple (finite or infinite) is usually called a *trajectory* or an *orbit*.

If x is a counterexample, then x is the first element of an infinite tuple $\langle x, y, ... \rangle$ which does not contain 1. Of course, if x is a range element of C, then x can be an element other than the first in another counterexample tuple.

A counterexample tuple must be one of two types: either there is an infinitely-repeated finite cycle of elements (none of which is 1) in the infinite tuple having the counterexample x as first element, or else there is no such cycle, but there is no 1 in the infinite tuple having the counterexample x as first element — in other words, there is no upper bound to the elements of the infinite tuple.

Exponent, Exponent Sequence

If $C(x) = y$, with $y = (3x + 1)/2^a$, we say that *a is the exponent associated with x*. In more formal language, this can be expressed as $ord_2(3x + 1) = a$. Sometimes we simply write $e(x) = a$. The sequence $\{a_2, a_3, ..., a_i\}$, where $a_2, a_3, ..., a_i$ are the exponents associated with x, $C(x)$, ..., $C^{(i-1)}(x)$ respectively, is called an *exponent sequence*. We number exponents beginning with a_2 in order that the subscript corresponds to a level number in the corresponding tuple-set. See "Levels in Tuples and Tuple-sets" on page 6

We say that *x maps to y via a_i* if $C(x) = y$ and $ord_2(3x + 1) = a_i$. By extension, we say that x *maps to z* if z is the result of a finite sequence of iterations of C beginning with x, that is if the tuple $\langle x, y, ..., z\rangle$ exists.

Tuple-set[1]

Let $A = \{a_2, a_3, ..., a_i\}$ be a finite sequence of exponents, where $i \geq 2$. The *tuple-set T_A* consists of all and only the tuples that are associated with all successive approximations to A. Thus T_A consists of all and only the following tuples. (*Note*: First elements x in different tuples are different odd, positive integers. No two tuples in a tuple-set have the same first element.)

all tuples $\langle x\rangle$ such that x does not map to an odd, positive integer via a_2;

all tuples $\langle x, y\rangle$ such that x maps to y via a_2 but y does not map to an odd, positive integer via a_3;

1. The literature contains several results that establish properties of the 3x + 1 function that are equivalent to some of those for tuple-sets. However, the language is very different, and the definition of the 3x + 1 function that is used is not ours, but the original one, in which each division by 2 is a separate node in the tree graph representing the function.

all tuples $<x, y, y'>$ such that x maps to y via a_2 and y maps to y' via a_3, but y' does not map to an odd, positive integer via a_4;

...

all tuples $<x, y, y',..., y^{(i-3)}, y^{(i-2)}>$ such that x maps to y via a_2 and y maps to y' via a_3 and ... and $y^{(i-3)}$ maps to $y^{(i-2)}$ via the exponent a_i. (The longest tuple in an i-level tuple-set has i elements.)

In other words, for each i-level exponent sequence A:

there are tuples $<x>$ whose associated exponent sequence is a prefix of A for no exponent of A, and
there are other tuples $<x, y>$ whose associated exponent sequence is a prefix of A for the first exponent of A, and
there are other tuples $<x, y, y'>$ whose associated exponent sequence is a prefix of A for the first two exponents of A, and
...
there are other tuples $<x, y, z, ..., y^{(i-2)}>$ whose associated exponent sequence is a prefix of A for all $i - 1$ exponents of A.

Tuples are ordered in the natural way by their first elements.

The set of first elements of all tuples in a tuple-set is the set of odd, positive integers (see proof under "The Structure of Tuple-sets" on page 7). Thus, there is a countable infinity of tuples in each tuple-set.

For each $i \geq 2$, tuple-sets are a *partition* of the set of all i-level tuples.

Levels in Tuples and Tuple-sets

Let A be an i-level exponent sequence, $\{a_2, a_3, ..., a_i\}$. The reason subscripts of exponents begin with 2, rather than with 0 or 1, is so that they correspond to levels in each tuple-set. (No tuple-set has only one level, because that would mean it is associated with no exponent sequence.) Let T_A be the tuple-set determined by A. Then, by definition of tuple-set, there exist j-level tuples in T_A, where $1 \leq j \leq i$, that is, tuples $t = <x, y, ..., z>$, where x is the 1-level element of t, y is the 2-level element of t, ..., and z is the j-level element of t. We say that T_A is an i-level tuple-set, and we sometimes speak of the set of j-level tuple-elements in T_A, where $1 \leq j \leq i$.

For $2 \leq j \leq i$, two tuples are said to be *consecutive at level j* if no j-level or higher-level tuple exists between them.

Example of Tuple-set

As an example of (part of) a tuple-set: in Fig. 1, where $A = \{a_2, a_3, a_4\} = \{1, 1, 2\}$ and where we adopt the convention of orienting tuples vertically on the page, the tuple-set T_A includes:

the tuple $<1>$, because $e(1) = 2 \neq (a_2 = 1)$;
the tuple $<3, 5>$, because $e(3) = (a_2 = 1)$, but $e(5) = 4 \neq (a_3 = 1)$;
the tuple $<5>$, because $e(5) = 4 \neq (a_2 = 1)$;

the tuple <7, 11, 17, 13> because $e(7) = 1$ ($a_2 = 1$) and $e(11) = 1$ ($a_3 = 1$) and $e(17) = 2$ ($a_4 = 2$);
etc.

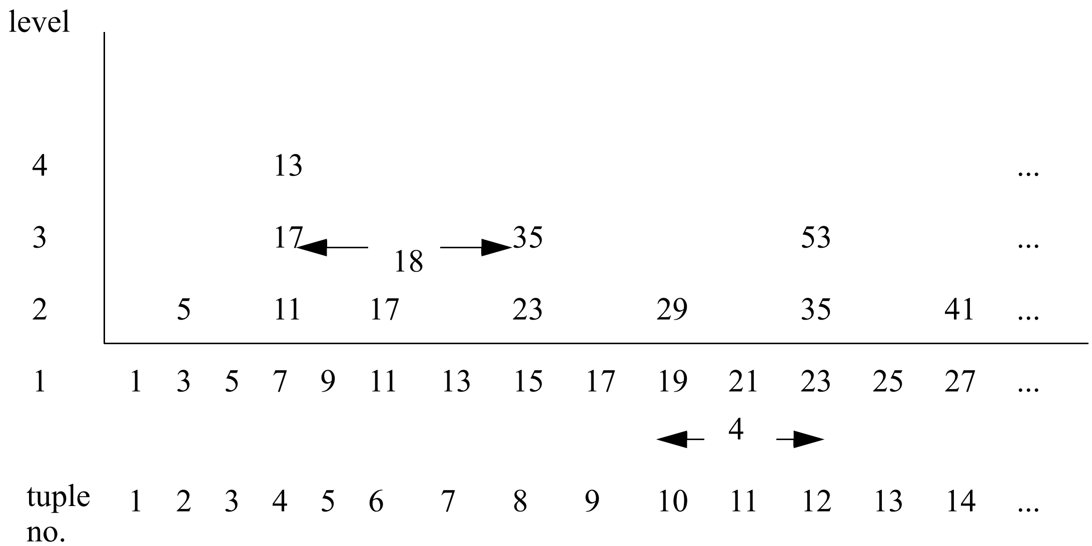

Fig. 1. Part of the tuple-set T_A associated with the sequence $A = \{1, 1, 2\}$

The number 18 between the arrows at level 3 and the number 4 between the arrows at level 1 are the values of the level 3 and level 1 distance functions, respectively, established by Lemma 1.0 (see "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 8).

In each i-level tuple-set T_A, where $i \geq 2$, for each odd, positive integer x there exists a tuple whose first element is x. The tuple may be one-level ($<x>$), or 2-level ($<x, y>$), or ... or i-level ($<x, y, y', ..., y^{(i-3)}, y^{(i-2)}>$). Thus each tuple-set is non-empty.

Graphical Representation of the Set of All Tuple-sets

It is clear from the definition of *tuple-set* that the set of all tuple-sets can be represented by an infinitary tree in which each node is a tuple-set. We can imagine the tuple-set (which contains an infinity of tuples) extending into the page.

The Structure of Tuple-sets

It is important for the reader to understand that the structure of each tuple-set is unchanged by the presence or absence of counterexample tuples. Regardless if counterexample tuples exist or not, the set of first elements of all tuples in each tuple-set is always the same, namely, the set of odd, positive integers. *Proof:* Let x be any odd, positive integer and let $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, be any exponent sequence. Then there are exactly two possibilities:

(1) x maps to a y in a single iteration of the $3x + 1$ function, C, via the exponent a_2, or
(2) x does not map to a y in a single iteration of C via the exponent a_2.

But if (1) is true, then a tuple containing at least two elements, with x as the first, is in T_A; if (2) is true, then the tuple <x> is in T_A. There is no third possibility.□

For each tuple-set, the first element of the first tuple is 1, the first element of the second tuple is 3, the first element of the third tuple is 5, etc.

It can never be the case that, if counterexample tuples exist, then somehow there are "more" tuples in a tuple-set than if there are no counterexample tuples[1].

Furthermore, the distance functions defined in "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 8 are the same regardless if counterexample tuples exist or not.

Extensions of Tuple-sets

Since there is a tuple-set for each finite sequence A of exponents, it follows that each tuple-set T_A has an extension via the exponent 1, and an extension via the exponent 2, and an extension via the exponent 3, ... In other words, if $A = \{a_2, a_3, ..., a_i\}$, then there is a tuple-set $T_{A'}$, where $A' = \{a_2, a_3, ..., a_i, 1\}$, and a tuple-set $T_{A''}$, where $A'' = \{a_2, a_3, ..., a_i, 2\}$, and a tuple-set $T_{A'''}$, where $A''' = \{a_2, a_3, ..., a_i, 3\}$, ...

All this is true whether or not the tuple-set T_A and/or any of its extensions contains counterexample tuples or not.

For further details on extensions of tuple-sets, see "How Tuple-sets 'Work'" and the proof that there exists an extension for each tuple-set ("Lemma 3.0 Statement and Proof") in our paper, "Are We Near a Solution to the 3x + 1 Problem?" on occampress.com.

Lemma 1.0: the "Distance" Functions *d*(*i*, *i*) and *d*(1, *i*)

(a) *Let* $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, *be a sequence of exponents, and let* $t_{(r)}$, $t_{(s)}$ *be tuples consecutive at level*[2] *i in* T_A. *Then d(i, i) is given by*:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) *Let* $t_{(r)}$, $t_{(s)}$ *be tuples consecutive at level i in* T_A. *Then d(1, i) is given by:*

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3})\ldots(2^{a_i})$$

Proof: see "Lemma 1.0: Statement and Proof" on page 17

It follows from part (a) of the Lemma that the set of all i-level elements of all i-level first tuples in all i-level tuple-sets is $\{z \mid 1 \leq z < 2 \cdot 3^{i-1}\}$, where z is an odd, positive integer not divis-

1. To make this statement more precise: in no tuple-set does there ever exist a first element of a tuple, regardless how large that first element is, such that there are more tuples in that tuple-set having smaller first elements if counterexamples exist, than if counterexamples do not exist.
2. For $2 \leq j \leq i$, two tuples are consecutive at level j if no j-level or higher-level tuple exists between them (see "Levels in Tuples and Tuple-sets" on page 6.

ible by 3.

Remark: Relationships similar to those described in parts (a) and (b) of the Lemma hold for successive j-level tuples, where $2 \leq j < i$. The following table shows these relationships for $(i - j)$-level elements of tuples consecutive at level $(i - j)$ in an i-level tuple-set, where $0 \leq j \leq (i - 1)$. The distances are easily proved using Lemma 1.0.

Table 1: Distances between elements of tuples consecutive at level *i*

Level	Distance between $(i - j)$-level elements of tuples consecutive at level $(i - j)$, where $0 \leq j \leq (i - 1)$
i	$2 \cdot 3^{i-1}$
$i - 1$	$2 \cdot 3^{i-2} \cdot 2^{a_i}$
$i - 2$	$2 \cdot 3^{i-3} \cdot 2^{a_{i-1}} 2^{a_i}$
$i - 3$	$2 \cdot 3^{i-4} \cdot 2^{a_{i-2}} 2^{a_{i-1}} 2^{a_i}$
...	...
2	$2 \cdot 3 \cdot 2^{a_3} \ldots 2^{a_{i-1}} 2^{a_i}$
1	$2 \cdot 2^{a_2} 2^{a_3} \ldots 2^{a_{i-1}}\ 2^{a_i}$

Further details can be found in the section, "Remarks About the Distance Functions" in our paper, "Are We Near a Solution to the 3x + 1 Problem?", on occampress.com.

Lemma 2.0

Assume a counterexample exists. Then for all i ≥ 2*, each i-level tuple-set contains an infinity of i-level counterexample tuples and an infinity of i-level non-counterexample tuples*.

Proof: "Lemma 2.0: Statement and Proof" on page 22

Recursive "Spiral"s: The Structure of the 3x + 1 Function in the "Backward", or Inverse, Direction

The 1-Tree

We define a tree called the *1-tree* as follows (see Fig. 4 below):

The set of all odd, positive integers that map to 1 in a *single* iteration[1] of the $3x + 1$ function is Level 1 in the tree; This set is {1, 5, 21, 85, 341, ... }. (Here and in all subsequent steps, we do not count iterations that take 1 to 1.)

The set of all odd, positive integers that map to 1 in *two* iterations of the $3x + 1$ function is Level 2 in the tree;

The set of all odd, positive integers that map to 1 in *three* iterations of the $3x + 1$ function is Level 3 in the tree;

...

Thus the 1-tree contains all odd, positive integers that map to 1 in a finite number of iterations of the function.

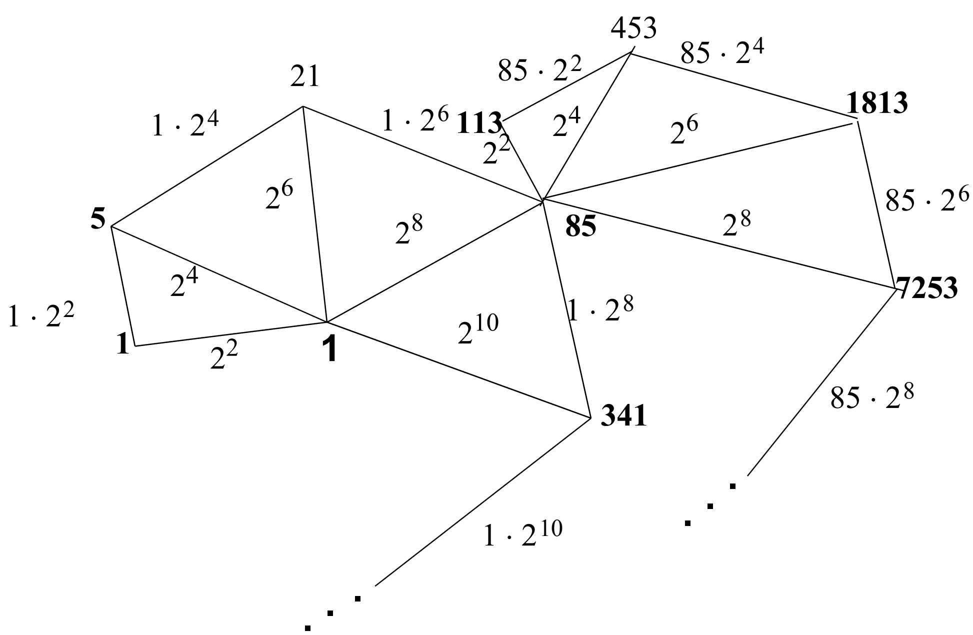

Fig. 4. Recursive "spirals" structure of odd, positive integers that map to 1.

Bold-faced numbers are range elements (21 and 453 are multiples of 3, hence not range elements). Partial "spirals" surrounding the base elements 1 and 85 are shown. The line connecting 1813 to 85 is marked with a 2^6 because $(3 \bullet 1813 + 1)/ 2^6 = 85$. The line connecting 453 to 1813 is marked $85 \bullet 2^4$ because $453 + 85 \bullet 2^4 = 1813$. The quantity $85 \bullet 2^4 = 3 \bullet 453 + 1$, and similarly for the difference between successive elements of a "spiral" in all "spiral"s. These facts follow from the fact that if x, y are consecutive elements of a "spiral", with $x < y$, then $y = 4x + 1$.

1. We remind the reader that throughout this paper we use, not the original definition of the $3x + 1$ function, but Crandall's equivalent definition, in which all successive divisions by 2 are collapsed into a single exponent of 2. See "Statement of Problem" on page 3.

Not all elements of a "spiral" are mapped to by even exponents. For example, the "spiral" of numbers (not shown) mapping to 5 has odd exponents, as does the "spiral" mapping to 341. It is easily shown that successive elements of a "spiral" have the pattern ... *e*, *o*, 3, *e*, *o*, 3, *e*,..., meaning that there is

an element mapped by all even exponents, then
an element mapped to by all odd exponents, then
a multiple-of-3, which is not mapped to by any odd, positive integer, then
an element mapped by all even exponents, then
an element mapped to by all odd exponents, then
a multiple-of-3, which is not mapped to by any odd, positive integer, then
an element mapped by all even exponents, then
...

(The first element of a "spiral" can have the *e*, *o*, or 3 property, but thereafter, the above pattern repeats for the entire "spiral".)

The tree is not strictly an infinitary tree, because if a node is not a range element of the $3x + 1$ function, that is, if a node is a multiple-of-3, then it has no descending nodes (no odd, positive integers map to a multiple-of-3).

We call the set of odd, positive integers that map to a range element y in one iteration of the function, a *recursive "spiral"*, or just a *"spiral"* for short. Thus , for example, {1, 5, 21, 85, 341, ... } is a "spiral". It maps to 1 in one iteration of the 3x + 1 function. It is easily shown that the elements of a "spiral" map to y either by all odd exponents, or by all even exponents (which is the case with {1, 5, 21, 85, ... }). It is likewise easily shown that if x is an element of a "spiral", then the next "spiral" element is $4x + 1$.

Each node in the 1-tree is a "spiral" element. From now on, we will use the term *"spiral" element*.

Each "spiral" element y that is a range element of the $3x + 1$ function is the root of a y-tree. The definition of the y-tree is the same as that of the 1-tree, with y replacing 1.

The odd, positive integers between successive "spiral" elements we call an *interval* in the "spiral". Thus, for example, in the "spiral" {1, 5, 21, 85, 341, ... }, 3 is the only element in the first interval; 7, 9, 11, 13, 15, 17, 19, are the elements of the second interval, etc.

If a counterexample exists, then it is an element of an interval, but it is never a "spiral" element. The tree or trees generated by counterexamples are of course separate (disjoint) from the 1-tree. There are "spiral"s, "spiral" elements, and intervals in counterexample trees just as there are in the 1-tree.

Let y denote a "spiral" element in {1, 5, 21, 85, ... } that is a range element. Then if x is a non-counterexample, it must be an element of a y-tree.

Suppose we ask, "What is the location (the "address") of an element x in the 1-tree?" A simple answer is: the tuple $\langle x, ..., 1\rangle$!

So, to summarize: In the case of the 1-tree, all "spiral" elements are non-counterexamples. Some interval elements we know are non-counterexamples. If a counterexample exists, then it is an element of an interval, but never of a "spiral" in the 1-tree. In the case of a counterexample tree, all spiral elements are counterexamples. Some interval elements are counterexamples, others are non-counterexamples.

There is exactly one 1-tree. This means that an odd, positive integer maps to 1 regardless if counterexamples exist or not. Thus, for example, 13 (an element of the 5-tree) maps to 1 (in two iterations of the function) today. If a proposed proof of the $3x + 1$ Conjecture is accepted as valid tomorrow, 13 will still map to 1. And if the Conjecture is proved *false* tomorrow, 13 will *still* map to 1. (A proof that there is only one 1-tree is the proof of Lemma 8.8 in our paper, "Are We Near a Solution to the 3x + 1 Problem?", on occampress.com. That Lemma is accompanied by several aids to overcoming the counterintuitive aspect of the Lemma.)

The reader may find the difference between the following two statements helpful:

(1) If x is a *non-counterexample*, then x is a non-counterexample whether or not counterexamples exist.

But

(2) If x is a *counterexample*, then x is most certainly *not* a counterexample whether or not counterexexamples exist!

If the $3x + 1$ Conjecture is true, then the 1-tree contains all odd, positive integers. If the Conjecture is false, then the 1-tree contains only a proper subset of the odd, positive integers.

Understanding the 1-Tree Relative to Successive Non-Counterexamples Beginning With 1

1. It is known by computer test[1] that all successive odd, positive integers less than approximately $5.76 \bullet 10^{18}$ are non-counterexamples. Call this set W. So let us consider the "spiral" $S = \{1, 5, 21, 85, ... \}$, all of whose elements map to 1 in one iteration of the $3x + 1$ function, and proceed as follows.

2. Obviously, 1 maps to 1, and so we color 1 green in S.

Moving to the next odd, positive integer to the right of 1,which is 3, we can easily find that 3 maps to 1. So we color 3 green in the first interval of S.

Moving to the next odd, positive integer to the right of 3, which is 5, we know that 5 maps to 1, because all elements of S map to 1. So we color 5 green in S. Since 5 is a range-element of the 3x + 1 function, , a "spiral" maps to it in one iteration of the function. That "spiral" is $\{3, 13, 53, ... \}$. We color all elements of the "spiral" green in intervals of S. We also color all other elements in the 5-tree green. We then color green in intervals of S, all elements of the 5-tree.

Moving to the next odd, positive integer to the right of 5, which is 7, we proceed similarly. But from now on, we also color green all elements of upward extensions of 7 if they are not already colored green. and we color green the same interval elements and elements of S.

Etc.

If we come to an odd, positive integer that is already green (because it is a member of an earlier y-tree) we move to the next odd, positive integer to the right.

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1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than $20 \bullet 2^{58} \approx 5.76 \bullet 10^{18}$, which is greater than $3.33 \bullet 10^{16} \approx 2 \bullet 3^{(35 - 1)}$, have been tested and found to be non-counterexamples.

3. If all odd, positive integers (including those that are greater than the largest odd, positive integer in *W* (see step 1), are either already green, or are elements of *S*, then we have a proof of the $3x + 1$ Conjecture.

The reader is urged to contemplate the fact that the portion of the 1-tree (including intervals) that is green is *exactly* what the portion of the 1-tree looks like if counterexamples do not exist — or if they do exist (see previous sub-section regarding fact that there is exactly one 1-tree).

One fact that seems promising for a proof of the $3x + 1$ Conjecture is that each successive element of each ‘spiral” is an element of an infinity of successive intervals in the “spiral” $S = \{1, 5, 21, 85, 341, ... \}$. Thus, consider the elements of the “spiral” $\{3, 13, 53, 213, ... \}$.

Many relevant results and ideas will be found in our paper, “Are We Near a Solution to the $3x + 1$ Problem?”, on occampress.com, in the sub-section, “ ‘Spiral’s, Intervals and Levels” of the section, “Strategy of ‘Filling-in’ of Intervals in the Base Sequence Relative to 1”.

In the case of the 1-tree for the $3x – 1$ function, the first counterexample, 5, occurs in the first interval in the “spiral” $\{1, 3, 11, 43, ... \}$ that maps to 1 in one iteration of the $3x – 1$ function — namely, in the interval whose elements are 5, 7, 9. (The element 7 is also a counterexample, because <5, 7, 5, ... > is an infinite cycle that never yields 1.)

. . . .

Theorem: The 3*x* + 1 Conjecture is true.
Proof:

(*Note* 1: A second proof of the Conjecture is given in “Appendix F — Second Proof of the 3x + 1 Conjecture” on page 41, and a third — which we believe to be our best — in “Appendix H — Third Proof of the 3x + 1 Conjecture” on page 44).

(*Note* 2: we ask the reader to inform us of the first sentence that the reader believes contains an error, and what that error is.)

1. It is easily shown that, for each $i \geq 2$, the set E_i of i-level elements in *first* i-level tuples in all i-level tuple-sets is the set of odd, positive integers less than $2 \bullet 3^{(i-1)}$ that are not divisible by 3 (by part (a) of “Lemma 1.0: the “Distance” Functions d(i, i) and d(1, i)” on page 8). Thus, for example, E_2 is the set of all the odd, positive integers less than $2 \bullet 3^{(2-1)} = 6$, that are not divisible by 3, and these integers are 1 and 5.

2. By computer test, it is known that $E_2, E_3, E_4, \ldots$, up to at least E_{35} each consists solely of non-counterexamples[1].

3.

(1) For *each* 35-level tuple-set T_A, the sequence S of 35-level elements in the sequence of 35-level tuples is given by $y + n(2 \bullet 3^{(35-1)})$, where $n \geq 0$ and y is the 35-level element of the first 35-level tuple in the tuple-set T_A. The sequence S is the sequence if counterexamples do not exist, and it is also the sequence if counterexamples exist. As we stated in step 2, y is a non-counterexample element,

Proof: Follows from part (a) of “Lemma 1.0: the “Distance” Functions d(i, i) and d(1, i)” on page 8. The Distance Functions are not themselves functions of the truth or falsity of the 3*x* + 1 Conjecture.□

Note: the fact that all elements of E_{35} are non-counterexamples is emphatically *not* the case for the 3*x* – 1 function, where one of the elements, 5, of E_2 is already a counterexample. Thus there exists a first 2-level tuple, namely <7, 5>, in a 2-level tuple-set that is a counterexample tuple. Each subsequent E_i contains counterexamples, each of which is the i-level element of the first i-level tuple in an i-level tuple-set. Each of these tuples is therefore a counterexample tuple. *So our proof cannot be used to prove the false 3x – 1 Conjecture.*

4. Since each 35-level tuple-set— and indeed each i-level tuple-set, where $i \geq 2$ — has an extension via each possible exponent, namely, via 1, 2, 3, ..., we can use an inductive argument beginning with (1) to arrive at the conclusion that the set of all non-counterexample tuples if counterexamples do not exist, is the same as the set of all non-counterexample tuples if counterexamples exist.

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1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than $20 \bullet 2^{58} \approx 5.76 \bullet 10^{18}$, which is greater than $3.33 \bullet 10^{16} \approx 2 \bullet 3^{(35-1)}$, have been tested and found to be non-counterexamples. These include the set of all 35-level elements of all 35-level first tuples in all 35-level tuple-sets .

5.We must now ask if counterexamples can exist in T_A in j-level tuples, where $j < 35$. The answer is No, because each j-level tuple in T_A is a 35-level tuple in some other 35-level tuple-set, and T_A is any 35-level tuple-set.

6. So we must conclude from step 4 that the set of counterexamples is the empty set, and therefore that the $3x + 1$ Conjecture is true.. □

Remark 1

The reader might enjoy answering — or attempting to answer — question (A), below, which arises from the following facts:

Let the tuple $t = <x, ..., 1, 1, ..., 1>$, which is clearly a non-counterexample tuple (x is a non-counterexample). Let A be the i-level exponent sequence associated with t. Then t is the first i-level tuple in the tuple-set T_A. The 1-level (first) element of t is x, the i-level element of t is 1.

The 1-level (first) element of the nth i-level tuple in T_A is given by $x + (n - 1)(2 \cdot (2^{a_2})(2^{a_3})\ldots(2^{a_i}))$, and the i-level element of the nth i-level tuple in T_A is given by $1 + (n - 1)(2 \bullet 3^{i-1})$, where $n \geq 1$ (by parts (b) and (a) of "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 8).

(A)

How does T_A differ if (1) counterexamples exist, and (2) counterexamples do not exist?

Remark 2

The reader might enjoy reading at least some of the appendices below that give other possible proofs of the $3x + 1$ Conjecture.

Remark 3

A wealth of additional results and possible strategies is available in our paper, "Are We Near a Solution to the $3x + 1$ Problem?" on occampress.com.

References

Lagarias, J., (1985), "The $3x + 1$ Problem and Its Generalizations", *American Mathematical Monthly*, **93**, 3-23.

Wirsching, Günther J.. *The Dynamical System Generated by the $3n + 1$ Function*, Springer-Verlag, Berlin, Germany, 1998.

Appendix A — Statement and Proof of Each Lemma

Lemma 1.0: Statement and Proof

Definition: let T_A be an i-level tuple-set, where $i \geq 2$. Let $t(r)$, $t(s)$ denote tuples consecutive at level i, with $r < s$ in the natural ordering of tuples by first elements. Let $t(r)(h)$, $t(s)(h)$ denote the elements of $t(r)$, $t(s)$ at level h, where $1 \leq h \leq i$. Then we call $|t_{(s)(h)} - t_{(r)(h)}|$ the *distance* between $t(r)$ and $t(s)$ at level h. We denote this distance by $d(h, i)$ and call d the *distance functions* (one function for each h).

Lemma 1.0

(a) Let $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, *be a sequence of exponents, and let* $t_{(r)}$, $t_{(s)}$ *be tuples consecutive at level i in* T_A. *Then d(i, i) is given by*:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) *Let* $t_{(r)}$, $t_{(s)}$ *be tuples consecutive at level i in* T_A. *Then d(1, i) is given by:*

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3})\ldots(2^{a_i})$$

Thus, in "Fig. 1. Part of the tuple-set TA associated with the sequence A = {1, 1, 2}" on page 7, the distance $d(3, 3)$ between $t_{8(3)} = 35$ and $t_{4(3)} = 17$ is $2 \cdot 3^{(3-1)} = 18$. The distance $d(1, 2)$ between $t_{12(1)} = 23$ and $t_{10(1)} = 19$ is $2 \cdot 2^1 = 4$.

Proof:

The proof is by induction.

Proof of Basis Step for Parts (a) and (b) of Lemma 1.0:

Let $t_{(r)}$ and $t_{(s)}$ be the first and second 2-level tuples, in the standard linear ordering of tuples based on their first elements, that are consecutive at level $i = 2$ in the 2-level tuple-set T_A, where $A = \{a_2\}$. (See Fig. 2 (1).)

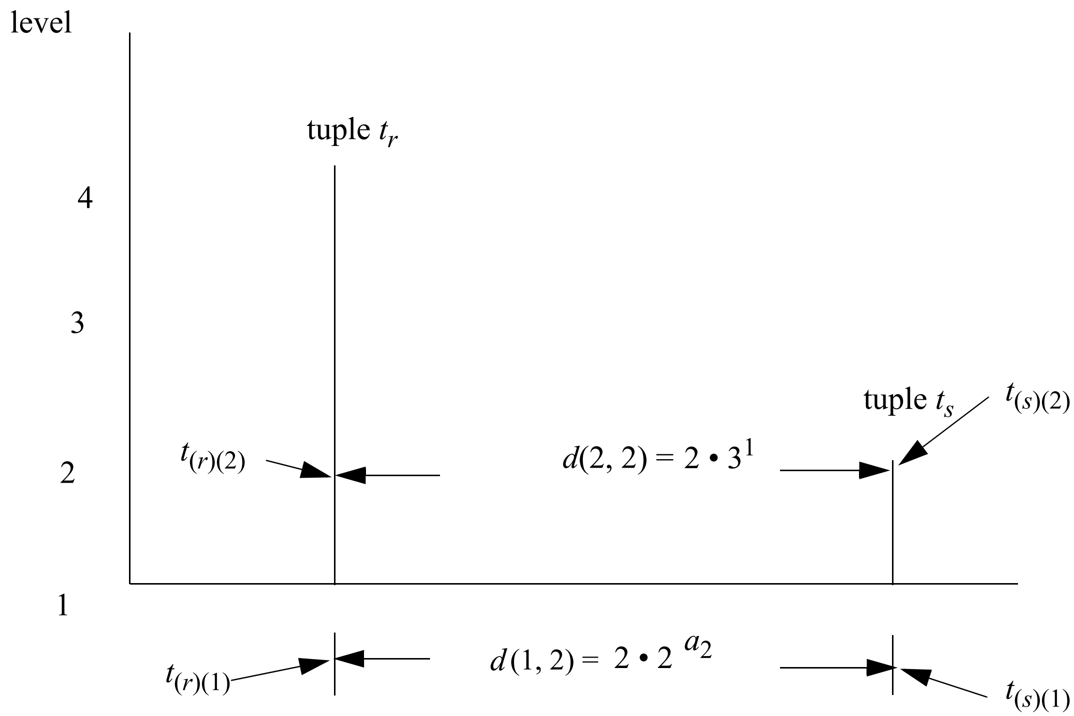

Fig. 2 (1). Illustration for proof of Basis Step of Lemma 1.0.

Then we have:

$$\frac{3t_{(r)(1)} + 1}{2^{a_2}} = t_{(r)(2)} \tag{1.1}$$

and since, by definition of $d(1, 2)$,

$$t_{(s)(1)} = t_{(r)(1)} + d(1, 2)$$

we have:

$$\frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}} = t_{(s)(2)} \tag{1.2}$$

Therefore, since, by definition of $d(i, i)$,

$$t_{(r)(2)} + d(2, 2) = t_{(s)(2)}$$

we can write, from (1.1) and (1.2):

$$\frac{3t_{(r)(1)} + 1}{2^{a_2}} + d(2, 2) = \frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}}$$

By elementary algebra, this yields:

$$2^{a_2} d(2, 2) = 3 \cdot d(1, 2)$$

Now $d(2, 2)$ must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level i, it must be the smallest such even number, whence it follows that $d(2, 2)$ must = 3 • 2, and necessarily

$$d(1, 2) = 2 \bullet 2^{a_2}$$

A similar argument establishes that $d(2, 2)$ and $d(1,2)$ have the above values for every other pair of tuples consecutive at level 2.

Thus we have our proof of the Basis Step for parts (a) and (b) of Lemma 1.0.

Proof of Induction Step for Parts (a) and (b) of Lemma 1.0

Assume the Lemma is true for all levels j, $2 \le j \le i$ and that T_A is an i-level tuple-set, where A = $\{a_2, a_3, ..., a_i\}$.

Let $t_{(r)}$ and $t_{(s)}$ be tuples consecutive at level i, and let $t_{(r)}$ and $t_{(f)}$ be tuples consecutive at level i +1. (See Fig. 2 (2).)

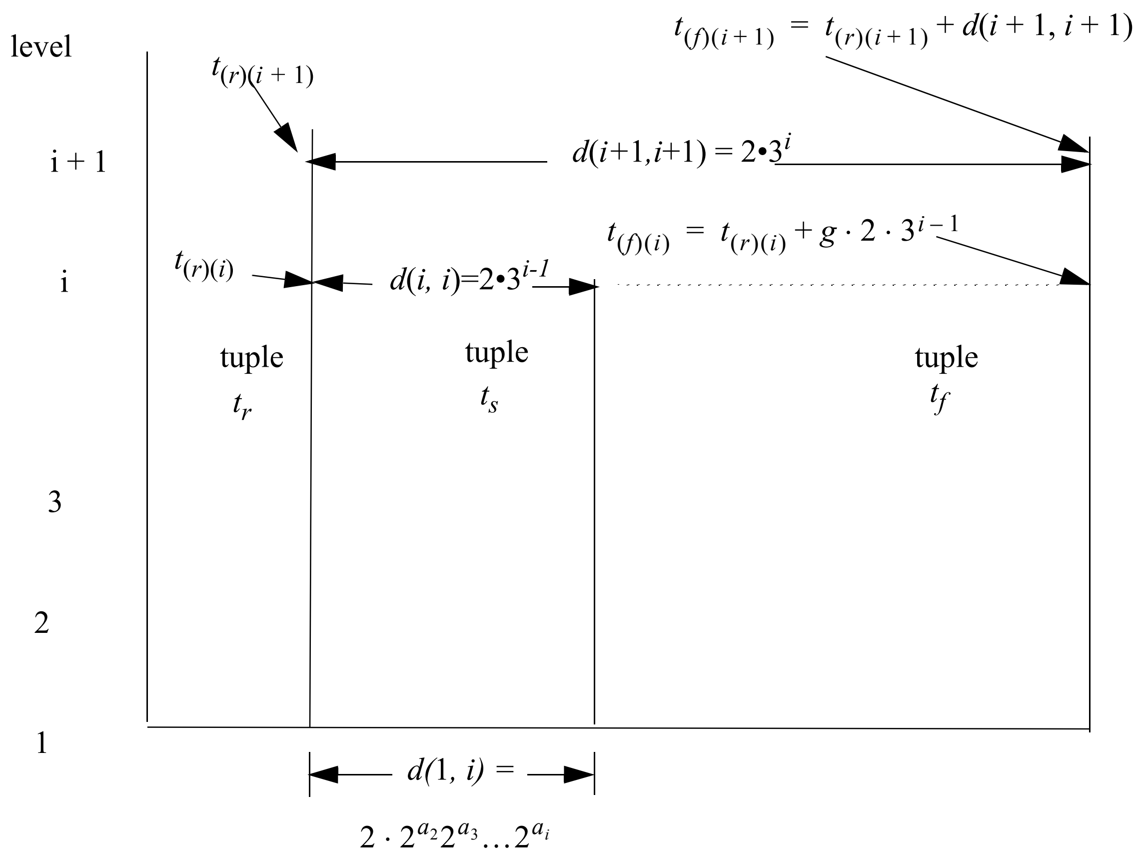

Fig. 2 (2). Illustration for proof of Induction Step of Lemma 1.0.

Then we have:

$$\frac{3t_{(r)(i)} + 1}{2^{a_{i+1}}} = t_{(r)(i+1)}$$

and since, by definition of $d(i, i)$,

$$t_{(f)(i)} = t_{(r)(i)} + g \cdot d(i, i)$$

for some $g \geq 1$, we have:

$$\frac{3(t_{(r)(i)} + g \cdot d(i, i)) + 1}{2^{a_{i+1}}} = t_{(f)(i+1)}$$

Thus, since

$$t_{(r)(i+1)} + d(i+1, i+1) = t_{(f)(i+1)}$$

we can write:

$$\frac{3t_{t_{(r)(i)}} + 1}{2^{a_{i+1}}} + d(i+1, i+1) = \frac{3(t_{t_{(r)(i)}} + gd(i, i)) + 1}{2^{a_{i+1}}}$$

This yields, by elementary algebra:

$$2^{a_{(i+1)}} d(i+1, i+1) = 3 \cdot gd(i, i)$$

As in the proof of the Basis Step, $d(i+1, i+1)$ must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level $i+1$, it must be the smallest such even number. Thus $d(i+1, i+1) = 3 \bullet d(i, i)$, and

$$g \cdot d(i, i) = 2^{a_{i+1}} d(i, i) \quad .$$

Hence

$$g = 2^{a_{i+1}}$$

Now g is the number of tuples consecutive at level i that must be "traversed" to get from $t_{(r)}$ to $t_{(f)}$. By inductive hypothesis, d(1, i) for *each* pair of these tuples is:

$$d(1, i) = 2 \cdot 2^{a_2} \cdot 2^{a_3} \cdot \ \ldots \cdot 2^{a_i}$$

hence, since

$$g = 2^{a_{i+1}}$$

we have

$$d(1, i+1) = d(1, i) \cdot 2^{a_{i+1}} .$$

A similar argument establishes that $d(i+1, i+1)$ and $d(1, i+1)$ have the above values for every pair of tuples consecutive at level $i+1$.

Thus we have our proof of the Induction Step for parts (a) and (b) of Lemma 1.0. The proof of Lemma 1.0 is completed. □

Lemma 2.0: Statement and Proof

Assume a counterexample exists. Then for all $i \geq 2$, each i-level tuple-set contains an infinity of i-level counterexample tuples and an infinity of i-level non-counterexample tuples.

Proof:

1. Assume counterexamples exist. Then:

There is a countable infinity of non-counterexample range elements.

Proof: Each non-counterexample maps to a range element, by definition of *range element*.

Each range element is mapped to by an infinity of elements ("Lemma 6.0: Statement and Proof" on page 24). A countable infinity of these are range elements (proof of "Lemma 7.0: Statement and Proof" on page 27).

There is a countable infinity of counterexample range elements.

Proof: same as for non-counterexample case.

2. For each finite exponent sequence A, and for each range element y, non-counterexample or counterexample, there is an x that maps to y via A possibly followed by a buffer exponent ("Lemma 7.0: Statement and Proof" on page 27). The presence of the buffer exponent does not change the fact that x is the first element of a tuple associated with the exponent A. □

Lemma 3.0: Statement and Proof

Exactly one set J of odd, positive integers maps to 1, regardless if counterexamples exist or not. [1]

Proof: .

The set J =

{odd, positive integers y | y maps to 1 in *one* iteration of the $3x + 1$ function} ∪
{odd, positive integers y | y maps to 1 in *two* iterations of the $3x + 1$ function} ∪
{odd, positive integers y | y maps to 1 in *three* iterations of the $3x + 1$ function} ∪
...

If an odd, positive integer y maps to 1, then it maps to 1 regardless if counterexamples exist or not. For, certainly y will continue to map to 1 if counterexamples do not exist. If counterexamples exist, and y does not continue to map to 1, that implies that the proof that counterexamples

1. Some readers consider the phrase "regardless if counterexamples exist or not" as unnecessary. The reason we include it is that when it was not present, other readers assumed that the meaning of the Lemma was: exactly one set J of odd, positive integers maps to 1 (if counterexamples do not exist), and exactly one (different) set J of odd, positive integers maps to 1 if counterexamples exist. This meaning is incorrect.
A second reason for including the phrase will become clear when the reader reads our proofs of the $3x + 1$ Conjecture.

exist somehow changes the definition of the 3x + 1 function or the laws of arithmetic. But such changes are absurd. Thus the Lemma is true. □

Lemma 4.0: Statement and Proof

No multiple of 3 is a range element.

Proof :
If

$$\frac{3x+1}{2^a} = 3m$$

then $1 \equiv 0 \bmod 3$, which is false. □

Lemma 5.0: Statement and Proof

Each odd, positive integer (except a multiple of 3) is mapped to by a multiple of 3 in one iteration of the 3x + 1 function.

Proof:

Since the domain of the 3x + 1 function is the odd, positive integers, the only relevant generators are $3(2k + 1)$, $k \geq 0$. We show that, for each odd, positive integer y not a multiple of 3, there exists a k and an a such that

$$y = \frac{(3(3(2k+1))+1)}{2^a} \quad , \tag{11.1}$$

where a is necessarily the largest such a, since y is assumed odd.

Rewriting (11.1), we have:

$$y2^{a-1} - 5 = 9k \ . \tag{11.2}$$

Without loss of generality, we can let $y \equiv r \bmod 18$, where r is one of 1, 5, 7, 11, 13, or 17 (since y is odd and not a multiple of 3, these values of r cover all possibilities mod 18). Or, in other words, for some q, r, $y = 18q + r$. Then, from (11.2) we can write:

$$18(2^{a-1})q + (2^{a-1})r - 5 = 9k \ . \tag{11.3}$$

Since the first term on the left-hand side is a multiple of 9, $(2^{a-1})r - 5$ must also be if the equation is to hold. We can thus construct the following table. (Certain larger a also serve equally well, but those given suffice for purposes of this proof.)

Table 2: Values of *r*, *a*, for Proof of Lemma

r	a	$(2^{a-1})r - 5$
1	6	27
5	1	0
7	2	9
11	5	171
13	4	99
17	3	63

Given q and r (hence y), we can use r to look up a in the table, and then solve (11.3) for integral k, thus producing the multiple of 3 that maps to y in one iteration of the $3x + 1$ function. □

Lemma 6.0: Statement and Proof

(a) Each range element y is mapped to, in one iteration of the 3x + 1 function, by every exponent of one parity only. Furthermore,

(b) For each of the two parities, there exists a range element that is mapped to by every exponent of that parity.

Proof of part (a):

Steps 1 and 2 are slightly edited versions of proofs by Jonathan Kilgallin and Alex Godofsky. Any errors are entirely ours. Step 3 is a slightly edited version of a proof by Michael Klipper. Any errors are entirely ours.

1. We first show that if y is mapped to by the exponent a, then y is mapped to by every exponent greater than a that is of the same parity as a.

Let y be a range element, and let x map to y via the exponent a. Then

$$y = \frac{3x+1}{2^a}$$

We wish to show that there exists an x' such that x' maps to y via the exponent 2^{a+2}. That is, we wish to show that there exists an x' such that

$$y = \frac{3x' + 1}{2^{a+2}}$$

Rewriting, this gives

$$x' = \frac{2^{a+2}y - 1}{3}$$

Substituting for y yields

$$x' = \frac{2^{a+2}\left(\frac{3x+1}{2^a}\right) - 1}{3}$$

Simplifying, this gives $x' = 4x + 1$. Since x is an odd, positive integer, clearly x' is as well.

Thus, by induction, if y is mapped to via the exponent a , it is mapped to by every exponent greater than a of the same parity. □

2. Next we show that if y is mapped to by the exponent a which is greater than 2, then it is mapped to by every exponent less than a that is of the same parity as a.

Let y be a range element, and let x map to y via the exponent a where $a > 2$. Then

$$y = \frac{3x + 1}{2^a}$$

We wish to show that there exists an x' such that x' maps to y via the exponent 2^{a-2}. That is, we wish to show that there exists an x' such that

$$y = \frac{3x' + 1}{2^{a-2}}$$

Rewriting, this gives

$$x' = \frac{2^{a-2}y - 1}{3}$$

Substituting for y yields

$$x' = \frac{2^{a-2}\left(\frac{3x+1}{2^a}\right) - 1}{3}$$

Simplifying yields

$$x' = \frac{x-1}{4}$$

3. We must now show that $x' = (x - 1)/4$ is an odd, positive integer. This means we must show that $(x - 1) = 4(2k + 1)$ for some $k \geq 0$, or that $(x - 1) = 8k + 4$, hence that $x = 8k + 5$. Thus, we must prove $x \equiv 5 \bmod 8$.

We know that x maps to y via a, where $a \geq 3$. Thus, $y = (3x + 1)/2^a$, so $2^a y = 3x + 1$. Because $a \geq 3$, $2^a y$ is a multiple of 8. Thus, $(3x + 1) \equiv 0 \bmod 8$, and $3x \equiv 7 \bmod 8$. This readily implies $x \equiv 5 \bmod 8$.

4. Thus, by induction, if y is mapped to via the exponent a, where $a > 2$, then it is mapped to by every exponent less than a of the same parity. □

Proof of part (b):

We now show that for each of the two parities there exists a range element that is mapped to by every exponent of that parity.

1. Fix a range element y, and suppose that x maps to y via the exponent a. Now a is either even or odd, hence $a = 2n + h$, where h is either 0 or 1. Since $y = (3x + 1)/2^a$, it follows that $(2^a)y = 3x+1$. Reduce the equation mod 3, and we get $(2^h)y \equiv 1 \bmod 3$, by the following reasoning: $(2^a)y \equiv 1 \bmod 3$ implies $(2^{2n+h})y \equiv 1 \bmod 3$ implies $2^{2n}\,2^h y \equiv 1 \bmod 3$ implies $2^h y \equiv 1 \pmod 3$ because $2^{2n} = 4^n \equiv 1 \bmod 3$.

2. Since y is fixed, either $y \equiv 1$ or $y \equiv 2 \bmod 3$. (We know that y, a range element, is not a multiple of 3 by "Lemma 4.0: Statement and Proof" on page 23). If $y \equiv 1 \bmod 3$, then we have $2^h(1) \equiv 1 \bmod 3$, which implies that h must be 0. If $y \equiv 2 \bmod 3$, then we have $(2^h)(2) \equiv 1 \bmod 3$, implying that h must be 1. □

Lemma 7.0: Statement and Proof

Let y be a range element of the 3x + 1 function. Then for each finite exponent sequence A, there exists an x that maps to y via A possibly followed by a "buffer" exponent. (For example, if y is mapped to by even exponents, and our exponent sequence A ends with an odd exponent, then there must be a "buffer" exponent following A, and similarly if y is mapped to by odd exponents and A ends with an even exponent. However, there are other cases in which a "buffer" exponent is required.)

Proof:

1. Each range element y is mapped to by all exponents of one parity ("Lemma 6.0: Statement and Proof" on page 24).

2. Each range element y is mapped to by a multiple of 3 ("Lemma 5.0: Statement and Proof" on page 23).

Each range element is mapped to by an infinity of range elements ("Lemma 5.0: Statement and Proof" on page 23).

3. Let y be a range element and let $S = \{s_1, s_2, s_3, ... \}$ be the set of all odd, positive integers that map to y in one iteration of the $3x + 1$ function. In other words, S is the set of all elements in a "spiral". Furthermore, let the s_i be in increasing order of magnitude. It is easily shown that $s_{i+1} = 4s_i + 1$.

(In Fig. 18, $y = 13$, $S = \{17, 69, 277, 1109, ... \}$

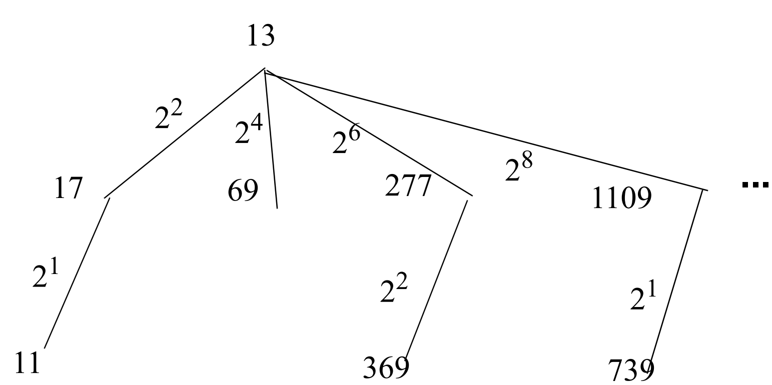

Fig. 18

(*Note*: for a graphical representation of part of the tree having 1 as its root instead of 13, see "Recursive "Spiral"s: The Structure of the 3x + 1 Function in the "Backward", or Inverse, Direction" in our paper, "Are We Near a Solution to the $3x + 1$ Problem?", on occampress.com.)

4. If s_i is a multiple of 3, then $4s_i + 1$ is mapped to, in one iteration of the $3x + 1$ function, by all exponents of even parity.

To prove this, we need only show that x is an integer in the equation

$$4(3u) + 1 = \frac{3x+1}{2^2}$$

Multiplying through by 2^2 and collecting terms we get

$$(48u) + 4 = 3x + 1$$

and clearly x is an integer.

5. If s_j is mapped to by all even exponents, then $4s_j + 1$ is mapped to, in one iteration of the $3x + 1$ function, by all exponents of odd parity.

(The proof is by an algebraic argument similar to that in step 4.)

6. If s_k is mapped to by all odd exponents, then $4s_k + 1$ is a multiple of 3.

(The proof is by an algebraic argument similar to that in step 4.)

7. The Lemma follows by an inductive argument that we now describe.

Let y be a range element. It is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. As a consequence of steps 1 through 6, we can represent an infinite sub-sequence of the sequence by

...3, 2, 1, 3, 2, 1, ...

where

"3" means "this odd, positive integer is a multiple of 3 and therefore is not mapped to by any odd, positive integer";

"2" means "this odd, positive integer is mapped to by all even exponents";

"1" means "this odd, positive integer is mapped to by all odd exponents".

Each type "2" and type "1" odd, positive integer is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. We can represent an infinite sub-sequence of the sequence by

...3, 2, 1, 3, 2, 1, ...

where each integer has the same meaning as above.

Temporarily ignoring the case in which a buffer exponent is needed, it should now be clear that, for each range element y, and for each finite sequence of exponents B, we can find a finite path down through the infinitary tree we have just established, starting at the root y. The path will end in an odd, positive integer x. Let A denote the path B taken in reverse order. Then we have our result for the non-buffer-exponent case. The buffer-exponent case follows from the fact that the buffer exponent is one among an infinity of exponents of one parity. Thus y is mapped to by an infinite sequence of odd, positive integers. We then simply apply the above argument.. □

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Appendix B — A Possible Strategy for A Second Proof of the 3x + 1 Conjecture

In the above proof of the Conjecture, we argued in what we might call the "vertical" direction, that is, the direction of increasing i. We showed that the set of all 35-level elements of all 35-level tuples in all 35-level tuple-sets is the same regardless if counterexamples exist or not, and from that, via an elementary inductive argument, we showed that the contents of all tuple-sets are the same regardless if counterexamples exist or not, which implies that counterexamples do not exist.

In the following possible strategy, we argue in what we might call the "horizontal" direction.

1, Assume counterexamples exist, and consider the set of all first 35-level tuples in the set of all 35-level tuple-sets. By what we established in the above proof of the Conjecture, we know that all these tuples are non-counterexample tuples. By Lemma 8.8 in our paper, "Are We Near a Solution to the 3x + 1 Problem?", on occampress.com, we know that all these tuples are the same regardless if counterexamples exist or not.

2. Now, moving in the "horizontal" direction in any of these tuple-sets, that is, in the direction of increasing first elements of tuples, starting from the first 35-level tuple, we see from Table 1, "Distances between elements of tuples consecutive at level i," on page 9, that each successive tuple remains the same regardless if counterexamples exist or not, because the Table is not a function of the truth or falsity of the $3x + 1$ Conjecture.

So we must conclude that all the tuples in the tuple-set remain the same regardless if counterexamples exist or not. But this implies that the contents of all tuple-sets remain the same, regardkess if counterexamples exist or not, and therefore we must conclude, as we did in our proof of the Conjecture, that counterexamples do not exist.

Appendix C — Possible Proof, and Possible Strategies for a Proof, of the 3*x* + 1 Conjecture Based on the 1-Tree

For definitions and background on the 1-tree, see “The 1-Tree” on page 10.

Proof of 3*x* + 1 Conjecture Based on 1-Tree

This proof is now in “Appendix F — Second Proof of the 3x + 1 Conjecture” on page 41.

Possible Strategy for a Proof Based on 1-Tree

1. We must remember that there is exactly one 1-tree, whether or not counterexamples exist (see “The 1-Tree” on page 10). If the set of all “spiral” elements in the 1-tree is the set of all odd, positive integers, then counterexamples do not exist. Otherwise, counterexamples exist.

2. If a counterexample exists, it is the first element of a 2-tuple in a 2-level tuple-set. However, by what we have said regarding the table below (see “Partial List of Facts Pertaining to the 1-Tree” on page 34), this means that the counterexample is an element of a counterexample “spiral”. The “spiral” maps to a range-element y of the $3x + 1$ function in a single iteration of the function, and therefore y is the root of a y-tree. The y-tree contains an infinity of counterexample “spiral”s.

If a counterexample exists, there is an infinity of counterexamples (“Lemma 2.0” on page 9, and the fact that, by the ... 2, 1, 3, 2, 1, 3, ..., pattern in “spiral” elements (see “Partial List of Facts Pertaining to the 1-Tree” on page 34)) we know that there is an infinity of counterexample range-elements. Hence there is an infinity of y-trees, where y is a counterexample range element).

3. There are no counterexample tuples in the set of all tuple-sets if the set contains only non-counterexample tuples.

It is easily shown that in order for there to be an infinity of tuples associated with each finite exponent sequence there must be an infinity of non-counterexample tuples in each tuple-set that are extended by the exponent 1, and an infinity that are extended by the exponent 2, etc. (See “Extensions of Tuple-sets” on page 8.) However, this property cannot survive the omission of a countable infinity of “spiral”s.

It seems, therefore, that we must conclude that the existence of a counterexample results in the failure of non-counterexample tuples to be associated with *each* finite exponent sequence, contradicting “Lemma 2.0” on page 9.

If this reasoning can be made precise and valid, we have a proof of the $3x + 1$ Conjecture.

(*Note*: we must reconcile the omission of non-counterexample “spiral”s from tuple-sets, with the fact that the 1-tree is the same, whether or not counterexamples exist.)

Second Possible Strategy for a Proof Based on 1-Tree

1. Let S denote the “spiral” {1, 5, 21, 85, 341, ... }in the 1-tree. (For background on “spiral”s

see "Recursive "Spiral"s: The Structure of the 3x + 1 Function in the "Backward", or Inverse, Direction" on page 10.)

This "spiral" is the set of odd, positive integers that map to 1 in one iteration of the $3x + 1$ function. The set of odd, positive integers lying between "spiral" elements is called an *interval*. Thus, {3} is the first interval in *S*, {7, 9, 11, 13, 15, 17, 19} is the second interval, etc.

2. By computer test[1], we know that all odd, positive integers less than $5 \cdot 10^{18}$,are non-counterexamples. These integers occupy more than the first 30 successive intervals in *S*.

Call the largest odd, positive integer less than $5 \cdot 10^{18}$, *w*.

3. A basic fact governing "spiral"s is that the elements of each "spiral" other than *S*, occupy an infinity of successive intervals in *S*. Thus, consider the "spiral" {3, 13, 53, 213, ... } and the intervals in the "spiral" *S*..

We remark in passing that each "spiral" element *x*, and the range element *y* it maps to in one iteration of the $3x + 1$ function, is a tuple $<x, y>$ in a tuple-set. Since the set of elements of a "spiral" map to their range element either by all odd exponents or by all even exponents, it is easy to see that the set of all elements of all "spiral"s in the 1-tree is the set of all first elements of all non-counterexample 2-tuples in all 2-level tuple-sets.

The set of all (finite) upward paths (that is, paths in the direction of the root, 1, of the 1-tree, or away from 1 in the upward direction) is the set of all non-counterexample tuples in the set of all tuple-sets.

4.Therefore we can go through 1, 2, 3, ..., *w*. Call the set of these odd, positive integers, *W*. We can color green each interval element *y* in *W* that is a range element of the $3x + 1$ function, and then color green each element of the *y*-tree (same structure as the 1-tree, except with *y* as its root — see the above-mentioned section on "spiral"s). We can then color green the elements of all *extensions* of each *y* in *W*, stopping at 1.

We can then color green each interval element *v* in W that is not a range-element (that is, which is a multiple-of-3), and all *extensions* of each such *v*, stopping at 1.

We can then color green each *interval element in S* that equals an element that we have already colored green. Step 3 assures us that infinities of interval elements in *S* greater than *w* will be colored green.

5. Counterexamples occupy intervals only. They are never "spiral" elements.

If counterexamples exist, they can never themselves fill an interval in *S* as non-counterexamples less than *w* have done. The reason is that this would prevent non-counterexample "spiral" elements from having "room" in the interval, as these elements must, by step 3.

But after the first counterexample, *non-counterexamples* can never themselves fill an interval

1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$, which is greater than $3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35 - 1)}$, have been tested and found to be non-counterexamples.

in *S* as non-counterexamples have done. The reason is that this would prevent *counterexample* elements from having room in the interval, as these elements must, by step 3.

So after the first counterexample, all subsequent intervals in *S* must be shared between non-counterexamples and counterexamples. There can never again be an interval containing solely non-counterexamples, and there can never be one containing solely counterexamples.

It seems hard to believe that the behavior of the $3x + 1$ function would change so radically once the first counterexample has appeared — especially since a counterexample is simply another odd, positive integer that, if it and all other counterexamples were *non*-counterexamples, would result in an infinite succession of intervals containing solely non-counterexamples.

We might ask, informally, "How does the function 'know' what to do when it is given an odd, positive integer that may or may not be a counterexample?" Keeping in mind that, as far as the function is concerned, a counterexample is simply an odd, positive integer that is not in the 1-tree, one answer (invalid!) is the following.

For each odd, positive integer the function is given, it asks:

1. Is this integer in the 1-tree?
   If the answer is yes, then the function asks,
   Have I been given an integer *not* in the 1-tree before?
   If the answer is yes, then the function simply proceeds with its mixed-intervals-only rule;
   If the answer is no, then the function continues its practice of filling successive intervals entirely with non-counterexamples.
2. If the answer to 1. is no, then the function asks,
   Have I been given an integer *not* in the 1-tree before?
   If the answer is yes, then the function simply proceeds with its mixed-intervals-only rule;
   If the answer is no, then the function begins the permanent application of its mixed-intervals-only rule.

But of course there is nothing like this question/answer sequence in the definition of the $3x + 1$ function.

Whether all this can lead to a proof of the $3x + 1$ Conjecture, remains to be seen.

Third Possible Strategy for a Proof Based on 1-Tree

We can color green all "spiral" elements that must be in the 1-tree as a result of all consecutive odd, positive integers in the range 1 through *w* being non-counterexamples (this is what we did in "Second Possible Strategy for a Proof Based on 1-Tree" on page 31).

These "spiral" elements are part of the 1-tree *if counterexamples do not exist*.

We now argue that the rest of the "spiral" elements of the 1-tree must likewise be part of the 1-tree if counterexamples do not exist, for there is only one such "rest of the elements". Therefore, the 1-tree is the 1-tree if counterexamples do not exist.

But one could argue, no, the rest of the elements could be those if counterexamples do exist.

To which one could reply, but then the 1-tree is the same in both cases, and hence counterexamples if they exist are the same as non-counterexamples, which is a contradiction. Therefore counterexamples do not exist.

Remark

A similar argument can be applied to the smallest counterexample y_c if counterexamples exist. This counterexample must occupy a position in an interval in the "spiral" $S = \{1, 5, 21, 85, .. \}$. By assumption, all "spiral" elements and all interval elements less than y_c are non-counterexamples.

Now if y_c is a range element[1], then $y_c$is the root of a y_c-tree. But if y_c were *not* a counterexample then the y_c tree would be the same as if y_c were a counterexample, because the definition of the $3x + 1$ function is not subject to the truth or falsity of the $3x + 1$ Conjecture.

The reader might reply that it is *extensions* of *yc* that demonstrate counterexample behavior. But all these extensions must consist of range elements, and each range element occupies a position in an interval in the "spiral" *S*. And so what we said in the previous paragraph applies to each of these range elements.

We tentatively conclude that there is no difference between non-counterexample and counterexamples and hence that counterexamples do not exist.

Partial List of Facts Pertaining to the 1-Tree

It is a major challenge to find a proof of the $3x + 1$ Conjecture based on the 1-tree. In this section, we set forth a list of facts that may enable the creative mathematician to arrive at such a proof. Of course, the reader should not overlook "Proof of 3x + 1 Conjecture Based on 1-Tree" on page 31.

• There is exactly one 1-tree, whether or not counterexamples exist (Lemma 8.8 in our paper, "Are We Near a Solution to the $3x + 1$ Problem?", on occampress.com). The question is, Does the 1-tree contain all odd, positive integers (in which case the Conjecture is true) or does it contain only a proper subset of the odd, positive integers (in which case the Conjecture is false).

• The reader may find the difference between the following two statements helpful:

(1) If x is a *non-counterexample*, then x is a non-counterexample whether or not counterexamples exist.

But

(2) If x is a *counterexample*, then x is most certainly *not* a counterexample whether or not counterexexamples exist!

• If x is an element of a "spiral", then $4x + 1$ is the next element.

Each "spiral" maps to a range element y in one iteration of the $3x + 1$ function either by all odd exponents, or by all even exponents.

The successive elements of a "spiral" are mapped to in accordance with a rule that can be expressed as ... 2, 1, 3, 2, 1, 3, ..., where "2" means "is mapped to by all even exponents", "1" means "is mapped to by all odd exponents", and "3" means "is not mapped to because element is a multiple-of-x, hence not a range element".

Thus, for example, in the "spiral" $S = \{1, 5, 21, 85, ... \}$, 1 is mapped to by all even exponents, 5 is mapped to by all odd exponents, 21 is not mapped to because it is a multiple-of-3, 85 is

1. We can make appropriate adjustments in our argument if y_c is not a range element.

mapped to by all even exponents, ...

• The elements of each "spiral" other than *S*, occupy an infinity of successive intervals in *S*, one element per interval. Thus, consider the "spiral" {3, 13, 53, 213, ... }. The same applies recursively down through the 1-tree, except that if a "spiral" *s* maps via *even* exponents to *y* in another "spiral *r* in one iteration of the $3x + 1$ function, then the elements of *s* skip one interval in *r* before occupying an infinity of successive intervals in *r*, one element per interval.

• Each "spiral" element *x*, and the range element *y* it maps to in one iteration of the $3x + 1$ function, is a tuple <*x*, *y*> in a tuple-set. Since the set of elements of a "spiral" map to their range element either by all odd exponents or by all even exponents, it is easy to see that the set of all elements of all "spiral"s in the 1-tree is the set of all first elements of all non-counterexample 2-tuples in all 2-level tuple-sets.

The set of all (finite) upward paths (that is, paths in the direction of the root, 1, of the 1-tree, or away from 1 in the upward direction) is the set of all non-counterexample tuples in the set of all tuple-sets.

• The following table shows the first three 2-tuples in the 2-level tuple-sets T_A for $A = \{1\}$, $\{3\}$, and $\{5\}$. (The reader might find it more natural to imagine the exponent 1 tuple-set as being foremost, with the exponent 3 tuple-set parallel and directly behind the exponent 1 tuple-set, and the exponent 5 tuple-set as parallel and directly behind the exponent 3 tuple-set, etc.)

In brief, "spiral" elements run *vertically upward*, tuple-set elements run *horizontally. to the right.*

Observe the first three elements of the "spiral" that maps to 5 in one iteration of the $3x + 1$ function, namely, the elements 3, 13, 53 running vertically (and upward in the Table).

Observe the first three elements of the "spiral" that maps to 11 in one iteration of the $3x + 1$ function, namely, the elements 7, 29, 117 running vertically (and upward).

Observe the first three elements of the "spiral" that maps to 17 in one iteration of the $3x + 1$ function, namely, the elements 11, 45, 181 running vertically (and upward).

In keeping with the rule governing successive elements of "spiral"s, namely, that if *x* is a "spiral" element, then $4x + 1$ is the next element, we observe that $3(4) + 1 = 13$, etc.; $4(7) + 1 = 29$, etc., and $11(4) + 1 = 45$, etc.

But observe also, in keeping with "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 8 :

the (horizontal) sequence of level-2 elements in each 2-level tuple-set is 5, 6, 11, ...

The (horizontal) difference between level-1 elements in the tuple-set T_A where $A = \{1\}$ is $2(2^1)$,

The (horizontal) difference between level-1 elements in the tuple-set T_A where $A = \{3\}$ is $2(2^3)$,

The (horizontal) difference between level-1 elements in the tuple-set T_A where $A = \{5\}$ is $2(2^5)$,

Table 3: Partial View of Relationship Between 2-Level Tuple-sets and Odd-Exponent "Spiral"s

Exponent				
5	5	11	17	...
	53	117	181	...
3	5	11	17	...
	13	29	45	...
1	5	11	17	...
	3	7	11	...

Similar 2-level tuple-sets and corresponding "spiral" elements exist for even exponents, that is, for 2-level tuple-sets T_A where $A = \{2\}, \{4\}, \{6\},,,$.

• If counterexamples exist, it is not because a sub-tree of the 1-tree has somehow been "broken off" and become a counterexample tree. The structure of the 1-tree is the same, whether or not counterexamples exist.

• If counterexamples exist, then the intervals in each "spiral" — not just the "spiral" {1, 5, 21, 85, ... } — contain an infinity of counterexamples ("Lemma 2.0" on page 9).

• It is not possible to tell, from a single "spiral", if counterexamples exist or not. For each range element y of the $3x + 1$ function, there is exactly one "spiral", whether or not counterexamples exist. The function is not itself a function of the existence or non-existence of counterexamples.

• Is there a contradiction in the fact that, in the tree(s) of counterexamples, the intervals of each "spiral" contain all the non-counterexamples?

• It us possible to traverse from any point ("spiral" element) y in the 1-tree to any other point ("spiral" element) z in the 1-tree via the four movements: one element to the right or to the left in a "spiral" (the latter is possible only if y is not the first element of the "spiral"), up one element to the range-element u that the "spiral" maps to in one iteration of the $3x + 1$ function, or down one element to the range element v that maps to y in one iteration of the function (there is no such v if y is a multiple-of-3, since then v is not a range element).

• Suppose we ask, “What is the location (the “address”) of an element x in the 1-tree?” A simple answer is: the tuple $<x, ..., 1>$!

Appendix D — For Professional Mathematicians Only

Understandable Reluctance of Mathematicians to Read This Paper

There has been an understandable reluctance on the part of professional mathematicians to give serious attention to this paper, or its predecessors It seems clear to us that the main reason is mathematicians' difficulty in believing that such an extraordinarily difficult problem can have been solved by a non-mathematician (our degree is in computer science, and we have spent most of our working life doing research in the computer industry). This skepticism is reinforced by the fact that there have been many false claims of solutions to the $3x + 1$ Problem, the overwhelming majority of which having been made by non-mathematicians.

But we must point out that the occasionally-heard remark, "Nothing of importance in mathematics has ever come from outside the university", is, in fact, false, considering that some of the best of the best worked entirely outside the university — Descartes, Pascal, Fermat, Leibniz, and Galois, to name only the most famous.

We must also not fail to mention another reason for mathematicians' reluctance to read this paper, and that is the online presence of obsolete criticisms of the paper. For example, Stack Overflow has a website containing criticisms of a proof in a 2015 version of the paper. Not only were the criticisms false, but the proof that was criticized has long since been removed from the paper. This website appears next to the website containing this paper, and thus unquestionably discourages potential readers — especially mathematicians — from reading this paper. Yet despite many pleas on our part, the managers of the website have refused to delete the criticisms or to add a note to the website stating that the criticisms do not apply to the current version of the paper. Nor have they explained to us the reason for their refusals. Apparently we have no recourse in this matter, except to encourage others to boycott the Stack Overflow websites, and to write to the organization explaining the reason for the boycott. The email address is team@stackoverflow.com, the item no. is 201708202111462820.

It appears that the managers of the above website have no experience of actually doing research. They believe that if a paper is published online and contains an error, that means that the author is incapable of correcting the error, and that his underlying ideas do not deserve any attention. But errors are almost inevitable in the course of attempting to solve very difficult problems. We remind the reader that Wiles' first proposed proof of the Taniyama–Shimura–Weil Conjecture in the early 90s, which implied a proof of Fermat's Last Theorem, contained an error that took Wiles, with the help of the mathematician Richard Taylor, more than a year to repair. The important question obviously was, Do the underlying ideas in this paper offer hope for correcting the error? And the answer was yes.

We have been struck by the eagerness with which readers of this paper look for anything they can regard as an error, and the indifference they display to understanding, and thinking about, the underlying ideas.

If You Do Not Accept Our Proofs of the 3x + 1 Conjecture...

If you do not accept our proofs of the Conjecture, or any of the possible strategies for a proof that are set forth in the above appendices, we urge you to at least peruse our paper, "Are We Near a Solution to the $3x + 1$ Conjecture?" on occampress.com. This paper contains a wealth of results, insights, possible strategies for a proof, plus a section on what we have called "$3x + 1$-like functions". We will welcome comments.

We are confident that at least two publishable, significant papers can be produced from the material in our $3x + 1$ papers, and that this is true *even if* the proof of the Conjecture in the present paper and all the possible strategies in the appendices, are faulty and cannot be repaired.

We feel that the two structures we have discovered that underlie the $3x + 1$ function, namely, tuple-sets and recursive "spiral"s, are of fundamental importance, and should be brought to the attention of the entire $3x + 1$ research community.

Difficulty, So Far, In Getting This Paper Published

Not surprisingly, so far, no journal that we know of is willing to even consider this paper for possible publication. The reason seems to be that editors cannot believe that such a difficult problem might have been solved by a non-mathematician.

Incentives for Mathematicians to Take This Paper Seriously and to Spread the Word About It

Unquestionably, if this paper contains a solution to the $3x + 1$ Problem, or can easily be modified to contain a solution, considerable prestige will be gained by the first mathematicians who promote the paper. Of course, mathematicians who believe that the proof of the Conjecture is correct, and/or that at least one of the possible strategies in the appendices look promising, but do not want to risk their reputations by saying so, especially given that the author of the paper is not an academic mathematician, will not be recipients of that prestige. We are offering three incentives:

(1) Any reasonable consulting fee;

(2) Generous mention in the Acknowledgments when the paper is published. (But no name will be mentioned without the prior written approval of the mathematician concerned.)

(3) An offer of shared authorship to the first mathematician who makes a significant contribution to the paper prior to publication.

In any case, all communications we receive about this paper will be kept strictly confidential.

Appendix E —A Brief Summary of Our First Proof of the 3*x* + 1 Conjecture

The $3x + 1$ Problem asks if repeated iterations of the function $C(x) = (3x + 1)/(2^a)$ always terminate in 1 Here x is an odd, positive integer, and $a = ord_2(3x + 1)$, the largest positive integer such that the denominator divides the numerator. The conjecture that the function always eventually terminates in 1 is the *3x + 1 Conjecture*.

(*Note*: the reader is asked to inform us of the first sentence that the reader believes contains an error, and what that error is.)

The following is a summary of our solution to the Problem — that is, of our first proof of the $3x + 1$ Conjecture.

1. Using our definition[1] of the $3x + 1$ function, we define an infinite set of infinite i-level tuple-sets, where $i \geq 2$, and where a tuple is the result of a finite sequence of iterations of the function. Thus, for example, <17, 13, 5, 1> is a tuple in a tuple-set.

2. By computer test, we know that the 35-level elements in all first 35-level tuples in all 35-level tuple-sets are non-counterexamples. This fact holds regardless if counterexamples exist or not -- that is, no known non-counterexample can somehow become a counterexample, under any circumstances.

3. The distance function (part (a) of "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 8) gives the value of the ith element of the nth i-level tuple in an i-level tuple-set, where $n \geq 2$. This function is not a function of the truth or falsity of the $3x + 1$ Conjecture.

4. Therefore, from step 2, the value of the 35th-level element of the nth 35-level tuple in any 35-level tuple-set is the same, whether or not counterexamples exist.

5. But since each $(i + 1)$ - level tuple-set is an extension of i-level tuples in an i-level tuple-set, we see that the value of the 36th-level element in the nth 36-level tuple in any 36-level tuple-set is the same, whether or not counterexamples exist. And so on for all i-level tuple-sets, where $i \geq 36$.

6. But then we must arrive at the conclusion that the set of all non-counterexample tuples if counterexamples do not exist, is the same as the set of all non-counterexample tuples if counterexamples exist. Therefore the set of counterexamples is the empty set, and the $3x + 1$ Conjecture is true.

1. It is actually Crandall's alternate, but equivalent, definition of the function

Appendix F — Second Proof[1] of the 3x + 1 Conjecture

The $3x + 1$ Problem asks if repeated iterations of the function $C(x) = (3x + 1)/(2^a)$ always terminate in 1 Here x is an odd, positive integer, and $a = ord_2(3x + 1)$, the largest positive integer such that the denominator divides the numerator. The conjecture that the function always eventually terminates in 1 is the *3x + 1 Conjecture*.

(*Note*: we ask the reader to inform us of the first sentence that the reader believes contains an error, and what that error is.)

Brief Description of the 1-Tree

(*Note*: proofs of statements are given in "The 1-Tree" on page 10 and "Partial List of Facts Pertaining to the 1-Tree" on page 34.

Each range element of the $3x + 1$ function is mapped to by an infinity of odd, positive integers. Thus, 1, the root of the 1-tree, is mapped to by elements of the set { 1, 5, 21, 85, … , } where if x is the nth element of the sequence, then $4x + 1$ is the $(n + 1)$th element (a general rule in all such sets).

A range element is mapped to by either all even, positive exponents, or by all odd, positive exponents. Thus $(3(1) + 1)/2^2 = 1$, $(3(5) + 1)/2^4 = 1$, $(3(21) + 1)/2^6 = 1$, etc. All the exponents are even.

But 5 is mapped to by elements of the set {3, 13, 53, … }: $(3(3) + 1)/2^1 = 5$, $(3(13) + 1)/2^3 = 5$, $(3(53) + 1)/2^5 = 5$, etc. All the exponents are odd.

The infinity of elements that map to a range element consists of an infinity of range elements and an infinity of multiples of 3. No multiple of 3 is a range element.

The nth level of the 1-tree consists of all odd, positive integers that map to 1 in n iterations of the $3x + 1$ function. Thus the elements of the set S ={ 1, 5, 21, 85, … , } are elements of the 1st level. Elements of the set {3, 13, 53, … } are elements of the 2nd level, because each element maps to 5, and then 5 maps to 1.

Suppose we ask, "What is the location (the "address") of an element x in the 1-tree?" A simple answer is: the tuple $<x, ..., 1>$!

Proof of the 3x + 1 Conjecture

1. The set { 1, 5, 21, 85, … , } has the smallest first element (namely, 1) that maps to a range element (1) in one iteration of the $3x + 1$ function via all even exponents. The set {3, 13, 53, … } has the smallest first element (namely, 3) that maps to a range element (5) in one iteration of the $3x + 1$ function via all odd exponents. Both sets map to 1.

1. Our first proof is given under "Theorem: The 3x + 1 Conjecture is true." on page 14. Our third — and, we believe, our best proof — is given in "Appendix H — Third Proof of the 3x + 1 Conjecture" on page 44.

2. All consecutive odd, positive integers, starting at 1, and less than at least $20 \bullet 2^{58} \approx 5.76 \bullet 10^{18}$, have been shown to map to 1. (See results of computer tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html.)

3. *There exists exactly one 1-tree, whether or not counterexamples exist*. The reason is a generalization of the fact, for example, that 13 maps to 1 today, it will map to 1 if the Conjecture is proved true tomorrow, and it will map to 1 if the Conjecture is proved *false* tomorrow. The reason for this is that the $3x + 1$ function is not itself a function of the truth or falsity of the $3x + 1$ Conjecture.

The fact that there is exactly one 1-tree may seem counter-intuitive, but it isn't:

(A) There is one and only one set J of non-counterexamples.

(B) There are now two possibilities: (a) J consists of all the odd, positive integers, or (b) J does not.

(C) If (a) is true, then counterexamples do not exist. If (b) is true, then they do.

4. As we said above under "Brief Description of the 1-Tree" on page 41, the set $S = \{1, 5, 21, 85, ... \}$ is the set of odd, positive integers that map to 1 in one iteration of the $3x + 1$ function. An *interval* between elements of the set is the set of odd, positive integers that lie between successive elements of S.

If a counterexample does not exist, then each element of each interval is a non-counterexample. If a counterexample exists, then some elements of some intervals are counterexamples (this is the case with the $3x – 1$ function).

5. We now color green all elements of the 1-tree that are known, by computer test (see step 2) to map to 1, or are connected, via branches of the 1-tree, to elements that are known, by computer test, to map to 1.

The green portion of the 1-tree is exactly how this portion of the 1-tree looks if counterexamples do not exist. We now assert:

The rest of the 1-tree also looks the way the 1-tree looks if counterexamples do not exist, and therefore counterexamples do not exist.

Proof:

(I) As we have said, 1 is mapped to, in one iteration of the $3x + 1$ function, by each element of the set $S = \{1, 5, 21, 85, ... \}$, where if x is an element of the sequence of elements, $4x + 1$ is the next element.

(II) As we also have said, we define an *interval* between elements of S to be the odd, positive integers lying between two successive elements of S. Thus 3 is the only element of the first interval, 7, 9, 11, 13, 15, 17, 19 are all the elements of the second interval, etc.

(III) Assume a counterexample exists. Then it must be in an interval of S, since each odd, positive integer is either an element of S, or an element of an interval of S.

Let x_c be the smallest counterexample. Then $x_c – 2$, which we will call x_{nc}, must be a non-counterexample.

So if counterexamples exist, then $x_{nc} + 2$ is a counterexample.

If counterexamples do not exist, then $x_{nc} + 2$ is a non-counterexample.

Yet $x_{nc} + 2$ is the same odd, positive integer in each case!

Therefore either an odd, positive integer, namely, $x_{nc}+ 2$, can be a non-counterexample or a counterexample, which contradicts the first paragraph of step 3, or counterexamples do not exist, and the rest of the 1-tree looks the way the 1-tree looks if counterexamples do not exist. We conclude that counterexamples do not exist. □

Remark 1

The above proof does not apply to the $3x - 1$ Conjecture because the above proof is based on the fact that no counterexample to the $3x + 1$ Conjecture is known after computer tests of all consecutive odd, positive integers from 1 to at least 10^{18}, whereas 5 and 7 are known counterexamples to the $3x - 1$ Conjecture. (These two integers cycle indefinitely, and thus never arrive at the value 1.)

In the $3x - 1$ function, the set { 1, 3, 11, 43, … , }[1] has the smallest first element (namely, 1) that maps to a range element (1) in one iteration of the $3x + 1$ function via all *odd* exponents. The set {7, 27, 107, … } has the smallest first element (namely, 7) that maps to a range element (5) in one iteration of the $3x + 1$ function via all *even* exponents. However, *only the first set maps to 1*. The second set maps to 7, which is a counterexample, hence all elements of the set are counterexamples.

In the case of the $3x + 1$ function, the analogous two sets, and all others that we currently know about, map to 1.

Remark 2

The strategy in the above proof is essentially the same as the strategy for our first proof of the $3x + 1$ Conjecture (see "Theorem: The 3x + 1 Conjecture is true." on page 14). The only difference is that in the above case, the underlying structure of the $3x + 1$ function is y-trees (in particular, the 1-tree), whereas in the other case, the underlying structure is tuple-sets.

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1. In the case of the $3x - 1$ function, if x is an element of the infinite set that maps to a range element y in one iteration of the function, then $4x - 1$ is the value of the next element in the set.

Appendix H — Third Proof of the 3*x* + 1 Conjecture

The following proof of the 3*x* + 1 Conjecture is related to our first two proofs of the Conjecture, but is simpler and clearer.

(*Note*: we ask the reader to inform us of the first sentence that the reader believes contains an error, and what that error is.)

1. *Definitions*: an *anchor tuple* is the first *i*-level tuple in an *i*-level tuple-set. An *anchor* is the *i*-level element of the anchor tuple in an *i*-level tuple-set.

(a) If *x* is a range element of the 3*x* + 1 function, then *x* is eventually — for some $i \geq 2$ — an anchor

Proof: If *x* exists, then for some $i \geq 2$, $x < 2 \cdot 3^{(i-1)}$. Therefore, *x* is an anchor (part (a) of "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 8). □

It follows trivially that *x* is also an anchor for all greater *i*. ("Once an anchor, always an anchor.")

(b) If *x* is a non-counterexample anchor, then it is a non-counterexample anchor whether or not counterexamples exist.

Proof: The arithmetic defining the 3*x* + 1 function is not itself a function of the truth or falsity of the 3*x* + 1 Conjecture. □

Thus, for example, 13 is a non-counterexample (maps to 1) today, and if the Conjecture is proved true tomorrow, it will be a non-counterexample tomorrow, and if the Conjecture is proved false tomorrow it will *still* be a non-counterexample.

(Actually, the statement (b) holds for non-counterexamples in general, not just non-counterexample anchors.)

2. At this point, it is reasonable to assume that there are two possible sets of anchors: one containing counterexamples if counterexamples exist, and one not containing counterexamples, if counterexamples do not exist.

However this assumption is false.

(1) There is one and only one set of anchors, regardless if counterexamples exist or not.

Proof:

(a) The "distance" between consecutive *i*-level elements of an *i*-level tuple-set is $2 \cdot 3^{(i-1)}$ (follows from part (a) in "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 8).

Thus, for example, the distance between the first and second 2-level elements of any 2-level tuple-set having 1 as first element, namely, between the elements 1 and 7, is $2 \cdot 3^{(2-1)} = 2 \cdot 3^1 = 6$. The distance between the second and third elements, that is, between the elements 7 and 13, is likewise 6.

(b) Each *i*-level anchor is less than $2 \cdot 3^{(i-1)}$ (follows from part (a) in "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 8). Of course, each anchor is greater than 0, by definition of the domain of the $3x + 1$ function.

(c) For each $i \geq 2$, the number of anchors in all i-level tuple-sets is $2 \cdot 3^{((i-1)-1)}$ (follows from part (a) in "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 8).

Thus, for example, the number of anchors in all 2-level tuple-sets is $2 \cdot 3^{((2-1)-1))} = 2 \cdot 3^0 = 2$. These anchors are 1 and 5. It is easy to show that 1 is mapped to by all even exponents, and 5 is mapped to by all odd exponents. Those are the only two possibilities for the anchors of 2-level tuple-sets.

The number of anchors in all 3-level tuple-sets is $2 \cdot 3^{((3-1)-1))} = 2 \cdot 3^1 = 6$. These anchors are 1, 5, 7, 11, 13, 17.

(d) The set of $(i + 1)$-level anchors comes into being as follows:

If a is an i-level anchor then a is an $(i + 1)$-level anchor, because if a is less than $2 \cdot 3^{(i-1)}$, as it must be if a is an i-level anchor, then a is certainly less than $2 \cdot 3^{((i+1)-1)}$.

Since the i-level tuple-set element $a + 1 \cdot (2 \cdot 3^{(i-1)})$ is less than $2 \cdot 3^{((i+1)-1)}$, the element is an $(i + 1)$-level anchor.

Since the i-level tuple-set element $a + 2 \cdot (2 \cdot 3^{(i-1)})$ is less than $2 \cdot 3^{((i+1)-1)}$, the element is an $(i + 1)$-level anchor.

No other element of an i-level tuple-set is less than $2 \cdot 3^{((i+1)-1)}$, and therefore no other element of an i-level tuple-set is an $(i + 1)$-level anchor.

The reader can see an example of this increase in anchors from level 2 to level 3 in step 2 (c).

(e) The process we have described is unique. It yields all i-level anchors for all $i \geq 2$. There is thus one and only one set of anchors. In other words:

If counterexamples do not exist, then the set of all anchors is exactly the set that results from the process we have described. Call that set S.

If counterexamples exist, then the set of all anchors is exactly the set that results from the process we have described. In other words, if counterexamples exist, then the set of all anchors is the same set S. □

3. Computer tests[1] have shown the Conjecture to be valid for all consecutive odd positive integers up to at least $10^{18} + 1$, which includes all the anchors (each of which is a non-counterexample anchor) from level 2 through level 35.

(This is not true in the case of the $3x - 1$function since the first counterexample in the case of that function is 5.)

Furthermore, there is an infinity of non-counterexample anchors at levels greater than 35.

1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than $20 \bullet 2^{58} \approx 5.76 \bullet 10^{18}$, which is greater than $3.33 \bullet 10^{16} \approx 2 \bullet 3^{(35-1)}$, have been tested and found to be non-counterexamples. These include the set of all 35-level anchors.

Proof: each range element, hence each non-counterexample anchor at levels 2 through 35 is mapped to by an infinity of odd, positive integers (see Lemma 13.0 in our paper, “Are We Near a Solution to the 3x + 1 Problem?”, on occampress.com), and range element there is an infinity of them in that infinity of odd, positive integers is also mapped to by an infinity of odd, positive integers, etc.. The fact that there is an infinity of these range elements in each case means that an infinity of them are greater than the largest anchor at level 35. □

The unique process for generating anchors (step 2) then continues to generate anchors for all levels beyond 35. The set of anchors so generated for each level is the same whether or not counterexamples exist (the process is unique). So, in particular, we can regard the process as generating the set of all non-counterexample anchors.

If counterexamples exist, the set of anchors so generated is the same as the set of anchors if counterexamples do not exist. Each anchor is an element of an infinite tuple. Non-counterexample infinite tuples are, by definition, of the form <*x*, ..., 1, 1, 1, ... >, whereas counterexample infinite tuples are of the form <*y*, ... >, with no element equal to 1.

And so if counterexamples exist, then some counterexample anchors are the same as non-counterexample anchors, which is absurd. Therefore the $3x + 1$ Conjecture is true.

Another way of stating our argument here is:

The set of all tuple-sets (structure and contents) is the same, whether or not counterexamples exist. Therefore there is no difference between the set of all counterexamples and the set of all non-counterexamples. Therefore, counterexample tuples behave exactly the same as non-counterexample tuples, which is absurd. Therefore counterexamples do not exist, and the Conjecture is true. □

Remark

Suppose that the anchors were all and only those odd, positive integers that map to 1. Suppose, further, that if counterexamples exist, they never become anchors. Then there would be no difficulty: the set of anchors would be fixed, whether or not counterexamples existed, and they would all map to 1.

However, the simple argument in step 1 shows that if counterexamples exist, they must eventually be anchors. And so there is, in reality, a difficulty: how to reconcile this fact with the fact that the set of anchors is fixed whether or not counterexamples exist. Our proof, above, shows one way to reconcile this fact.

Another possible proof might be the following, although we make no claims at this point.

1. It is possible (at the time of this writing) that no counterexamples exist — that is, that all anchors are non-counterexample anchors.

(This does not apply in the case of the $3x - 1$ function, since 5 is known to be a counterexample to the $3x - 1$ Conjecture.)

2. There is one and only one set of non-counterexamples, whether or not counterexamples exist.

Proof: The arithmetic defining the $3x + 1$ function is not itself a function of the truth or falsity of the $3x + 1$ Conjecture. □

Thus, for example, 13 maps to 1 today; if the Conjecture is proved true tomorrow it will map to 1; and if the Conjecture is proved false tomorrow, it will *still* map to 1.)

3. If counterexamples exist, it follows from step 2 that there must be "more" anchors than the ones containing non-counterexamples, namely, the anchors containing counterexamples.

4. But by step 2 in the main proof in this Appendix, there is one and only one set of anchors, whether or not counterexamples exist. So there must be anchors that are "empty" if counterexamples do not exist, and "full" (with counterexamples) if counterexamples exist.

But there is no such thing as an "empty" anchor. Therefore we have a contradiction. Hence counterexamples do not exist, and the $3x + 1$ Conjecture is true.