

## **A Solution to the $3x + 1$ Problem**

by

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*Important Note:* In order to understand the First Proof and the Third Proof of the  $3x + 1$  Conjecture, it is only necessary to read pp. 1-12, and then the short section on anchor tuples on p. 14.

In order to understand the Second Proof of the  $3x + 1$  Conjecture, it is only necessary to read pp. 30-34 of “Section 2. Recursive ‘Spiral’s” in the first file of the paper, “The Structure of the  $3x + 1$  Function: An Introduction” on [occampress.com](http://occampress.com), or, in this paper, “On the Inverse of the  $3x + 1$  Function” on page 16, then the statement of “Lemma 12.0: Statement and Proof” on page 37, then the statement of “Lemma 8.8” on page 16, then the proof of the Conjecture on p.. 17.

The reader can safely assume, *initially*, that all referenced lemmas are true, since their proofs have been checked and deemed correct by several mathematicians.

Key words:  $3x + 1$  Problem,  $3n + 1$  Problem, Syracuse Problem, Ulam’s Problem, Collatz Conjecture, computational number theory, proof of termination of programs, recursive function theory

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## **Abstract**

We present two proofs of the  $3x + 1$  Conjecture, which asserts that repeated iterations of the function  $C(x) = (3x + 1)/(2^a)$  always terminate in 1. Here  $x$  is an odd, positive integer, and  $a$  is the largest positive integer such that the denominator divides the numerator. For the first proof, we define a structure called “tuple-sets” which represents the  $3x + 1$  function in the “forward” direction. We then show that the structure is the same whether or not counterexamples exist, and infer that counterexamples do not exist.

For the second proof, we define a structure (an infinitary tree) representing the  $3x + 1$  function in the inverse or “backward” direction. We then show that, as a result of computer tests, a large upper part of this tree contains only non-counterexamples. Thus this part is the same as it would be if counterexamples did not exist. From this fact, we show that the entire tree is the same as if counterexamples did not exist, and conclude that counterexamples do not exist.

## Introduction

### Statement of Problem

For  $x$  an odd, positive integer, set

$$C(x) = \frac{3x + 1}{2^{\text{ord}_2(3x+1)}}$$

where  $\text{ord}_2(3x + 1)$  is the largest exponent of 2 such that the denominator divides the numerator. Thus, for example,  $C(17) = 13$ ,  $C(13) = 5$ ,  $C(5) = 1$ . The  $3x + 1$  Problem, also known as the  $3n + 1$  Problem, the Syracuse Problem, Ulam's Problem, the Collatz Conjecture, Kakutani's Problem, and Hasse's Algorithm, asks if repeated iterations of  $C$  always terminate at 1. The conjecture that they do is hereafter called the  $3x + 1$  Conjecture. We call  $C$  the  $3x + 1$  function; note that  $C(x)$  is by definition odd.

Other equivalent formulations of the  $3x + 1$  Problem are given in the literature; we base our formulation on the  $C$  function (following Crandall) because, as we shall see, it brings out certain structures that are not otherwise evident.

### Summary of Research on the Problem

As stated in (Lagarias 1985), "The exact origin of the  $3x + 1$  problem is obscure. It has circulated by word of mouth in the mathematical community for many years. The problem is traditionally credited to Lothar Collatz, at the University of Hamburg. In his student days in the 1930's, stimulated by the lectures of Edmund Landau, Oskar Petron, and Issai Schur, he became interested in number-theoretic functions. His interest in graph theory led him to the idea of representing such number-theoretic functions as directed graphs, and questions about the structure of such graphs are tied to the behavior of iterates of such functions. In the last ten years [that is, 1975-1985] the problem has forsaken its underground existence by appearing in various forms as a problem in books and journals..."

As far as we have been able to determine, our approach to a solution of the Problem is original. We have been unable to find papers in the literature that suggest that other researchers have had the same basic idea as ours, namely, that which is described in "Abstract" on page 2 and in the next section, and, in more detail, in the first two paragraphs under "Tuple-Sets" on page 5.

### Brief Description of Solution Strategy

(1) We begin with a partitioning of the set of all finite sequences of iterations of  $C$  into "tuple-sets". Each tuple-set  $T_A$  consists of the set of all tuples that are associated with a finite exponent sequence  $A$  and its prefixes. Thus, for example, the tuple  $\langle 7, 11, 17, 13 \rangle$  is associated with the exponent sequence  $A = \{1, 1, 2\}$ , because 7 maps to 11 (in one iteration of  $C$ ) via the exponent 1, 11 maps to 17 (in one iteration of  $C$ ) via the exponent 1, and 17 maps to 13 (in one iteration of  $C$ ) via the exponent 2. The tuple-set  $T_A$  contains all tuples associated with  $A$ , plus all tuples associated with each non-empty prefix of  $A$ . There are always an infinity of tuples in a tuple-set.

(2) Next, we define, and prove the validity of, a simple function that establishes the difference (“distance”) between the values of successive odd, positive integers at each level  $j$  in each tuple-set (parts (a) and (b) of “Lemma 1.0” on page 10).

(3) We then show that if counterexamples exist, each tuple-set contains an infinity of counterexample tuples *and* an infinity of non-counterexample tuples (“Lemma 5.0” on page 14).

(4) With this simple machinery, and the assumption that counterexamples exist, we are able to show that there is no difference in the set of all tuple-sets if counterexamples exist, and if counterexamples do not exist (steps 1 - 2 of “Proof of the  $3x + 1$  Conjecture” on page 18). This fact then gives us a contradiction that proves the  $3x + 1$  Conjecture (step 3).

### **On the Format of this Paper**

Because the tuple-set structure lies at the heart of our solution to the Problem, and thus is referred to in most proofs of lemmas, and because defining this structure requires defining quite a few terms, we have placed all these definitions at the beginning of our exposition.

To enhance ease and rapidity of understanding, we merely state lemmas in the course of the exposition. All proofs are given in “Appendix A — Statement and Proof of Each Lemma” on page 25. The page number of each proof is given in the text. Lemma numbering is not necessarily consecutive, since, for the benefit of readers who are acquainted with previous versions of this paper or with our paper, “Are We Near a Solution to the  $3x + 1$  Problem?” on [occampress.com](http://occampress.com), we have retained numbering of lemmas in those papers.

Referenced equations in each lemma are numbered  $n.1$ ,  $n.2$ , etc., where  $n$  is the number of the lemma.

## Tuple-Sets

In the first part of this paper, we describe a structure called *tuple-sets* that underlies all finite sequences of iterations of the  $3x + 1$  function,  $C$ . We have placed virtually all definitions in this first part of the paper because the terms defined are used repeatedly in the lemmas and proofs given later.

A tuple-set can be briefly, and informally, described as follows. (A formal definition is given under “Tuple-set” on page 7.) Consider the sequence of two iterations of  $C$ :  $C(17) = 13$  (via the exponent 2 in the definition of  $C$ ) followed by  $C(13) = 5$  (via the exponent 3 in the definition of  $C$ ). This sequence of iterations can be represented by the tuple  $\langle 17, 13, 5 \rangle$ . The tuple-set  $T_A$  defined by the 2-level exponent sequence  $A = \{2, 3\}$  contains the tuple  $\langle 17, 13, 5 \rangle$ . But in addition it contains all other tuples that are determined by the exponent sequence  $\{2\}$  but not by  $\{2, 3\}$  — in other words, all other tuples that are determined by “approximations” to, or prefixes of,  $A$ . For example, the tuples  $\langle 33, 25 \rangle$  and  $\langle 81, 61, 23 \rangle$  are in  $T_A$ , because  $\langle 33, 25 \rangle$  is associated with the exponent sequence  $\{2\}$  but 25 does not map to another odd positive integer via the exponent 3, and  $\langle 81, 61, 23 \rangle$  is associated with the exponent sequence  $\{2, 3\}$ .

We then show that each  $i$ -level tuple-set, where  $i \geq 2$ , has a unique first  $i$ -level tuple (called an *anchor* tuple) that (like all tuples) must be either a non-counterexample tuple or a counterexample tuple, but cannot be both. In the second part of this paper, we show how a basic result — that if counterexamples exist, then every tuple-set contains an infinity of counterexample tuples and an infinity of non-counterexample tuples — enables us to prove that the set of all tuple-sets if counterexamples exist is the same as the set of all tuple-sets if counterexamples do not exist, which implies that counterexamples do not exist, and hence that the  $3x + 1$  Conjecture is true..

We now proceed with our definitions.

## Iteration

An *iteration* takes an odd, positive integer,  $x$ , to another odd, positive integer,  $y$ , via one application of the  $3x + 1$  function,  $C$ . Thus, in one iteration  $C$  takes 17 to 13 because  $C(17) = 13$ .

## Tuple

A *tuple* is a sequence of zero or more successive iterations of  $C$ , that is, if the sequence is finite,

$$(C^k(x))_{k \geq 0} = (x, C(x), C^2(x), \dots, C^k(x))$$

or, if the sequence is infinite,

$$(C^\infty(x)) = (x, C(x), C^2(x), \dots)$$

A finite sequence is not required to end with a 1, and an infinite sequence is not required to end with an infinity of successive 1's. If an infinite sequence does not end with an infinity of successive 1's, then it consists of counterexamples to the  $3x + 1$  Conjecture.

A finite tuple is denoted<sup>1</sup>  $\langle x, y, y', \dots, y^{(n)} \rangle$ . We say that  $x$  maps to  $y^{(n)}$ . For example,  $\langle 5, 1 \rangle$  and  $\langle 11, 17, 13 \rangle$  are finite tuples. An infinite tuple is denoted  $\langle x, y, y', \dots \rangle$ . For example,  $\langle 5, 1, 1, 1, \dots \rangle$  and  $\langle 11, 17, 13, 5, 1, 1, 1, \dots \rangle$  are infinite tuples.

Let  $t = \langle x, y, y', \dots, y^{(n)} \rangle$  be a finite tuple. Then the tuple  $t' = \langle x, y, y', \dots, y^{(n)}, y^{(n+1)} \rangle$  is an *extension* of  $t$ . An extension of an extension of  $t$  we likewise call an extension of  $t$ , etc. By definition of the function  $C$ , every finite tuple has an infinite number of extensions. In the case of a sequence of iterations of  $C$  that eventually yield 1, the corresponding infinite tuple is  $\langle x, y, y', \dots, 1, 1, 1, \dots \rangle$ . A tuple consisting of an infinite number of extensions is an *infinite tuple*. We denote an infinite tuple by  $\bar{t}$ .

Clearly, since the domain of  $C$  consists of the odd, positive integers, each odd, positive integer is the first element of an infinite tuple.

If  $\bar{t}$  is an infinite tuple, we denote the first  $i$  levels of  $\bar{t}$  (that is, the first  $i$  elements of  $\bar{t}$ ), by  $\bar{t}(i)$ , and we call  $\bar{t}(i)$  a *prefix* of  $\bar{t}$ . For example, if  $\bar{t} = \langle 17, 13, 5, 1, 1, 1, \dots \rangle$ , then  $\bar{t}(1) = 17$ , and  $\bar{t}(4) = \langle 17, 13, 5, 1 \rangle$ . Thus every finite tuple is a prefix of an infinite tuple and every prefix of an infinite tuple is a finite tuple. The term *tuple* standing alone, without the qualifier “infinite”, denotes a finite tuple, that is, the prefix of an infinite tuple, unless context clearly indicates the reference is to an infinite tuple.

In the literature on the  $3x + 1$  Problem, tuples are sometimes called “trajectories” or “orbits”.

Each tuple element except, possibly, the first, is an odd, positive integer that is not a multiple of 3. The element is odd by definition of the  $3x + 1$  function,  $C$ , and is not a multiple of 3 by “Lemma 9.0: Statement and Proof” on page 34.

## Non-Counterexample and Counterexample

If  $x$  is the first element of an infinite tuple  $\langle x, \dots, 1, 1, 1, \dots \rangle$ , then  $x$  is called a *non-counterexample*. Otherwise,  $x$  is called a *counterexample*. Thus, a counterexample never yields 1 under repeated iterations of the  $3x + 1$  function.

## Exponent

If  $C(x) = y$ , with  $y = (3x + 1)/2^a$ , we say that  $x$  maps under iteration to  $y$  (or  $x$  maps directly to  $y$ ) via the exponent  $a$ , and that  $a$  is the exponent associated with  $x$ . By abuse of language, we sometimes speak of  $a$  as *mapping directly to  $y$* . We sometimes omit the word *directly* when context makes clear that it is implied. The sequence  $\{a_2, a_3, \dots, a_i\}$ , where  $a_2, a_3, \dots, a_i$  are the exponents associated with  $x, C(x), \dots, C^{(i-1)}(x)$  respectively, is called an *admissible vector* in (Wirsching 1998). We call the sequence an *exponent sequence*. We define the function  $e(x)$  to be the exponent associated with  $x$ . We sometimes refer to  $y$  as a *range element*. It is easily shown that  $y$  cannot be a multiple of 3 (see “Lemma 9.0: Statement and Proof” on page 34). An element  $x$  of the domain of the  $3x + 1$  function, whether multiple of 3 or not, we sometimes refer to as a *domain element*.

Clearly, an exponent is a positive integer.

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1. In a tuple, “ $x^{(n)}$ ”, “ $y^{(n)}$ ”, etc., denotes  $x$  followed by  $n$  prime symbols,  $y$  followed by  $n$  prime symbols, etc.

## Symbols for Exponent Sequences and for Tuples

It is important that the reader keep clearly in mind the symbols we use for two different types of sequences. Curly braces denote exponent sequences. Thus, for example, we write  $\{a_2, a_3, \dots, a_i\}$  for an  $i$ -level exponent sequence, and  $\{a_2, a_3, \dots\}$  for an infinite exponent sequence. Angle brackets denote tuples, that is, the results of iterations of the  $3x + 1$  function. Thus, for example, we write  $\langle x, y, \dots, z \rangle$  or  $\langle x, y, y', \dots, y^{(n)} \rangle$  for finite tuples, and  $\langle x, y, y', \dots \rangle$  for infinite tuples.

## Exponent Sequence Associated With a Tuple

As we established under “Exponent” on page 6, associated with every non-empty finite sequence of iterations of the function  $C$  — hence with every tuple — is an exponent sequence. We speak of the exponent sequence *associated with* a finite tuple. If  $t$  is a tuple, then we denote the exponent sequence associated with  $t$  by  $A(t)$ . Thus, for example, if  $t = \langle 17, 13, 5, 1 \rangle$  then  $A(t) = \{2, 3, 4\}$  because 17 maps directly to 13 via the exponent 2, 13 maps directly to 5 via the exponent 3, and 5 maps directly to 1 via the exponent 4.

## Extension of an Exponent Sequence

Let  $A = \{a_2, a_3, \dots, a_i\}$  be a finite sequence of exponents, where  $i \geq 2$ . Then an exponent sequence  $A' = \{a_2, a_3, \dots, a_i, a_{i+1}\}$  is an *extension* of  $A$ . An extension of  $A'$  is also an extension of  $A$ , etc.

## Tuple-set

(The reader might find it helpful to refer to Fig. 1 in this sub-section while reading the following.)

Let  $A = \{a_2, a_3, \dots, a_i\}$  be a finite sequence of exponents, where  $i \geq 2$ . The *tuple-set*  $T_A$  consists of all and only the following tuples:

all tuples  $\langle x \rangle$  such that  $x$  does not map to an odd, positive integer via  $a_2$ ;

all tuples  $\langle x, y \rangle$  such that  $x$  maps to  $y$  via  $a_2$  (that is,  $e(x) = a_2$ ) but  $y$  does not map to an odd, positive integer via  $a_3$ ;

all tuples  $\langle x, y, y' \rangle$  such that  $x$  maps to  $y$  via  $a_2$  (that is,  $e(x) = a_2$ ) and  $y$  maps to  $y'$  via  $a_3$  (that is,  $e(y) = a_3$ ), but  $y'$  does not map to an odd, positive integer via  $a_4$ ;

...

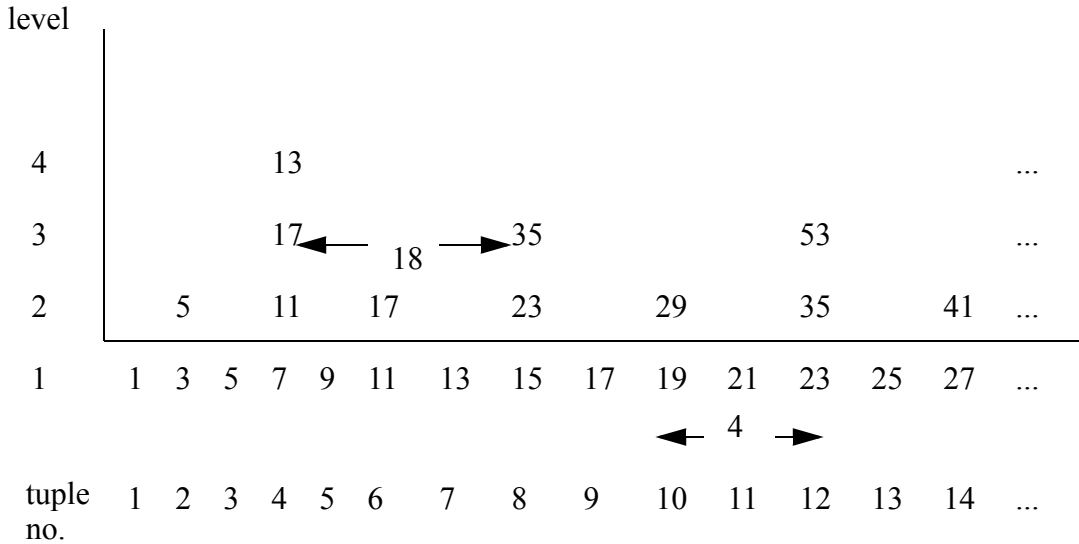
all tuples  $\langle x, y, y', \dots, y^{(i-3)}, y^{(i-2)} \rangle$  such that  $x$  maps to  $y$  via  $a_2$  (that is,  $e(x) = a_2$ ) and  $y$  maps to  $y'$  via  $a_3$  (that is,  $e(y) = a_3$ ) and ... and  $y^{(i-3)}$  maps to  $y^{(i-2)}$  via the exponent  $a_i$  (that is,  $e(y^{(i-3)}) = a_i$ ). (The longest tuple in an  $i$ -level tuple-set has  $i$  elements.)

There is a countable infinity of tuples in each tuple-set, as we establish in “Proof That Tuple-sets Exist as Defined” on page 12 and “Why There Are An Infinite Number of Tuples in Each Tuple-set” on page 13.

If  $A = \{a_2, a_3, \dots, a_i\}$  is a finite exponent sequence, then of an  $i$ -level tuple  $t$  in the tuple-set  $T_A$ , we say that  $t$  is *determined* by the exponent sequence  $A$  and that  $A$  is *associated with*  $t$ . Finally, we say that the tuple-set  $T_A$  is *determined* by the sequence  $A$ . To review: given a tuple, we speak of the exponent sequence *associated with* it; given an exponent sequence, we speak of the tuple or tuple-set it *determines*.

As an example of (part of) a tuple-set: in Fig. 1, where  $A = \{a_2, a_3, a_4\} = \{1, 1, 2\}$  and where we adopt the convention of orienting tuples vertically on the page, the tuple-set  $T_A$  includes:

- the tuple  $\langle 1 \rangle$ , because  $e(1) \neq a_2$ ;
- the tuple  $\langle 3, 5 \rangle$ , because  $e(3) = a_2 = 1$ , but  $e(5) = 4 \neq a_3 = 1$ ;
- the tuple  $\langle 15, 23, 35 \rangle$ , because  $e(15) = a_2 = 1$ , and  $e(23) = a_3 = 1$ , but  $e(35) = 1 \neq a_4 = 2$ .



**Fig. 1.**  
**Part of the tuple-set  $T_A$  associated with the sequence  $A = \{1, 1, 2\}$**

The number 18 between the arrows at level 3 and the number 4 between the arrows at level 1 are the values of the level 3 and level 1 distance functions, respectively, established by Lemma 1.0 (see “Lemma 1.0” on page 10).

In each  $i$ -level tuple-set  $T_A$ , where  $i \geq 2$ , for each odd, positive integer  $x$  there exists a tuple whose first element is  $x$ . The tuple may be one-level ( $\langle x \rangle$ ), or 2-level ( $\langle x, y \rangle$ ), or ... or  $i$ -level ( $\langle x, y, y', \dots, y^{(i-3)}, y^{(i-2)} \rangle$ ). Thus each tuple-set is non-empty.

Lemma 4.0 (see “Lemma 4.0: Statement and Proof” on page 31) establishes that a tuple-set  $T_A$  exists for each exponent sequence  $A$ .

**Note:** *our proofs will almost always involve only  $i$ -level tuples in  $i$ -level tuple-sets. We have included  $j$ -level tuples, where  $2 \leq j \leq i$ , in our definition of tuple-set because we feel that these tuples are necessary to fully describe the structure of tuple-sets. If the level of a tuple in an  $i$ -level tuple-set is not specified, the reader should assume that the tuple is  $i$ -level.*

## Ordering of Tuples in a Tuple-set

Tuples in a tuple-set  $T_A$  are linearly ordered by the natural order of their first elements. We denote a specific tuple in a tuple-set by  $t_{(r)}$ , where  $r \geq 1$ . If  $T_A$  is an  $i$ -level tuple-set, where  $i \geq 2$ , we denote the  $j$ th element of  $t_{(r)}$  (if it exists in  $T_A$ ) by  $t_{(r)(j)}$ , where  $1 \leq j \leq i$ .

The reader may find it helpful to imagine an  $i$ -level tuple-set, where  $i \geq 2$ , as a “picket fence” infinite to the right, with the tuples serving as the pickets, as suggested by Fig. 1 under “Tuple-set” on page 7.

## Level in a Tuple-set

A *level  $j$*  in a tuple-set is defined as follows. If  $A = \{a_2, a_3, \dots, a_i\}$ , where  $i \geq 2$ , is a finite sequence of exponents, for  $2 \leq j \leq i$  we shall call  $a_j$  the element at level  $j$  in the sequence, hence in the tuple-set  $T_A$ . Subscripts of exponents in an exponent sequence are numbered beginning with 2 instead of with 1 so that the last subscript then indicates the number of levels in the corresponding tuple-set. Thus, for example, if  $A = \{a_2\}$ , then  $T_A$  is a 2-level tuple-set; if  $A = \{a_2, a_3\}$ ,  $T_A$  is a 3-level tuple-set, etc. Level 1 is then the level containing the set of all possible tuple first elements  $\{1, 3, 5, 7, \dots\}$  in  $T_A$ , that is, the set of odd, positive integers. Thus, for example in the tuple  $\langle 17, 13, 5, 1 \rangle$ , 17 is at level 1, 13 is at level 2, 5 is at level 3, and 1 is at level 4. We denote the element at level  $j$  in the tuple  $t_{(r)}$  in a  $i$ -level tuple-set, where  $i \geq 2$ , by  $t_{(r)(j)}$ , where  $1 \leq j \leq i$ . (The element at level  $j$  is the  $j$ th element in the tuple.)

If a tuple has an element at level  $j$ , but none at level  $j + 1$ , we refer to the tuple as a  *$j$ -level tuple*. If the tuple also has an element at level  $j + 1$ , we sometimes refer to the tuple as a  $(\geq j)$ -level tuple.

In the case that  $A = \{a_2, a_3, \dots, a_i\}$ , where  $i \geq 2$ , we refer to  $A$  as an  *$i$ -level exponent sequence*. An  $i$ -level exponent sequence consists of  $(i - 1)$  exponents. The longest tuple in a tuple-set determined by an  $i$ -level exponent sequence is an  $i$ -level tuple.

## Tuples Consecutive at Level $j$

Tuples *consecutive at level  $j$* ,  $j \geq 2$ , are defined as follows. Let  $t_{(r)}, t_{(s)}$  be  $(\geq j)$ -level tuples in some  $i$ -level tuple-set  $T_A$ , where  $i \geq 2$ . If there is no  $(\geq j)$ -tuple between  $t_{(r)}$  and  $t_{(s)}$ , we say that  $t_{(r)}$  and  $t_{(s)}$  are *tuples consecutive at level  $j$* . Here, “between” means relative to the natural linear ordering of tuples based on their first elements.

Thus, for example, in Fig. 1, the tuples numbered 4 and 8 are consecutive at level 3.

## Extension of a Tuple-set

Let  $T_A$  be a tuple-set, where  $A = \{a_2, a_3, \dots, a_i\}$ . Then a tuple-set  $T_{A'}$ , where  $A' = \{a_2, a_3, \dots, a_i, a_{i+1}\}$  is an *extension* of  $T_A$ . A proof that there exists such an extension for each exponent  $a_{i+1}$  is given in Lemma 3.0 (see “Lemma 3.0: Statement and Proof” on page 30).

## Tuple-sets and Infinite Tuples

Tuples in a tuple-set are oriented vertically in accordance with our convention (see “Tuple-set” on page 7). Each tuple is a prefix of an infinite tuple (see “Tuple” on page 5). Therefore the infinite tuples whose  $i$ -level prefixes constitute the  $i$ -level tuples in an  $i$ -level tuple-set, are like-

wise oriented vertically, and thus occupy a single, vertical plane  $P_A$  that is infinite in the upward direction and to the right.

Let  $A = \{a_2, a_3, \dots, a_i\}$  be an  $i$ -level exponent sequence. Let  $T_A$  be the associated tuple-set. The  $i$ -level tuples in  $T_A$  are  $i$ -level prefixes of infinite tuples. Call the set of these infinite tuples,  $S$ . Let  $A' = \{a_2, a_3, \dots, a_i, a_{(i+1)}, \dots, a_{(i+k)}\}$  be an extension of  $A$ . Then the  $(i+k)$ -level tuples in  $T_{A'}$  are  $(i+k)$ -level prefixes of infinite tuples. These infinite tuples constitute a proper subset of  $S$ .

Each tuple-set that is an extension of  $T_A$  — each tuple in each such tuple-set — is contained in the single, vertical plane  $P_A$ .

## Each $i$ -Level Tuple-set Contains an Infinity of Tuples of Each Level $i$ or Less

### Lemma 0.0.

*Each  $i$ -level tuple-set, where  $i \geq 2$ , contains an infinity of tuples of each length  $j$ , where  $1 \leq j \leq i$ .*

**Proof:** see “Lemma 0.0: Statement and Proof” on page 25.

## Distance Functions on Tuple-sets

*Definition:* let  $T_A$  be an  $i$ -level tuple-set, where  $i \geq 2$ . Let  $t(r), t(s)$  denote tuples consecutive at level  $i$ , with  $r < s$  in the natural ordering of tuples by first elements. Let  $t(r)(h), t(s)(h)$  denote the elements of  $t(r), t(s)$  at level  $h$ , where  $1 \leq h \leq i$ . Then we call  $|t(s)(h) - t(r)(h)|$  the *distance* between  $t(r)$  and  $t(s)$  at level  $h$ . We denote this distance by  $d(h, i)$  and call  $d$  the *distance functions* (one function for each  $h$ ).

### Lemma 1.0

(a) Let  $A = \{a_2, a_3, \dots, a_i\}$ , where  $i \geq 2$ , be a sequence of exponents, and let  $t(r), t(s)$  be tuples consecutive at level  $i$  in  $T_A$ . Then  $d(i, i)$  is given by:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) Let  $t(r), t(s)$  be tuples consecutive at level  $i$  in  $T_A$ . Then  $d(1, i)$  is given by:

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3}) \dots (2^{a_i})$$

Thus, in Fig. 1 under “Tuple-set” on page 7, the distance  $d(3, 3)$  between  $t_{8(3)} = 35$  and  $t_{4(3)} = 17$  is  $2 \cdot 3^{(3-1)} = 18$ . The distance  $d(1, 2)$  between  $t_{12(1)} = 23$  and  $t_{10(1)} = 19$  is  $2 \cdot 2^1 = 4$ .

**Proof:** see “Lemma 1.0: Statement and Proof” on page 25.

**Remarks About the Distance Functions**

(1) Strictly speaking, we should include the sequence  $A$  of exponents as arguments of  $d(1, i)$ ,  $d(i, i)$ , but this notation would be cumbersome and, since typically this sequence is known, unnecessary.

(2) The distance functions make clear that, for each  $i$ -level sequence of exponents, there exists an infinity of  $i$ -level tuples produced by that sequence. (The equivalent of this statement is made in (Wirsching 1998) (p. 48).) The following table shows the distance relationships for  $(i - j)$ -level elements of tuples consecutive at level  $(i - j)$  in an  $i$ -level tuple-set, where  $0 \leq j \leq (i - 1)$ . The distances are easily proved using Lemma 1.0. (An example is given following the table.) We only use the distances at levels 1 and  $i$  in this paper.

**Table 1: Distances between elements of tuples  $t_{(r)}$ ,  $t_{(s)}$  consecutive at level  $i$**

Level	Distance between $(i - j)$ -level elements of tuples consecutive at level $(i - j)$ , where $0 \leq j \leq (i - 1)$
$i$	$2 \cdot 3^{i-1}$
$i - 1$	$2 \cdot 3^{i-2} \cdot 2^{a_i}$
$i - 2$	$2 \cdot 3^{i-3} \cdot 2^{a_{i-1}} 2^{a_i}$
$i - 3$	$2 \cdot 3^{i-4} \cdot 2^{a_{i-2}} 2^{a_{i-1}} 2^{a_i}$
...	...
2	$2 \cdot 3 \cdot 2^{a_3} \dots 2^{a_{i-1}} 2^{a_i}$
1	$2 \cdot 2^{a_2} 2^{a_3} \dots 2^{a_{i-1}} 2^{a_i}$

For example, let  $x$  be an element at level  $(i - 1)$  of an  $i$ -level tuple. Then, by the table, the element at level  $(i - 1)$  in the next  $i$ -level tuple (that is, in the next tuple consecutive at level  $(i - 1)$ ) =  $(x + 2 \cdot 3^{i-2} \cdot 2^{a_i})$ , and so it must be the case that

$$\frac{3(x + 2 \cdot 3^{i-2} \cdot 2^{a_i}) + 1}{2^{a_i}} = \frac{3x + 1}{2^{a_i}} + 2 \cdot 3^{i-1}$$

which, as the reader can check, is indeed the case.

(3) Lemma 1.0 makes clear that no two  $i$ -level tuples in a given  $i$ -level tuple-set have the same last element. In fact, the values of the last elements of  $i$ -level tuples in an  $i$ -level tuple-set always increase as one proceeds along the sequence of  $i$ -level tuples.

## **Every Possible 2-Level Tuple-set Exists**

### **Lemma 2.0**

*For each exponent  $a_2$ , a tuple-set  $T_A$ , where  $A = \{a_2\}$ , exists.*

**Proof:** See “Lemma 2.0: Statement and Proof” on page 30.

## **Every Possible Extension of Each $i$ -Level Tuple-set Exists**

### **Lemma 3.0**

*Each  $i$ -level tuple-set  $T_A$ , where  $A = \{a_2, a_3, \dots, a_i\}$  and  $i \geq 2$ , has an extension via each exponent  $a_{i+1}$ .*

**Proof:** See “Lemma 3.0: Statement and Proof” on page 30.

## **Proof That Tuple-sets Exist as Defined**

### **Lemma 4.0**

*For each exponent sequence  $A = \{a_2, a_3, a_4, \dots, a_i\}$ , where  $i \geq 2$ , there exists a tuple-set  $T_A$ .*

**Proof:** See “Lemma 4.0: Statement and Proof” on page 31.

Lemmas 2.0, 3.0 and 4.0 establish, as part of their proofs, that there are an infinite number of tuples in each tuple-set. A plausible question at this point is: Why should there be? The answer is given in “Why There Are An Infinite Number of Tuples in Each Tuple-set” on page 13.

## **On the Number of Tuple-sets**

### **Lemma 4.5**

*For each  $i \geq 2$ , the number of  $i$ -level tuple-sets is countably infinite.*

**Proof:** See “Lemma 4.5: Statement and Proof” on page 32.

## **On the Set of All $i$ -Level Elements of All $i$ -Level Tuple-sets**

### **Lemma 4.75**

*For each  $i \geq 2$ , the set of all  $i$ -level elements of all  $i$ -level tuples in all  $i$ -level tuple-sets is the set of all range elements of the  $3x + 1$  function.*

**Proof:** See “Lemma 4.75: Statement and Proof” on page 32.

## A Recursive Description of Any Tuple-set

Let  $x$  denote the set of odd, positive integers. Let  $y = C\{a_2 \bmod 2 \cdot 3^{(1-1)}\}(x)$  denote the set of range elements of the  $3x + 1$  function produced by the exponent  $a_2 \bmod 2 \cdot 3^{(1-1)}$  operating on all the elements of  $x$ . As we know from Lemma 1.0,  $y$  is one of two sets, namely, the set of all  $y \equiv 1 \bmod 2 \cdot 3^{(1-1)}$  (if  $a_2$  is even) or the set of all  $y \equiv 5 \bmod 2 \cdot 3^{(1-1)}$  (if  $a_2$  is odd).

We can repeat the process recursively, so that, if  $A = \{a_2, a_3, \dots, a_i\}$ , then

$$(1) \quad T_A = C\{a_i \bmod 2 \cdot 3^{((i-1)-1)}(\dots C\{a_3 \bmod 2 \cdot 3^{(2-1)}\}(C\{a_2 \bmod 2 \cdot 3^{(1-1)}\}(x))\dots).$$

The reason that this is a recursive description of the tuple-set  $T_A$  is that it is precisely the sequence of tuple-set extensions,

$$T_{\{a_2\}}, T_{\{a_2, a_3\}}, T_{\{a_2, a_3, a_4\}}, \dots, T_{\{a_2, a_3, a_4, \dots, a_i\}}$$

The reason we only need to consider the indicated finite set of exponents at each level is established by Lemmas 7.0 and 7.1 in the first part of the second file of the paper, “The Structure of the  $3x + 1$  Function: An Introduction” on the web site [occampress.com](http://occampress.com).

We remind the reader that if  $y''\dots''$  is a set mapped to by  $C\{a_i\dots\}(y''\dots'')$ , then we know by “Lemma 1.0” on page 10 that  $y''\dots''$  is a reduced residue class mod  $2 \cdot 3^{((i+1)-1)}$ .

Equation (1) describes the behavior of the  $3x + 1$  function over its entire domain, namely, the set of all odd, positive integers, regardless if counterexamples exist or not.

## Why There Are An Infinite Number of Tuples in Each Tuple-set

Every finite exponent sequence — that is, every finite sequence of positive integers — determines an  $i$ -level tuple-set (“Lemma 4.0: Statement and Proof” on page 31), where  $i \geq 2$ . The last element (that is, the  $i$ -level element) of each tuple maps directly to one and only one odd, positive integer via one and only one exponent. Consider the tuple-set  $T_A$  determined by the exponent sequence  $A = \{a_2, a_3, a_4, \dots, a_i\}$  where  $i \geq 2$ .  $T_A$  has an extension for *each* positive integer  $a_{i+1}$  (“Lemma 3.0: Statement and Proof” on page 30). But since the last element of each tuple in  $T_A$  maps directly to one and only one odd positive integer, and since by Lemma 3.0 (see “Lemma 3.0: Statement and Proof” on page 30) each tuple-set  $T_{A'}$ ,  $A' = \{a_0, a_1, a_2, \dots, a_i, a_{i+1}\}$ , likewise has an extension for each positive integer  $a_{i+2}$ , etc., it follows that, for *each*  $a_i$ , there exists an *infinity* of tuples in  $T_A$  whose last elements directly map to their respective odd, positive integers *via*  $a_i$ . In short, the reason there are an infinite number of  $i$ -level tuples in each  $i$ -level tuple-set is that (1) each  $i$ -level tuple-set has an infinity of extensions, namely, one for each exponent  $a_{i+1}$ , but (2) each tuple maps directly to one and only one odd, positive integer via one and only one exponent.

Thus, in each  $i$ -level tuple-set  $T_A$ , where  $i \geq 2$ , the countable infinity of  $i$ -level non-counterexample tuples consists of:

- an infinity that have an extension via the exponent 1, and
- an infinity that have an extension via the exponent 2, and
- an infinity that have an extension via the exponent 3, and
- ...

If counterexamples exist, the same is true for  $i$ -level counterexample tuples.

We now come to a lemma that is of crucial importance in our proof of the  $3x + 1$  Conjecture.

## **On Non-Counterexample and Counterexample Tuples in a Tuple-set**

### **Lemma 5.0**

*Assume a counterexample exists. Then for all  $i \geq 2$ , each  $i$ -level tuple-set contains an infinity of  $i$ -level counterexample tuples and an infinity of  $i$ -level non-counterexample tuples.*

**Proof:** see “Lemma 5.0: Statement and Proof” on page 32.

### **Remark 1**

This lemma establishes that there is no way to distinguish counterexamples from non-counterexamples on the basis of the *finite exponent sequences* associated with each. Of course, if a non-trivial cycle exists, then an infinite tuple  $\langle x_1, x_2, \dots, x_1, x_2, \dots, x_1, x_2, \dots \rangle$  exists, and thus the finite tuple  $\langle x_1, x_2, \dots, x_1 \rangle$  immediately tells us that a counterexample exists. But there is no requirement that a counterexample give rise to a non-trivial cycle. A counterexample can simply give rise to an infinite tuple in which no element recurs, and which has no element = 1.

To repeat: there is no way of telling from a *finite exponent sequence* that the sequence is associated with a counterexample. For example, the sequence  $\{a_2, a_3, \dots, a_2, a_3, \dots, a_2, a_3, \dots\}$ , in which  $\{a_2, a_3, \dots, a_2\}$  is repeated, say, a trillion times, does not imply the existence of a counterexample cycle.

### **Remark 2**

Lemma 5.0 implies that the set of all  $i$ -level non-counterexample tuples, where  $i \geq 2$ , is associated with the set of all  $i$ -level exponent sequences and, if counterexamples exist, then the set of all  $i$ -level counterexample tuples is likewise associated with the set of all  $i$ -level exponent sequences.

### **Lemma 9.7**

*(a) If counterexamples do not exist, then for all  $i$ -level tuple-sets  $A = \{a_2, a_3, \dots, a_i\}$ , where  $i \geq 2$ , if  $x$  is the first element of an  $i$ -level (necessarily non-counterexample) tuple in  $T_A$ , then the first element of the next  $i$ -level (necessarily non-counterexample) tuple is*

(1)

$$(x + (2 \cdot (2^{a_2})(2^{a_3}) \dots (2^{a_i})))$$

*(b) If counterexamples exist, then in each  $i$ -level tuple-set  $A = \{a_2, a_3, \dots, a_i\}$ , where  $i \geq 2$ , there exists an  $x$  which is the first element of an  $i$ -level non-counterexample tuple in  $T_A$  such that the first element of the next  $i$ -level non-counterexample tuple in  $T_A$  is greater than the value in (1).*

**Proof:**

Part (a) follows directly from part (b) of the distance function lemma, namely, “Lemma 1.0” on page 10. Part (b) follows from the fact that, if counterexamples exist, then, by “Lemma 5.0”

on page 14, each tuple-set contains an infinity of counterexample tuples and an infinity of non-counterexample tuples. Hence there must exist at least one non-counterexample tuple that is followed by at least one counterexample tuple. Hence the distance to the next non-counterexample tuple is greater than (1).

**Remark:** The Lemma shows that, informally, if counterexamples exist, non-counterexamples are “farther apart” from each other than if counterexamples do not exist.

We are now at the final stage of our preparation for the proof of the  $3x + 1$  Conjecture. This stage is concerned with the first  $i$ -level tuple in an  $i$ -level tuple-set. This tuple is called the *anchor tuple* of the tuple-set.

### **Anchor and Anchor Tuple**

Since tuples in a tuple-set are linearly ordered by the natural order of their first elements, in every  $i$ -level tuple-set, where  $i \geq 2$ , there is a unique first  $i$ -level tuple, which we call the *anchor tuple* of the tuple-set. The last element, that is, the  $i$ -level element, of the anchor tuple we call the *anchor* of the anchor tuple, sometimes referring to it (redundantly) as the  *$i$ -level anchor*. We emphasize that in each  $i$ -level tuple-set there is *only one* anchor tuple: it is the first  $i$ -level tuple in the tuple-set.

Each element of an anchor tuple except, possibly, the first, (like each element of any tuple) is an odd, positive integer that is not a multiple of 3. The element is odd by definition of the  $3x + 1$  function,  $C$ , and is not a multiple of 3 by “Lemma 9.0: Statement and Proof” on page 34.

#### **Lemma 6.0**

*Let  $t$  be the anchor tuple (by definition an  $i$ -level tuple) in an  $i$ -level tuple-set, where  $i \geq 2$ . Then the last element  $y$  of  $t$ , that is, the  $i$ -level element of  $t$  (this element being the anchor), is a number less than  $2 \cdot 3^{(i-1)}$ .*

**Proof:** see “Lemma 6.0: Statement and Proof” on page 33.

### **Definition of “Reduced Residue Class” and of “Complete Set of Reduced Residue Classes”**

If a residue class mod  $m$  is such that each element of the class is relatively prime to  $m$ , then we call the class a *reduced residue class mod  $m$* . Thus, for example, the residue class mod 6 whose minimum element is 5 is a reduced residue class mod 6. The set of all reduced residue classes mod  $m$  we call a *complete set of reduced residue classes mod  $m$* .

#### **Lemma 7.0**

*(a) For each  $i$ -level tuple-set  $T_A$ , where  $A = \{a_2, a_3, \dots, a_i\}$ , the set of all  $i$ -level elements of all  $i$ -level tuples is a reduced residue class mod  $2 \cdot 3^{(i-1)}$ .*

(b) *The set of all such reduced residue classes, over all  $i$ -level tuple-sets  $T_A$ , is a complete set of reduced residue classes mod  $2 \cdot 3^{(i-1)}$ .*

**Proof:** see “Lemma 7.0: Statement and Proof” on page 33.

## **On the Inverse of the $3x + 1$ Function**

### **Lemma 8.8**

*Exactly one set  $J$  of odd, positive integers maps to 1, regardless whether counterexamples exist or not. In other words:*

*If counterexamples exist, then the set of odd, positive integers that map to 1 is  $J$ .*

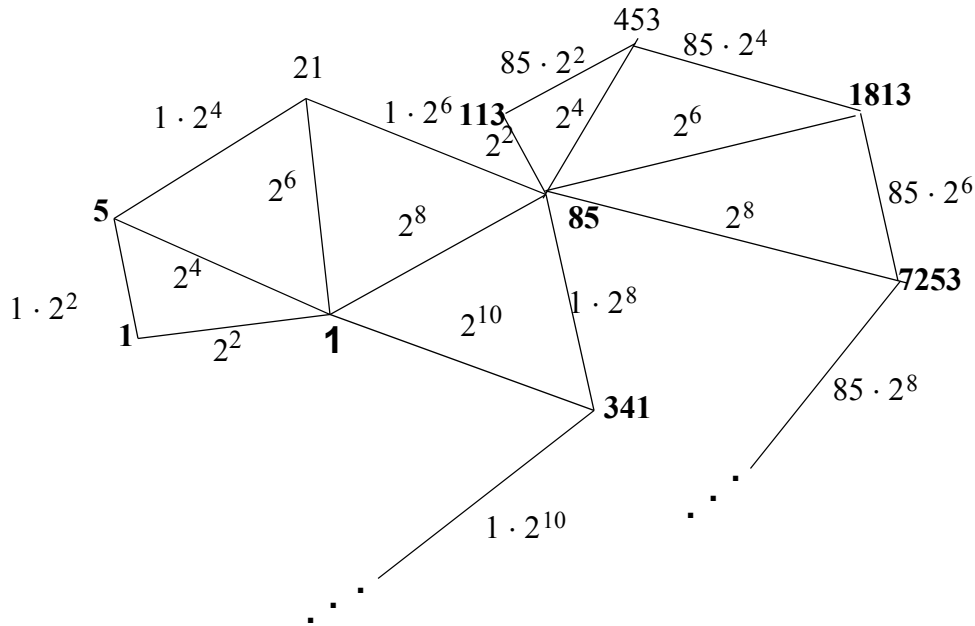
*If counterexamples do not exist, then the set of odd, positive integers that map to 1 is  $J$ .*

**Proof:**

The set  $J =$

{odd, positive integers  $y$  |  $y$  maps to 1 in *one* iteration of the  $3x + 1$  function}  $\cup$   
{odd, positive integers  $y$  |  $y$  maps to 1 in *two* iterations of the  $3x + 1$  function}  $\cup$   
{odd, positive integers  $y$  |  $y$  maps to 1 in *three* iterations of the  $3x + 1$  function}  $\cup$   
...

The set of odd, positive integers that map to 1 in one iteration of the  $3x + 1$  function is  $\{1, 5, 21, 85, 341, \dots\}$ . This set is called a “spiral” in “Section 2. Recursive ‘Spiral’s” in the first file of the paper “The Structure of the  $3x + 1$  Function: An Introduction” on the web site [www.occampress.com](http://www.occampress.com). We here reproduce, from that section, a graphical representation of part of the infinite set of recursive “spiral’s with base element 1.



**Fig. 4. Recursive “spirals” structure of odd, positive integers that map to 1.**

Bold-faced numbers are range elements (21 and 453 are multiples of 3, hence not range elements). Partial “spirals” surrounding the base elements 1 and 85 are shown. The line connecting 1813 to 85 is marked with a  $2^6$  because  $(3 \cdot 1813 + 1) / 2^6 = 85$ . The line connecting 453 to 1813 is marked  $85 \cdot 2^4$  because  $453 + 85 \cdot 2^4 = 1813$ . The exponents of 2 are not even in all “spiral”s, of course. For example, the “spiral” of numbers (not shown) mapping to 341 has odd exponents.

In the above-mentioned Section it is shown that:

If  $x$  is an element of a “spiral”, then  $4x + 1$  is the next element;

The “spiral” contains a countable infinity of multiples of 3. These cannot be range elements of the  $3x + 1$  function (by “Lemma 9.0: Statement and Proof” on page 34), that is, cannot be mapped to;

The “spiral” also contains a countable infinity of range elements of the function: each in turn is mapped to by another “spiral”, which yields, recursively, the set of odd, positive integers that map to 1 in two, three, four, ... iterations.

It is therefore clear that no odd, positive integer can be added to or removed from a “spiral”. Hence the set  $J$  is unique, regardless whether counterexamples exist or not.  $\square$

Readers who have difficulty believing that Lemma 8.8 is valid should consider the following:

Let  $S_1$  denote the singleton set containing the set of all odd, positive integers. Let  $S_2$  denote the set containing all proper subsets of the odd, positive integers. Then if counterexamples do not exist,  $J \in S_1$ ; if counterexamples exist, then  $J \in S_2$ .

## **Proof of the $3x + 1$ Conjecture**

### **Theorem. *The $3x + 1$ Conjecture is true.***

*Note:* if the reader believes that any part of the following proofs is obscure or employs invalid logic, we urge him or her to contact us and tell us the first sentence that is obscure or invalid. (See contact information on first page of this paper.)

### **First Proof**

1. Assume that counterexamples exist, and let  $T_A$  be any 2-level tuple-set. The anchor for any 2-level tuple-set is either 1 or 5 (these are the only two odd, positive integers relatively prime to  $2 \cdot 3^{(2-1)} = 6$ ). Both these numbers map to 1. Thus, as will become clear, our proof does not also apply to the  $3x - 1$  Conjecture, which is known to be false, because in the case of the  $3x - 1$  function, the first counterexample is 5. If our proof did apply to the  $3x - 1$  Conjecture, that would be a sign that our proof was wrong

By Lemma 5.0, we know that  $T_A$  contains a countable infinity of counterexample tuples and a countable infinity of non-counterexample tuples. Let  $t_j(2)$  be the 2-level element of the  $j$ th 2-level tuple in  $T_A$ , where  $j \geq 1$ . By part (a) of “Lemma 1.0” on page 10, for all  $j$ ,  $t_{(j+1)}(2) = t_j(2) + d(2, 2) = t_j(2) + 2 \cdot 3^{(2-1)} = t_j(2) + 6$ .

This is true regardless if counterexamples exist or not, since the first element  $x$  of the tuple  $t_{(j+1)}(2)$  since Lemma 1.0 does not depend on the existence or non-existence of counterexamples. Now, if counterexamples exist, there is a smallest  $(j + 1)$  such that  $t_{(j+1)}(2)$  is a counterexample<sup>1</sup>. We designate the tuple  $t_{(j+1)}(2) = \langle u, v \rangle$ . Since the anchor for  $T_A$  must be either 1 or 5, and since both these numbers are non-counterexamples, we know that  $(j + 1) \geq 2$ . So, there is a non-counterexample 2-level tuple immediately preceding  $\langle u, v \rangle$ . We designate this non-counterexample tuple  $\langle x, y \rangle$ . Thus,  $t_j(2) = y$  and  $t_{(j+1)}(2) = v$ . By what we said in the previous paragraph,  $v = y + d(2, 2) = y + 2 \cdot 3^{(2-1)} = y + 6$ . But since  $x$  maps to  $y$  in one iteration of the  $3x + 1$  function regardless if counterexamples exist, and similarly  $u$  maps to  $v$  in one iteration of the  $3x + 1$  function regardless if counterexamples exist (the definition of the function is not dependent on whether or not counterexamples exist),  $v = y + 6$  is the same regardless if counterexamples exist.  $t_{(j+1)}(2) = t_j(2) + d(2, 2) = t_j(2) + 2 \cdot 3^{(2-1)} = t_j(2) + 6$ , and this is the same value that  $t_{(j+1)}(2)$  would have if counterexamples did not exist. Furthermore, our argument applies to all counterexamples  $t_{(j+1+k)}(2)$ , where  $k \geq 0$ . That is,  $t_{(j+1+k)}(2) = t_j(2) + k \cdot d(2, 2) = t_j(2) + k \cdot (2 \cdot 3^{(2-1)}) = t_j(2) + k \cdot 6$ , and this is the same value that  $t_{(j+1+k)}(2)$  would have if counterexamples did not exist

What we have said implies that  $T_A$  is the same regardless if counterexamples exist or not. The same argument holds for all 2-level tuple-sets. Thus we claim that all 2-level tuples are the same regardless if counterexamples exist or not.

---

1. By computer tests of the  $3x + 1$  function performed so far, we know that  $(j + 1)$  is greater than  $10^{15}$ .

2. But if all 2-level tuple-sets are the same regardless if counterexamples exist or not, then so are all 3-level tuple-sets the same. The reason is that each 2-level element of each 2-level tuple in each 2-level tuple-set maps to exactly one odd, positive integer  $y$ , and we have shown that  $y$  is the same regardless if counterexamples exist or not. And for the same reason, if all 3-level tuple-sets are the same regardless if counterexamples exist or not, then so are all 4-level tuple-sets the same. And similarly for all 5-level, 6-level, ... tuple-sets, and thus so are all  $i$ -level tuple-sets the same regardless if counterexamples exist or not, where  $i \geq 2$ .

Or, in other words, (paralleling the language of “Lemma 8.8” on page 16), *there is exactly one set of all tuple-sets, regardless whether counterexamples exist or not.*

Some readers believe that the previous sentence is sufficient for a proof of the  $3x + 1$  Conjecture, since it implies that the set of counterexamples is the null set. However, we will attempt to provide a more explicit final step in step 3. But before we do, we must attempt to make sure that the reader clearly understands the surprising result that steps 1 and 2 have established. We will begin by giving the simplest explanation we are capable of.

Suppose there were a function that we will call the *Toy  $3x + 1$  Function* and suppose it had only one odd, positive integer  $x$  as argument. (The function behaved the same as our function  $C$  on  $x$ .) The Toy  $3x + 1$  Conjecture then asserted that  $x$  was the first element of a  $k$ -level tuple, where  $k$  was fixed, that contained at least one element 1. In this case,  $x$  was called a non-counterexample. If, on the other hand,  $x$  was the first element of a  $k$ -level tuple no element of which was 1, then  $x$  was called a counterexample (to the Toy  $3x + 1$  Conjecture). So there were two possibilities:  $x$  was the first element of a  $k$ -level tuple  $\langle x, y, \dots, z \rangle$  at least one element of which was 1; or  $x$  was the first element of a  $k$ -level tuple  $\langle x, y', \dots, z' \rangle$ , no element of which was 1. The conclusion of step 2, applied to the Toy  $3x + 1$  function, implies that  $\langle x, y, \dots, z \rangle = \langle x, y', \dots, z' \rangle$  for all  $k$ .

A second explanation is the following. We begin by emphasizing that when we say *there is exactly one set of all tuple-sets, regardless whether counterexamples exist or not*, we mean more than that there is exactly one *structure* to the set of all tuple-sets, regardless whether counterexamples exist or not. That there is exactly one structure is obvious from the definition of *tuple-set* and the fact that the set of all first elements of all tuples in each tuple-set is the set of odd, positive integers. What we mean is the following.

Let  $U$  denote the set of all tuple-sets if counterexamples exist, and let  $V$  denote the set of all tuple-sets if counterexamples do not exist. We know, by definition of *tuple-set*, that the set of first elements of all tuples in each tuple-set is the set of odd, positive integers. Let  $T_{A,U}$  be any  $i$ -level tuple-set in  $U$ , where  $i \geq 2$ , that is defined by the exponent sequence  $A$ , and let  $T_{A,V}$  be the  $i$ -level tuple-set in  $V$  that is defined by the same exponent sequence  $A$ . Then for any odd, positive integer  $x$ , if the  $i$ -level tuple in  $T_{A,U}$  having  $x$  as first element, is  $\langle x, y, \dots, z \rangle$ , then the  $i$ -level tuple in  $T_{A,V}$  having  $x$  as first element is likewise  $\langle x, y, \dots, z \rangle$ .

We are now ready to give the more explicit concluding step of our proof of the  $3x + 1$  Conjecture. We give two versions, step 3 and step 3'.

3. Steps 1 - 2 have established that:

There is no difference between the set of all tuple-sets if counterexamples do not exist, and the set of all tuple-sets if counterexamples exist. In other words, there is exactly one set of all tuple-

sets, regardless whether counterexamples exist or not. Thus for each odd, positive integer  $x$ , and for each  $k \geq 1$ , the  $k$ -level tuple having  $x$  as first element if counterexamples do not exist, is the same as the  $k$ -level tuple having  $x$  as first element if counterexamples exist.

We assumed, at the start of our proof, that counterexamples exist. Let  $x$  be a counterexample. Then there is an infinite sequence of finite tuples  $\langle x \rangle$ ,  $\langle x, y \rangle$ ,  $\langle x, y, y' \rangle$ , etc. Since  $x$  is a counterexample, no tuple in the sequence contains 1. But by the conclusion of step 2, at least one element of one tuple in the sequence must be 1. This contradiction gives us our proof of the  $3x + 1$  Conjecture.  $\square$

**Remark 1:** Some readers claim that in step 3 we are assuming that counterexamples exist and do not exist, and that from that contradiction, we are asserting the validity of the Conjecture. This claim is false. Step 3 is merely an expression of what is implied by steps 1 and 2. It makes no additional assumptions. We invite the reader who disapproves of step 3 to tell us what he or she believes steps 1 and 2 imply.

**Remark 2:** Observe that the above argument passes the  $3x - 1$  Test — it cannot also apply to the  $3x - 1$  function because counterexamples (5, 7) are already known in the second interval in the corresponding “spiral” that maps to 1 via the  $3x - 1$  function. .

## Second Proof

1. Let us call the infinite set of recursive “spiral”s with base element 1, the *1-tree*. (See “On the Inverse of the  $3x + 1$  Function” on page 16. That section includes a graphical representation of part of the tree.)

By “Lemma 8.8” on page 16, we know that the 1-tree contains all and only the odd, positive integers that map to 1.

If the 1-tree contains all odd, positive integers, then there are no counterexamples. If it contains only a proper subset of the odd, positive integers, then counterexamples exist.

2. Let  $tr_{nc}$  denote the 1-tree if counterexamples do not exist, and let  $tr_c$  denote the 1-tree if counterexamples exist. Orient both trees vertically, with the root, 1, at the top. Superimpose  $tr_{nc}$  on  $tr_c$ . Since structurally the trees are the same, each branch of  $tr_{nc}$  will be superimposed on the corresponding branch in  $tr_c$ .

The left-hand side of each tree has a boundary, whereas the right-hand side does not. Let an odd, positive integer that maps to 1 in one iteration of the  $3x + 1$  function, be called a *level 1 integer*. And similarly for level 2, level 3, etc., integers. Let a level  $j$  integer, where  $j \geq 1$ , that is on the boundary of a tree be denoted a *level  $j$  boundary integer*.

Descending down along the boundary, of, say,  $tr_{nc}$ , we see that 5 is the level 1 boundary integer; 3 is the level 2 boundary integer; 13 is the level 3 boundary integer; 17 is the level 4 boundary integer; etc.

These boundary integers are the same in both trees, as will be all boundary integers having values up to at least  $2 \cdot 3^{(35-1)}$ , which computer tests tell us are all counterexamples. The rule is: if an odd, positive integer  $y$  is known to map to 1, for example, as a result of computer test, then  $y$

is present in both trees, as is the “spiral” that  $y$  generates, and all “spiral”s descending, recursively, from range elements in that “spiral”, and all range elements  $z$  that  $y$  maps to, etc.

3. If counterexamples exist, then there must be a first level  $k$  in which  $tr_c$  differs from  $tr_{nc}$ . All levels less than  $k$  are the same in both trees. Let  $y$  be a level  $k$  integer in  $tr_c$  that differs from the corresponding level  $k$  integer  $y'$  in the corresponding branch of  $tr_{nc}$ . But this is impossible, since  $y$  maps to a range element  $z$  in only one way (the rules governing the  $3x + 1$  function are not sensitive to the existence of counterexamples). So we must conclude that in fact  $y$  is the same as  $y'$ , and hence that  $tr_{nc}$  is the same as  $tr_c$ , from which we conclude that counterexamples do not exist (or, informally, that they are “the same as” non-counterexamples), and hence the  $3x + 1$  Conjecture is proved.  $\square$

**Remark:** Observe that the above argument passes the  $3x - 1$  Test — it cannot also apply to the  $3x - 1$  function because counterexamples (5, 7) are already known in the second interval in the corresponding “spiral” that maps to 1 via the  $3x - 1$  function. .

### Remarks Concerning Both Proofs

We are aware that the above proofs are controversial. Some of the objections that have been made to the First Proof, and our responses to them, are given in the paper, “Is It Legitimate to Begin a Sentence With ‘If Counterexamples Exit, Then...?’” on the web site [occampress.com](http://occampress.com). The essence of several of these objections is that it is illegitimate to consider the two cases, counterexamples exist and counterexamples do not exist. Thus, for example, the phrase “whether or not counterexamples exist” is misleading. “Lemma 8.8” on page 16, it is argued, should simply state that the set of odd, positive integers that map to 1 is given by the recursive structure... which can be represented by the diagram in Fig. 4. Informally, these objections assert, “Things simply are as they are. The use of phrases like ‘there are two possibilities’ applies only to us, the persons contemplating the Problem. There is only one possibility for the  $3x + 1$  function itself, namely, what in fact is the case. A proof of the  $3x + 1$  Conjecture should not even mention counterexamples. It should simply show that all odd, positive integers map to 1.” An attempt at such a proof is given in the section, “Strategy of ‘Filling-in’ of Intervals in the Base Sequence Relative to 1” in the first file of the paper, “The Structure of the  $3x + 1$  Function: An Introduction”.

However, we must point out that these objections would also apply if the value of the  $3x + 1$  function for each  $x$  were a random number. In that case, it would certainly be correct to assert, “Things simple are as they are.” Each value of the function would have no implication for other values. (Of course, it is hard to imagine the recursive structure of the function in that case.) But we could legitimately say, “counterexamples have no consequences”. On the other hand, as some of our lemmas show (e.g., “Lemma 5.0: Statement and Proof” on page 32), counterexamples most definitely do have consequences.

Perhaps the argument in the First Proof (and the Third Proof) will be more convincing if the reader considers a version of the  $3x + 1$  function that initially acts *simultaneously* on the entire set of odd, positive integers. Then, if the exponent is 1, the result is the *set* of range elements congruent to  $5 \pmod{2 \cdot 3^{(2-1)}} = 5 \pmod{6}$ . If the exponent is 2, the result is the *set* of range elements congruent to  $1 \pmod{2 \cdot 3^{(2-1)}} = 1 \pmod{6}$ .

We can designate this initial behavior of the  $3x + 1$  function as  $C_{\{1\}}(\mathbf{x}) = \mathbf{y}$  in the first case, and as  $C_{\{2\}}(\mathbf{x}) = \mathbf{y}'$  in the second case.

We then apply  $C$ , the set-argument version of the function, to the set  $\mathbf{y}$  or the set  $\mathbf{y}'$ , for any exponent  $a_3$ , and again arrive at a set of range elements, in this case, a set whose elements are congruent  $\text{mod } 2 \cdot 3^{(3-1)} = \text{mod } 18$ . And so on.

It should be clear that this process always yields the same results (the same sets of range elements) regardless if counterexamples exist or not.

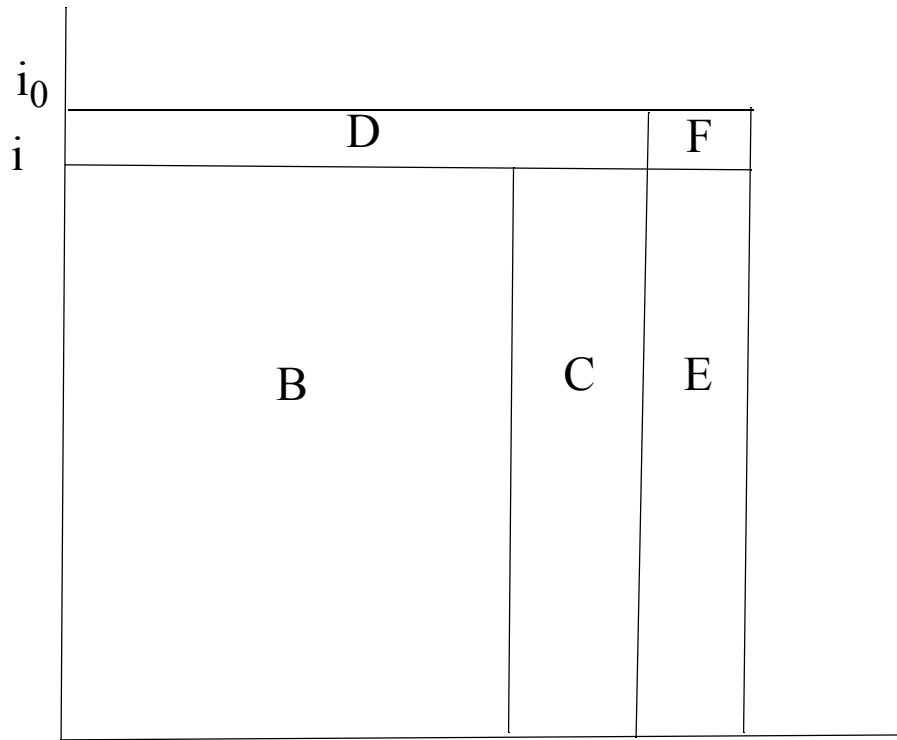
..

### Third Proof

This proof makes use of the same strategy as the previous proofs. In this case, as in “First Proof”, the proof shows that the contents of all tuple-sets are the same regardless if counterexamples exist or not.

1. Assume counterexamples exist. Then there must exist a minimum level  $i_0$  such that there are  $i_0$ -level counterexample anchor tuples. By computer tests, we know that  $i_0$  is greater than 35.

2. Consider the following diagram. (Labelled regions are explained immediately following.)



**Fig. 3.** Non-counterexample and counterexample tuples at levels  $i$  and  $i_0$ . (We have avoided the use of the label “A” in the diagram because in this paper “A” designates a single exponent sequence.)

Level  $i_0 = \text{level } i + 1$ .

B is the set of all  $i$ -level non-counterexample anchor tuples in all  $i$ -level tuple-sets. By step 1, this set is complete, that is, it is associated with the set of all  $i$ -level exponent sequences. Each of these  $i$ -level anchor tuples has an extension to level  $i_0$ .

C is the set of all  $i$ -level prefixes of  $i_0$ -level non-counterexample tuples that are not extensions of those in B.

D is the set of all anchors ( $i_0$  elements) of all  $i_0$  level non-counterexample anchor tuples. The set of all  $i_0$ -level non-counterexample anchor tuples is incomplete, because of the presence of  $i_0$ -level counterexample anchor tuples.

E is the set of all  $i$ -level prefixes of  $i_0$ -level counterexample anchor tuples. Each exponent sequence associated with one of these  $i$ -level prefixes has a duplicate in an exponent sequence associated with a non-counterexample tuple in B, because the set of all exponent sequences associated with tuples in B is complete.

F is the set of all anchors ( $i_0$  elements) of all  $i_0$ -level counterexample anchor tuples. The set of all  $i_0$ -level counterexample anchor tuples is incomplete, because of the presence of  $i_0$ -level non-counterexample anchor tuples.

3. Now, observing Fig. 3, and reading the definitions of the labels, and recalling that by definition of *tuple-set*, for each tuple-set  $T_A$ , the set of first elements of all tuples in  $T_A$  is always the set of odd, positive integers, we cannot escape the realization that Fig. 3 is exactly the same regardless if counterexamples exist or not. In particular, the set of  $i_0$ -level anchor tuples is the same regardless if counterexamples exist or not. We conclude, informally, “ $i_0$ -level counterexample anchor tuple” is just a name we can apply to *any*  $i_0$ -level anchor tuple. Furthermore, similar assertions can be made for the set of all  $(i_0 + j)$ -level anchor tuples, where  $j > 0$ . We conclude that counterexamples do not exist.  $\square$

## **Conclusion**

We have proved that the  $3x + 1$  Conjecture is true.

## **An Invitation to Skeptical Mathematicians**

We invite mathematicians who are skeptical of the validity of the above proofs, to read a brief description of what we at present regard as the most promising strategy for a proof of the  $3x + 1$  Conjecture apart from the strategy underlying the above proofs. See “Strategy of ‘Filling-in’ of Intervals in the Base Sequence Relative to 1” in our paper, “Are We Near a Solution to the  $3x + 1$  Problem?” on [occampress.com](http://occampress.com).

We believe that, from the material in this section, at least one mathematician in the world (and probably several) will be able to make a significant, publishable advance toward a proof of the Conjecture, if not actually prove it.

## References

Lagarias, J., (1985), “The  $3x + 1$  Problem and Its Generalizations”, *American Mathematical Monthly*, **93**, 3-23.

Wirsching, Günther J.. *The Dynamical System Generated by the  $3n + 1$  Function*, Springer-Verlag, Berlin, Germany, 1998.

## Appendix A — Statement and Proof of Each Lemma

### Lemma 0.0: Statement and Proof

#### Lemma 0.0

*Each  $i$ -level tuple-set, where  $i \geq 2$ , contains an infinity of tuples of each length  $j$ , where  $1 \leq j \leq i$ .*

**Proof:**

Let  $T_A$  be an  $i$ -level tuple-set, where  $i \geq 2$  and  $A = \{a_2, a_3, \dots, a_i\}$ .

There is an infinity of 1-level tuples  $\langle x \rangle$  because there is an infinity of odd, positive integers  $x$  that do not map to a range element  $y$  by the exponent  $a_2$ . (The contrary would violate part (a) of Lemma 1.0.)

For each  $j$ , where  $1 < j \leq i$ , there is an infinity of  $j$ -level tuples. This follows from Lemma 18.0 (see “Lemma 18.0: Statement and Proof” on page 42), which states that for each range element, and for each  $j$ -level exponent sequence  $A'$ , where  $A'$  is the  $j$ -level prefix of  $A$ , there exists an  $x$  that maps to  $y$  via  $A'$ , possibly followed by a buffer exponent. Thus for each range element, of which there are a countable infinity, there exists a  $j$ -level tuple associated with the exponent sequence  $A'$ . (The presence or absence of the buffer exponent does not change this fact.)  $\square$

### Lemma 1.0: Statement and Proof

*Definition:* let  $T_A$  be an  $i$ -level tuple-set, where  $i \geq 2$ . Let  $t(r), t(s)$  denote tuples consecutive at level  $i$ , with  $r < s$  in the natural ordering of tuples by first elements. Let  $t(r)(h), t(s)(h)$  denote the elements of  $t(r), t(s)$  at level  $h$ , where  $1 \leq h \leq i$ . Then we call  $|t(s)(h) - t(r)(h)|$  the *distance* between  $t(r)$  and  $t(s)$  at level  $h$ . We denote this distance by  $d(h, i)$  and call  $d$  the *distance functions* (one function for each  $h$ ).

#### Lemma 1.0

(a) Let  $A = \{a_2, a_3, \dots, a_i\}$ , where  $i \geq 2$ , be a sequence of exponents, and let  $t_{(r)}, t_{(s)}$  be tuples consecutive at level  $i$  in  $T_A$ . Then  $d(i, i)$  is given by:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) Let  $t_{(r)}, t_{(s)}$  be tuples consecutive at level  $i$  in  $T_A$ . Then  $d(1, i)$  is given by:

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3}) \dots (2^{a_i})$$

Thus, in “Fig. 1.” on page 8, the distance  $d(3, 3)$  between  $t_{8(3)} = 35$  and  $t_{4(3)} = 17$  is  $2 \cdot 3^{(3-1)} = 18$ . The distance  $d(1, 2)$  between  $t_{12(1)} = 23$  and  $t_{10(1)} = 19$  is  $2 \cdot 2^1 = 4$ .

**Proof:**

The proof is by induction.

**Proof of Basis Step for Parts (a) and (b) of Lemma 1.0:**

Let  $t_{(r)}$  and  $t_{(s)}$  be the first and second 2-level tuples, in the standard linear ordering of tuples based on their first elements, that are consecutive at level  $i = 2$  in the 2-level tuple-set  $T_A$ , where  $A = \{a_2\}$ . (See Fig. 2 (1).)

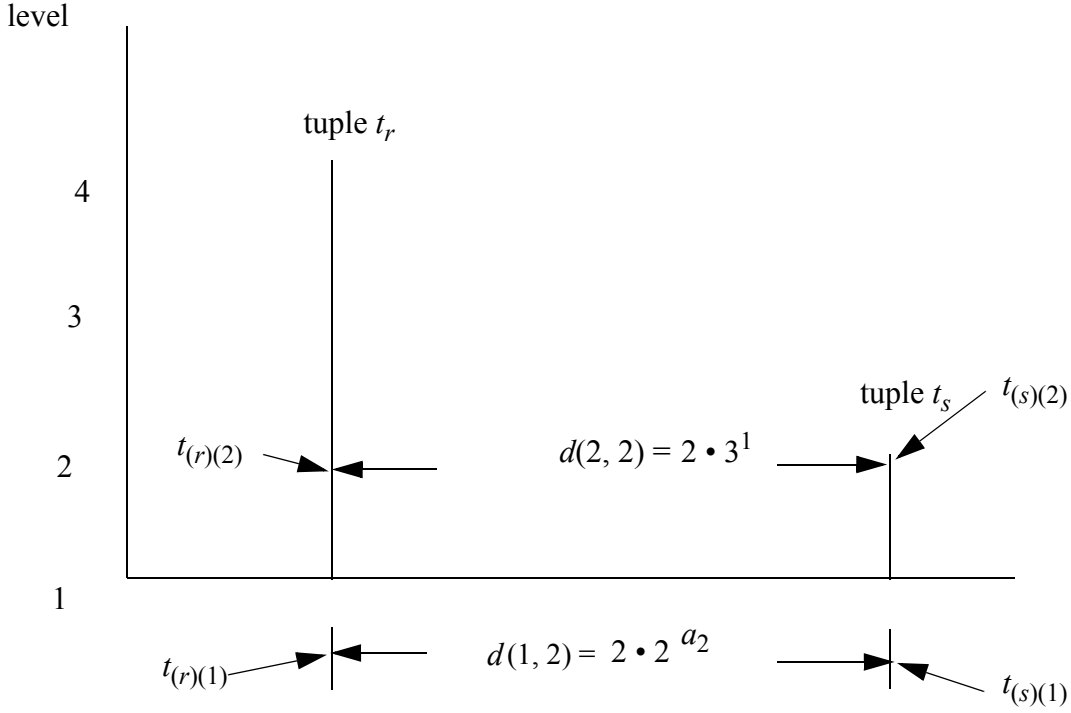


Fig. 2 (1). Illustration for proof of Basis Step of Lemma 1.0.

Then we have:

$$\frac{3t_{(r)(1)} + 1}{2^{a_2}} = t_{(r)(2)} \tag{1.1}$$

and since, by definition of  $d(1, 2)$ ,

$$t_{(s)(1)} = t_{(r)(1)} + d(1, 2)$$

we have:

$$\frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}} = t_{(s)(2)} \quad (1.2)$$

Therefore, since, by definition of  $d(i, i)$ ,

$$t_{(r)(2)} + d(2, 2) = t_{(s)(2)}$$

we can write, from (1.1) and (1.2):

$$\frac{3t_{(r)(1)} + 1}{2^{a_2}} + d(2, 2) = \frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}}$$

By elementary algebra, this yields:

$$2^{a_2}d(2, 2) = 3 \cdot d(1, 2)$$

Now  $d(2, 2)$  must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level  $i$ , it must be the smallest such even number, whence it follows that  $d(2, 2)$  must  $= 3 \cdot 2$ , and necessarily

$$d(1, 2) = 2 \cdot 2^{a_2}$$

A similar argument establishes that  $d(2, 2)$  and  $d(1, 2)$  have the above values for every other pair of tuples consecutive at level 2.

Thus we have our proof of the Basis Step for parts (a) and (b) of Lemma 1.0.

### **Proof of Induction Step for Parts (a) and (b) of Lemma 1.0**

Assume the Lemma is true for all levels  $j$ ,  $2 \leq j \leq i$  and that  $T_A$  is an  $i$ -level tuple-set, where  $A = \{a_2, a_3, \dots, a_i\}$ .

Let  $t_{(r)}$  and  $t_{(s)}$  be tuples consecutive at level  $i$ , and let  $t_{(r)}$  and  $t_{(f)}$  be tuples consecutive at level  $i + 1$ . (See Fig. 2 (2).)

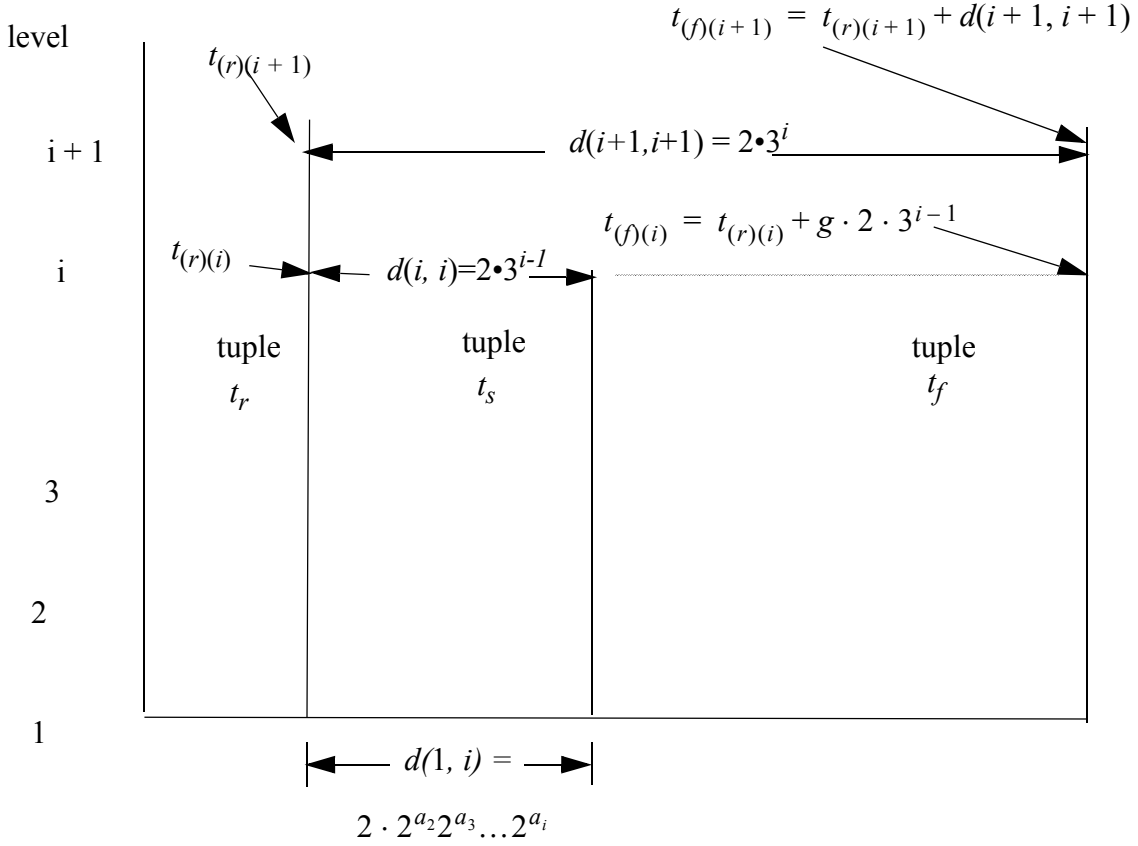


Fig. 2 (2). Illustration for proof of Induction Step of Lemma 1.0.

Then we have:

$$\frac{3t_{(r)(i)} + 1}{2^{a_{i+1}}} = t_{(r)(i+1)}$$

and since, by definition of  $d(i, i)$ ,

$$t_{(f)(i)} = t_{(r)(i)} + g \cdot d(i, i)$$

for some  $g \geq 1$ , we have:

$$\frac{3(t_{(r)(i)} + g \cdot d(i, i)) + 1}{2^{a_{i+1}}} = t_{(f)(i+1)}$$

Thus, since

$$t_{(r)(i+1)} + d(i+1, i+1) = t_{(f)(i+1)}$$

we can write:

$$\frac{3t_{(r)(i)} + 1}{2^{a_{i+1}}} + d(i+1, i+1) = \frac{3(t_{(r)(i)} + gd(i, i)) + 1}{2^{a_{i+1}}}$$

This yields, by elementary algebra:

$$2^{a_{i+1}}d(i+1, i+1) = 3 \cdot gd(i, i)$$

As in the proof of the Basis Step,  $d(i+1, i+1)$  must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level  $i+1$ , it must be the smallest such even number. Thus  $d(i+1, i+1) = 3 \cdot d(i, i)$ , and

$$g \cdot d(i, i) = 2^{a_{i+1}}d(i, i) \quad .$$

Hence

$$g = 2^{a_{i+1}}$$

Now  $g$  is the number of tuples consecutive at level  $i$  that must be “traversed” to get from  $t_{(r)}$  to  $t_{(f)}$ . By inductive hypothesis,  $d(1, i)$  for *each* pair of these tuples is:

$$d(1, i) = 2 \cdot 2^{a_2} \cdot 2^{a_3} \cdot \dots \cdot 2^{a_i}$$

hence, since

$$g = 2^{a_{i+1}}$$

we have

$$d(1, i+1) = d(1, i) \cdot 2^{a_{i+1}} \quad .$$

A similar argument establishes that  $d(i+1, i+1)$  and  $d(1, i+1)$  have the above values for every pair of tuples consecutive at level  $i+1$ .

Thus we have our proof of the Induction Step for parts (a) and (b) of Lemma 1.0. The proof of Lemma 1.0 is completed.  $\square$

## Lemma 2.0: Statement and Proof

*For each exponent  $a_2$ , a tuple-set  $T_A$ , where  $A = \{a_2\}$ , exists.*

### Proof:

By Lemma 12.0 (see “Lemma 12.0: Statement and Proof” on page 37) we know that each range element is mapped to by all exponents of one parity only. Then since 5 is mapped to by 3 via the exponent 1, we know that 5 is mapped to by all odd exponents. Since 1 is mapped to by 1 via the exponent 2, we know that 1 is mapped to by all even exponents. Both 1 and 5 are level-2 anchors, since each is less than  $2 \cdot 3^{2-1} = 6$ . Therefore each tuple  $\langle x, 5 \rangle$ , where  $x$  maps to 5 via the odd exponent  $a_2$  is the anchor tuple of a tuple-set, and each tuple  $\langle x', 1 \rangle$ , where  $x'$  maps to 1 via the even exponent  $a_2'$ , is the anchor tuple of a tuple-set. The result follows by Lemma 1.0 (a) and (b) (see “Lemma 1.0: Statement and Proof” on page 25), which assures us of an infinite number of tuples in each 2-level tuple-set.  $\square$

## Lemma 3.0: Statement and Proof

*Each  $i$ -level tuple-set, where  $i \geq 2$ , can be extended by each even or odd exponent  $a_{i+1}$ .*

### Proof:

By Lemma 2.0 (see “Lemma 2.0: Statement and Proof” on page 30), for each exponent  $a_2$ , a tuple-set  $T_{A'}$ , where  $A' = \{a_2\}$ , exists. So we show that for each exponent  $a_2 = a_{i+1}$ , the sequence of first elements of all tuples in  $T_{A'}$  has at least one element in common with the sequence of  $i$ -level elements in  $T_A$ .

The sequence of  $i$ -level elements in the  $i$ -level tuple-set  $T_A$  is given by

$$2 \cdot 3^{i-1}k + y \tag{3.1}$$

where  $k \geq 0$  and  $y$  is an  $i$ -level anchor, that is,  $y$  is an odd, positive integer that is less than or equal to, and relatively prime to,  $2 \cdot 3^{(i-1)}$ .

The sequence of 1-level elements of  $T_{A'}$  is given by

$$\frac{2^{a_2}y' - 1}{3} + j2 \cdot 2^{a_2} \tag{3.2}$$

where  $y' = 1$  or 5 is a 2-level anchor and  $j \geq 0$  (see “Lemma 1.0: Statement and Proof” on page 25). Specifically,  $y'$  is 1 if  $a_2 = a_{i+1}$  is even, and  $y'$  is 5 if  $a_2 = a_{i+1}$  is odd. The left-hand term of (3.2) gives the value of the first element  $x$  of the level-1 sequence of  $T_{A'}$ , because

$$\frac{3x + 1}{2^{a_2}} = y'$$

and an anchor, namely,  $y'$ , is the smallest  $i$ -level element (in this case 2-level element) of an  $i$ -level tuple-set. The right-hand term of (3.2) is  $j$  times the difference between successive first elements of  $T_{A'}$  (see “Lemma 1.0: Statement and Proof” on page 25).

Setting (3.1) equal to (3.2), we must prove that a solution  $j, k$  exists to the equation

$$2 \cdot 3^{i-1}k + y = \frac{2^{a_2}y' - 1}{3} + j2 \cdot 2^{a_2}$$

Multiplying through by 3, then dividing through by 2, which we can do since  $3y + 1$  is even, we get

$$3^i k + \frac{3y+1}{2} = 2^{a_2-1} y' + 3j2^{a_2}$$

Rearranging terms, we have

$$3^i k - 3j2^{a_2} = -\frac{3y+1}{2} + 2^{a_2-1} y' \quad (3.3)$$

or

$$3(3^{i-1}k - j2^{a_2}) = -\frac{3y+1}{2} + 2^{a_2-1} y'$$

The right-hand side of the equation must be a multiple of 3, and so we can divide both sides by 3 and write:

$$3^{i-1}k - 2^{a_2}j = U$$

This is an equation of the form

$$au + bv = c$$

and a basic fact of Diophantine Equations states that such an equation has a solution  $u, v$  if and only if  $(a, b)$  divides  $c$ . In our case,

$$(3^{i-1}, 2^{a_2}) = 1$$

and so (3.3) has a solution  $j, k$ .

Lemma 1.0 (see “Lemma 1.0: Statement and Proof” on page 25) then assures us of an infinity of  $i$ -level elements in  $T_A$  that have extensions via the exponent  $a_2 = a_{i+1}$ , thus creating the tuple-set  $T_{A''}$ , where  $A'' = \{a_2, a_3, \dots, a_i, a_{i+1}\}$ .  $\square$

### **Lemma 4.0: Statement and Proof**

*For each exponent sequence  $A = \{a_2, a_3, \dots, a_i\}$ , where  $i \geq 2$ , there exists a tuple-set  $T_A$  determined by  $A$ .*

**Proof:**

The proof is by induction.

***Basis Step:***

By Lemma 2.0 (see “Lemma 2.0: Statement and Proof” on page 30) we know that there is a 2-level tuple-set for each exponent  $a_2$ .

***Inductive Step:***

Assume the Lemma is true for all  $j$ -level exponent sequences  $2 \leq j \leq i$ . But then by Lemma 3.0 (see “Lemma 3.0: Statement and Proof” on page 30) it is true for all tuple-sets determined by  $(i + 1)$ -level exponent sequences.  $\square$

**Lemma 4.5: Statement and Proof**

*For each  $i \geq 2$ , the number of  $i$ -level tuple-sets is countably infinite.*

**Proof:**

Each  $i$ -level exponent sequence is a string of one or more of the symbols 1, 2, 3, ..., 8, 9, “;”. (Strings involving “,,,...”, however, that is, involving two or more commas in succession, do not occur. Nor do strings that begin with “;”.) There is a countable infinity of such strings.  $\square$

**Lemma 4.75: Statement and Proof**

*For each  $i \geq 2$ , the set of all  $i$ -level elements of all  $i$ -level tuples in all  $i$ -level tuple-sets is the set of all range elements of the  $3x + 1$  function.*

**Proof:**

We use an inductive proof.

*Basis step*

The Lemma is certainly true for all 2-level tuple-sets, since the set of all first elements of all 2-level tuples in all 2-level tuple-sets is the domain of the  $3x + 1$  function, and the set of all second elements in all 2-level tuples in all 2-level tuple-sets is therefore the range of the  $3x + 1$  function.

*Inductive step*

Assume the Lemma is true for all levels  $i$ , where  $2 \leq i \leq k$ . Assume now that at least one range element is absent from the set of all  $(k + 1)$ -level elements of all  $(k + 1)$ -level tuples in all  $(k + 1)$ -level tuple-sets.

But it is easily shown (see proof in “Lemma 18.0: Statement and Proof” on page 42) that each range element is mapped to, in one iteration of the  $3x + 1$  function, by an infinity of range elements. Therefore an infinity of range elements must be absent from the set of all  $k$ -level elements of all  $k$ -level tuples in all  $k$ -level tuple-sets, contrary to the first assumption in our inductive step.

$\square$

**Lemma 5.0: Statement and Proof**

*Assume a counterexample exists. Then for all  $i \geq 2$ , each  $i$ -level tuple-set contains an infinity of  $i$ -level counterexample tuples and an infinity of  $i$ -level non-counterexample tuples.*

**Proof:**

1. Assume counterexamples exist. Then:

There is a countable infinity of non-counterexample range elements.

*Proof:* Each non-counterexample maps to a range element, by definition of *range element*.

Each range element is mapped to by an infinity of elements

(“Lemma 12.0: Statement and Proof” on page 37). A countable infinity of these are range elements (proof of “Lemma 18.0: Statement and Proof” on page 42).

There is a countable infinity of counterexample range elements.

*Proof:* same as for non-counterexample case.

2. For each finite exponent sequence  $A$ , and for each range element  $y$ , non-counterexample or counterexample, there is an  $x$  that maps to  $y$  via  $A$  possibly followed by a buffer exponent (“Lemma 18.0: Statement and Proof” on page 42). The presence of the buffer exponent does not change the fact that  $x$  is the first element of a tuple associated with the exponent  $A$ .  $\square$

**Lemma 6.0: Statement and Proof**

*Let  $t$  be the anchor tuple (by definition an  $i$ -level tuple) in an  $i$ -level tuple-set, where  $i \geq 2$ . Then the last element  $y$  of  $t$ , that is, the  $i$ -level element of  $t$  (this element being the anchor), is a number less than  $2 \cdot 3^{(i-1)}$ .*

**Proof:**

By definition of  *$i$ -level anchor tuple*,  $t$  is the first  $i$ -level tuple in an  $i$ -level tuple-set. Hence there are no  $i$ -level tuples to the left of  $t$  under our convention for ordering tuples from left to right in a tuple-set. By the distance function defined in part (a) of “Lemma 1.0” on page 10, the distance between the last elements of consecutive  $i$ -level tuples in an  $i$ -level tuple-set is  $2 \cdot 3^{(i-1)}$ . An argument similar to that used in the proof of part (a) of Lemma 1.0 (see “Lemma 1.0: Statement and Proof” on page 25), but in the “leftward” direction, shows that, if the value of the  $i$ -level element of an  $i$ -level tuple  $t$  in an  $i$ -level tuple-set is greater than  $2 \cdot 3^{(i-1)}$ , then there exists an  $i$ -level element of an  $i$ -level tuple  $t'$  to the left of  $t$ . But if there is no  $i$ -level tuple to the left of  $t$ , it follows that the last element  $y$  of  $t$  must be less than  $2 \cdot 3^{(i-1)}$ .  $\square$

**Lemma 7.0: Statement and Proof**

*(a) For each  $i$ -level tuple-set  $T_A$ , where  $A = \{a_2, a_3, \dots, a_i\}$ , the set of all  $i$ -level elements of all  $i$ -level tuples is a reduced residue class mod  $2 \cdot 3^{(i-1)}$ .*

*(b) The set of all such reduced residue classes, over all  $i$ -level tuple-sets  $T_A$ , is a complete set of reduced residue classes mod  $2 \cdot 3^{(i-1)}$ .*

**Proof:**

*Part (a):* Let  $T_A$  be an  $i$ -level tuple-set. Since the first  $i$ -level tuple  $t$  in  $T_A$  is an anchor tuple, the last element  $y$  of  $t$  is an anchor. By Lemma 6.0 (see “Lemma 6.0: Statement and Proof” on page 33),  $y$  is an odd, positive integer not divisible by 3 that is less than  $2 \cdot 3^{i-1}$  — in other words,  $y$  is the minimum element of a reduced residue class mod  $2 \cdot 3^{i-1}$ .

*Part (b):* The set of all  $i$ -level elements of all  $i$ -level tuples in all  $i$ -level tuple-sets is the set of range elements of the  $3x + 1$  function (“Lemma 4.75” on page 12). This set includes the set  $U$  of range elements that are less than  $2 \cdot 3^{i-1}$ . Since a range element is an odd, positive integer that is not a multiple of 3, the set  $U$  consists of all minimum reduced residues mod  $2 \cdot 3^{i-1}$  — that is, the complete set of minimum reduced residues. The result follows from the fact that the distance between  $i$ -level elements of successive  $i$ -level tuples in an  $i$ -level tuple-set is  $2 \cdot 3^{i-1}$  (“Lemma 1.0: Statement and Proof” on page 25).  $\square$

### **Lemma 8.0: Statement and Proof**

*For each odd, positive integer  $x$  there exists a minimum  $i = i_0$  such that for each  $i \geq i_0$ ,  $x$  is the first element of the first  $i$ -level tuple in some  $i$ -level tuple-set, that is,  $x$  is the first element of an  $i$ -level anchor tuple in some  $i$ -level tuple-set. In terms of infinite tuples, this lemma states: if  $x$  is an odd, positive integer, then in the infinite tuple  $\bar{t} = \langle x, y, y', \dots \rangle$ , there exists a minimum level  $i_0$  such that:*

- $\bar{t}(i_0)$  is the  $i_0$ -level anchor tuple in an  $i_0$ -level tuple-set;
- $\bar{t}(i_0 + 1)$  is the  $(i_0 + 1)$ -level anchor tuple in an  $(i_0 + 1)$ -level tuple-set;
- $\bar{t}(i_0 + 2)$  is the  $(i_0 + 2)$ -level anchor tuple in an  $(i_0 + 2)$ -level tuple-set;
- etc.

( Of course, the  $(i_0 + k + 1)$ -level tuple-set, where  $k \geq 0$ , must be an extension of the  $(i_0 + k)$ -level tuple-set by the same exponent by which the anchor tuple is extended. )

**Proof:**

The following proof is an edited version of a proof by Sanjai Gupta. Any errors it contains are entirely our own.

Let  $x$  be an odd, positive integer. Then  $x$  is the first element of an infinite tuple  $\bar{t} = \langle x, y, \dots \rangle$ . With each increment of  $i$ ,  $i \geq 2$ , the element of  $\bar{t}$  at level  $i$  increases by at most a factor of 2, since for each exponent except 1,  $C(y) < y$ , and for exponent 1,  $C(y) \leq 2y$ . However, with each increment of  $i$ ,  $2 \cdot 3^{(i-1)}$  increases by a factor of 3. Therefore, a level  $i = i_0$  must eventually be reached such that the element  $y'$  of  $\bar{t}$  at level  $i$  is less than  $2 \cdot 3^{(i-1)}$ . But then by definition  $y'$  is an anchor, and hence the prefix  $\langle x, y, \dots, y' \rangle$  is an anchor tuple. By the informal rule, “once an anchor tuple, always an anchor tuple”, the final part of our result follows.  $\square$

### **Lemma 9.0: Statement and Proof**

*No multiple of 3 is a range element.*

**Proof :**

If

$$\frac{3x + 1}{2^a} = 3m$$

then  $1 \equiv 0 \pmod{3}$ , which is false.  $\square$

**Lemma 10.0: Statement and Proof**

*Each odd, positive integer (except a multiple of 3) is mapped to by a multiple of 3 in one iteration of the  $3x + 1$  function.*

**Proof:**

Since the domain of the  $3x + 1$  function is the odd, positive integers, the only relevant generators are  $3(2k + 1)$ ,  $k \geq 0$ . We show that, for each odd, positive integer  $y$  not a multiple of 3, there exists a  $k$  and an  $a$  such that

$$y = \frac{(3(3(2k + 1)) + 1)}{2^a} , \tag{11.1}$$

where  $a$  is necessarily the largest such  $a$ , since  $y$  is assumed odd.

Rewriting (11.1), we have:

$$y2^{a-1} - 5 = 9k . \tag{11.2}$$

Without loss of generality, we can let  $y \equiv r \pmod{18}$ , where  $r$  is one of 1, 5, 7, 11, 13, or 17 (since  $y$  is odd and not a multiple of 3, these values of  $r$  cover all possibilities mod 18). Or, in other words, for some  $q$ ,  $r$ ,  $y = 18q + r$ . Then, from (11.2) we can write:

$$18(2^{a-1})q + (2^{a-1})r - 5 = 9k . \tag{11.3}$$

Since the first term on the left-hand side is a multiple of 9,  $(2^{a-1})r - 5$  must also be if the equation is to hold. We can thus construct the following table. (Certain larger  $a$  also serve equally well, but those given suffice for purposes of this proof.)

**Table 2: Values of  $r$ ,  $a$ , for Proof of Lemma**

$r$	$a$	$(2^{a-1})r - 5$
1	6	27
5	1	0
7	2	9
11	5	171

**Table 2: Values of  $r, a$ , for Proof of Lemma**

$r$	$a$	$(2^a - 1)r - 5$
13	4	99
17	3	63

Given  $q$  and  $r$  (hence  $y$ ), we can use  $r$  to look up  $a$  in the table, and then solve (11.3) for integral  $k$ , thus producing the multiple of 3 that maps to  $y$  in one iteration of the  $3x + 1$  function.  $\square$

**Lemma 11.0: Statement and Proof**

*For each range element  $y$  there exists an infinity of  $x$  that map directly to  $y$ . Specifically, If*

$$\frac{3x + 1}{2^a} = y$$

*Then, for each  $n \geq 1$ ,*

$$\frac{3(x + (2^{a+2(0)} + 2^{a+2(1)} + \dots + 2^{a+2(n-1)})y) + 1}{2^{a+2(n)}} = y$$

**Proof:**

The proof is a matter of straightforward algebra.

From the antecedent, we have:

$$x = \frac{2^a y - 1}{3}$$

Substituting into the left-hand side of the consequent, multiplying the term in parentheses by 3, cancelling two 1's, and factoring out  $(2^a)(y)$  yields:

$$\frac{2^a y (1 + 3(2^0 + 2^2 + 2^4 + \dots + 2^{2(n-1)}))}{2^{a+2(n)}}$$

The  $2^a$ s cancel, the term  $(1 + 3(\dots))$  is easily shown to equal  $2^{2(n)}$ , the  $2^{2(n)}$  in numerator and denominator cancel, and we are left with  $y$ , which gives us our result.  $\square$

**Remark**

Lemma 11.0 and Lemma 10.0 (see “Lemma 10.0: Statement and Proof” on page 35) imply that if a counterexample exists, then there is an infinity of counterexamples.

**Lemma 12.0: Statement and Proof**

*(a) Each range element  $y$  is mapped to, in one iteration of the  $3x + 1$  function, by every exponent of one parity only. Furthermore,*

*(b) For each of the two parities, there exists a range element that is mapped to by every exponent of that parity.*

**Proof of part (a):**

Steps 1 and 2 are slightly edited versions of proofs by Jonathan Kilgallin and Alex Godofsky. Any errors are entirely ours. Step 3 is a slightly edited version of a proof by Michael Klipper. Any errors are entirely ours.

1. We first show that if  $y$  is mapped to by the exponent  $a$ , then  $y$  is mapped to by every exponent greater than  $a$  that is of the same parity as  $a$ .

Let  $y$  be a range element, and let  $x$  map to  $y$  via the exponent  $a$ . Then

$$y = \frac{3x + 1}{2^a}$$

We wish to show that there exists an  $x'$  such that  $x'$  maps to  $y$  via the exponent  $2^{a+2}$ . That is, we wish to show that there exists an  $x'$  such that

$$y = \frac{3x' + 1}{2^{a+2}}$$

Rewriting, this gives

$$x' = \frac{2^{a+2}y - 1}{3}$$

Substituting for  $y$  yields

$$x' = \frac{2^{a+2} \left( \frac{3x+1}{2^a} \right) - 1}{3}$$

Simplifying, this gives  $x' = 4x + 1$ . Since  $x$  is an odd, positive integer, clearly  $x'$  is as well.

Thus, by induction, if  $y$  is mapped to via the exponent  $a$ , it is mapped to by every exponent greater than  $a$  of the same parity.  $\square$

2. Next we show that if  $y$  is mapped to by the exponent  $a$  which is greater than 2, then it is mapped to by every exponent less than  $a$  that is of the same parity as  $a$ .

Let  $y$  be a range element, and let  $x$  map to  $y$  via the exponent  $a$  where  $a > 2$ . Then

$$y = \frac{3x+1}{2^a}$$

We wish to show that there exists an  $x'$  such that  $x'$  maps to  $y$  via the exponent  $2^{a-2}$ . That is, we wish to show that there exists an  $x'$  such that

$$y = \frac{3x'+1}{2^{a-2}}$$

Rewriting, this gives

$$x' = \frac{2^{a-2}y - 1}{3}$$

Substituting for  $y$  yields

$$x' = \frac{2^{a-2} \left( \frac{3x+1}{2^a} \right) - 1}{3}$$

Simplifying yields

$$x' = \frac{x-1}{4}$$

3. We must now show that  $x' = (x-1)/4$  is an odd, positive integer. This means we must show that  $(x-1) = 4(2k+1)$  for some  $k \geq 0$ , or that  $(x-1) = 8k+4$ , hence that  $x = 8k+5$ . Thus, we must prove  $x \equiv 5 \pmod{8}$ .

We know that  $x$  maps to  $y$  via  $a$ , where  $a \geq 3$ . Thus,  $y = (3x+1)/2^a$ , so  $2^a y = 3x+1$ . Because  $a \geq 3$ ,  $2^a y$  is a multiple of 8. Thus,  $(3x+1) \equiv 0 \pmod{8}$ , and  $3x \equiv 7 \pmod{8}$ . This readily implies  $x \equiv 5 \pmod{8}$ .

4. Thus, by induction, if  $y$  is mapped to via the exponent  $a$ , where  $a > 2$ , then it is mapped to by every exponent less than  $a$  of the same parity.  $\square$

**Proof of part (b):**

We now show that for each of the two parities there exists a range element that is mapped to by every exponent of that parity.

1. Fix a range element  $y$ , and suppose that  $x$  maps to  $y$  via the exponent  $a$ . Now  $a$  is either even or odd, hence  $a = 2n + h$ , where  $h$  is either 0 or 1. Since  $y = (3x+1)/2^a$ , it follows that  $(2^a)y = 3x+1$ . Reduce the equation mod 3, and we get  $(2^h)y \equiv 1 \pmod{3}$ , by the following reasoning:  $(2^a)y \equiv 1 \pmod{3}$  implies  $(2^{2n+h})y \equiv 1 \pmod{3}$  implies  $2^{2n} 2^h y \equiv 1 \pmod{3}$  implies  $2^h y \equiv 1 \pmod{3}$  because  $2^{2n} = 4^n \equiv 1 \pmod{3}$ .

2. Since  $y$  is fixed, either  $y \equiv 1$  or  $y \equiv 2 \pmod{3}$ . (We know that  $y$ , a range element, is not a multiple of 3 by “Lemma 9.0: Statement and Proof” on page 34). If  $y \equiv 1 \pmod{3}$ , then we have  $2^h(1) \equiv 1 \pmod{3}$ , which implies that  $h$  must be 0. If  $y \equiv 2 \pmod{3}$ , then we have  $(2^h)(2) \equiv 1 \pmod{3}$ , implying that  $h$  must be 1.  $\square$

**Lemma 13.0: Statement and Proof**

*There exists an explicit construction of the tuple-set whose exponent sequence is associated with a given tuple.*

**Proof:**

Let  $x$  be the first element of a tuple and let  $\{a_2, a_3, \dots, a_{n+1}\}$  be the sequence of exponents associated with the first  $n$  extensions of the tuple  $\langle x \rangle$ . The last element of the tuple is given by:

$$\frac{3^n x + r}{2^a}$$

where

$$a = \sum_{i=2}^n a_i$$

We find that  $r$  is most easily calculated by iterating from  $x = 0$ , then multiplying by the appropriate power of 2, as shown in the table at the end of this proof. We want the integral  $x$  that produce odd outputs:

$$\frac{3^n x + r}{2^a} = 2k + 1$$

which gives

$$3^n x - 2^{a+1} k = 2^a - r$$

This is a standard linear Diophantine equation. Since  $(3^n, 2^{a+1}) = 1$ , and 1 divides the right-hand side of the equation, the equation has a solution. One solution is:

$$x_0 = -(2^a - r) \left( \frac{2^2 \cdot 3^{n-1} \cdot (a+1) - 1}{3^n} \right)$$

$$k_0 = -(2^a - r) (2^{(2 \cdot 3^{n-1} - 1)(a+1)})$$

Note that the ratio in the expression for  $x_0$  is an integer because

$$2^2 \cdot 3^{n-1} \equiv 1 \pmod{3^n}$$

The general solution is:

$$x = x_0 + t \cdot (-2^{a+1})$$

$$k = k_0 - t \cdot 3^n$$

where  $t$  ranges over the integers. Thus, the  $x$ 's are the inputs that iterate with the specified exponents and

$$2k + 1 = 2k_0 - t \cdot 2 \cdot 3^n + 1$$

are the outputs.

**Table 3: Successive values of  $n$ , the  $x$  term, and  $r$  in proof of Lemma 14.0**

$n$	$x$ term	$r$	level of tuple element yielded, i.e., $i$ in $a_i$
1	$3^1x$	1	2
2	$3^2x$	$3^1 + 2^{a_2}$	3
3	$3^3x$	$3^2 + 3^1 2^{a_2} + 2^{a_2} 2^{a_3}$	4
4	$3^4x$	$3^3 + 3^2 2^{a_2} + 3^1 2^{a_2} 2^{a_3} + 2^{a_2} 2^{a_3} 2^{a_4}$	5
...	...	...	...

□

### **Lemma 14.0: Statement and Proof**

*For each range element  $y$ , and for each finite sum  $a$  of exponents, a domain element  $x$  exists that maps to  $y$  via a sum  $a'$  that contains  $a$ .*

**Proof:**

We are looking for an  $x$  such that the sequence of iterations represented by

$$\frac{3^n x + r}{2^a}$$

where  $n$ ,  $a$ , and  $r$  are defined as in Lemma 13.0 (see “Lemma 13.0: Statement and Proof” on page 39), lead to a computation that ends with  $y$ . The numbers  $n$ ,  $a$ , and  $r$  are determined by the exponent sequence we want. There also has to be an optional buffer iteration between the above and  $y$ , for example, to allow for parity constraints on the exponent leading to  $y$  (see “Lemma 11.0:

Statement and Proof” on page 36). Thus, for example, if  $y$  is mapped to by even exponents, and our exponent sequence  $a$  ends with an odd exponent, then there must be a buffer exponent following the sequence  $a$ . So, we want

$$\frac{3\left(\frac{3^n x + r}{2^a}\right) + 1}{2^j} = y$$

or

$$\frac{3^{n+1}x + 3r + 2^a}{2^{a+j}} = y$$

which gives

$$3^{n+1}x = (2^a y)2^j - 3r - 2^a \tag{15.1}$$

or

$$(2^a y)2^j \equiv 3r + 2^a \pmod{3^{n+1}}$$

We are looking for  $x$  and  $j$ . Since  $y$  is a range element, it cannot be a multiple of 3 (see “Lemma 9.0: Statement and Proof” on page 34). Therefore  $2^a y$  is relatively prime to  $3^{n+1}$ , as is  $3r + 2^a$ . Since  $2^j$ , where  $j \geq 0$ , is a member of a reduced residue class mod  $3^{n+1}$ , the congruence is solvable. Hence we can find  $j$ , and then, from (15.1),  $x$ .  $\square$

### Remarks

The result would hold for an arbitrary finite number of buffer exponents following the exponent sum  $a$ , since they do not change the fact that a tuple generating each exponent sequence whose sum is  $a$  is guaranteed by the proof.

A recursive proof of the Lemma is possible because the set of odd, positive integers mapping to a given range element  $y$  in one iteration of the  $3x + 1$  function  $C$  includes an infinite subset each element of which is mapped to by an infinity of even exponents, and an infinite subset each element of which is mapped to by an infinity of odd exponents.

### Lemma 18.0: Statement and Proof

*Let  $y$  be a range element of the  $3x + 1$  function. Then for each finite exponent sequence  $A$ , there exists an  $x$  that maps to  $y$  via  $A$  possibly followed by a “buffer” exponent. (For example, if  $y$  is mapped to by even exponents, and our exponent sequence  $A$  ends with an odd exponent, then there must be a “buffer” exponent following  $A$ , and similarly if  $y$  is mapped to by odd exponents and  $A$  ends with an even exponent. However, there are other cases in which a “buffer” exponent is required.)*

**Proof:**

1. Each range element  $y$  is mapped to by all exponents of one parity (“Lemma 12.0: Statement and Proof” on page 37).

2. Each range element  $y$  is mapped to by a multiple of 3 (“Lemma 10.0: Statement and Proof” on page 35).

Each range element is mapped to by an infinity of range elements (“Lemma 10.0: Statement and Proof” on page 35).

3. Let  $y$  be a range element and let  $S = \{s_1, s_2, s_3, \dots\}$  be the set of all odd, positive integers that map to  $y$  in one iteration of the  $3x + 1$  function. In other words,  $S$  is the set of all elements in a “spiral”. Furthermore, let the  $s_i$  be in increasing order of magnitude. It is easily shown that  $s_{i+1} = 4s_i + 1$ .

(In Fig. 18,  $y = 13$ ,  $S = \{17, 69, 277, 1109, \dots\}$ )

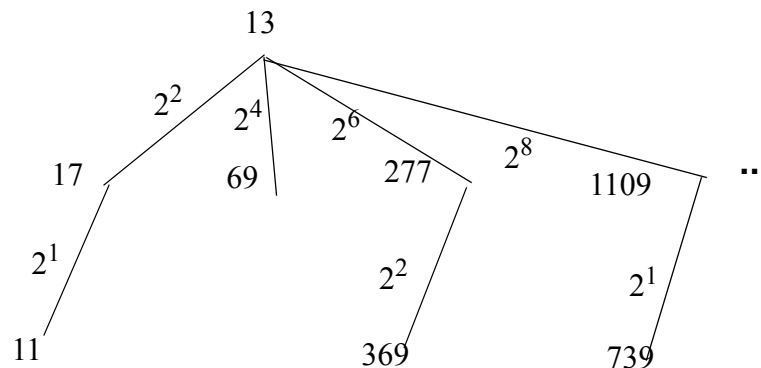


Fig. 18

4. If  $s_i$  is a multiple of 3, then  $4s_i + 1$  is mapped to, in one iteration of the  $3x + 1$  function, by all exponents of even parity.

To prove this, we need only show that  $x$  is an integer in the equation

$$4(3u) + 1 = \frac{3x + 1}{2^2}$$

Multiplying through by  $2^2$  and collecting terms we get

$$(48u) + 4 = 3x + 1$$

and clearly  $x$  is an integer.

5. If  $s_j$  is mapped to by all even exponents, then  $4s_j + 1$  is mapped to, in one iteration of the  $3x + 1$  function, by all exponents of odd parity.

(The proof is by an algebraic argument similar to that in step 4.)

6. If  $s_k$  is mapped to by all odd exponents, then  $4s_k + 1$  is a multiple of 3.

(The proof is by an algebraic argument similar to that in step 4.)

7. The Lemma follows by an inductive argument that we now describe.

Let  $y$  be a range element. It is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. As a consequence of steps 1 through 6, we can represent an infinite sub-sequence of the sequence by

...3, 2, 1, 3, 2, 1, ...

where

“3” means “this odd, positive integer is a multiple of 3 and therefore is not mapped to by any odd, positive integer”;

“2” means “this odd, positive integer is mapped to by all even exponents”;

“1” means “this odd, positive integer is mapped to by all odd exponents”.

Each type “2” and type “1” odd, positive integer is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. We can represent an infinite sub-sequence of the sequence by

...3, 2, 1, 3, 2, 1, ...

where each integer has the same meaning as above.

Temporarily ignoring the case in which a buffer exponent is needed, it should now be clear that, for each range element  $y$ , and for each finite sequence of exponents  $B$ , we can find a finite path down through the infinitary tree we have just established, starting at the root  $y$ . The path will end in an odd, positive integer  $x$ . Let  $A$  denote the path  $B$  taken in reverse order. Then we have our result for the non-buffer-exponent case. The buffer-exponent case follows from the fact that

*A Solution to the  $3x + 1$  Problem*

the buffer exponent is one among an infinity of exponents of one parity. Thus  $y$  is mapped to by an infinite sequence of odd, positive integers. We then simply apply the above argument..  $\square$

## Appendix B — $3x + C$ Functions

### Generalizations of the $3x + 1$ Function

During the course of our attempts to find a proof of the  $3x + 1$  Conjecture, we were occasionally encouraged to check if our proposed proof also constituted a proof of the  $3x - 1$  Conjecture. If the answer was Yes, then our proof must be wrong, since the  $3x - 1$  Conjecture is false (5 and 7 are counterexamples). We began referring to this check as the  $3x - 1$  Test.

But the existence of the  $3x - 1$  function encouraged us to investigate what we called  $3x + C$  functions, where  $C$  is an odd, positive integer (We have been told that the  $3x + 3^k$  function, where  $k$  is a positive integer, was first defined and investigated by Barry Brent in 1993. We have so far been unable to find anything about  $3x + C$  functions in the literature.) Some of these  $3x + C$  functions we now call  $3x + 1$ -like functions (see definition below).

Another generalization of the  $3x + 1$  function is  $3x + C$  functions whose domain includes the negative integers. The negative of the  $3x - 1$  function over the negative integers is the same as the  $3x + 1$  function over the negative integers, a fact that provides some insight into the nature of the  $3x + 1$  function. See, for example, “Why Are There Counterexamples to the  $3x - 1$  Conjecture?” on page 53.

A further generalization would be  $Ax + B$  functions, where  $A$  and  $B$  are integers.

Finally, for all the above functions, we can generalize the denominators.

### Definition of “ $3x + C$ Function” and the “ $3x + C$ Problem”

In the literature, a  $3x + C$  function  $F_C$  is defined as

$$F_C(x) = \frac{3x + C}{2^{\text{ord}_2(3x + C)}}$$

where  $C$  and  $x$  are odd, positive integers.

For each  $C$ , the  $3x + C$  Problem asks if for all  $x$ , repeated iterations of  $F_C$ , beginning with  $x$ , eventually terminate in 1. In some cases, for example, the  $3x - 1$  and  $3x + 5$  Problems, the answer is easily shown to be No. In other words, for these  $C$ , counterexamples to the  $3x + C$  Conjecture exist.

In the case of the  $3x - 1$  function, the smallest counterexample begins with 5, yielding the infinite cyclic tuple  $\langle 5, 7, 5, \dots \rangle$ . (In the  $3x + 1$  function, 5 is the first element of the non-counterexample 2-level anchor tuple  $\langle 5, 1 \rangle$ .) Thus 5 and 7 are counterexamples to the  $3x - 1$  Conjecture.

In the case of the  $3x + 5$  function, 5 is a counterexample because it yields the infinite cyclic tuple  $\langle 5, 5, \dots \rangle$ . Another counterexample is 19, yielding the infinite cyclic tuple  $\langle 19, 31, 49, 19, \dots \rangle$ . (In the  $3x + 1$  function, 19 is the first element of the non-counterexample 4-level anchor tuple  $\langle 19, 29, 11, 17 \rangle$ .)

Since any odd, positive integer that maps to a counterexample, is itself a counterexample, it turns out that, as the reader can verify, all odd, positive integers less than  $2 \cdot 3^{3-1} = 18$  except 1 and 9 are counterexamples!

## A Relationship Between $3x + C$ Tuples and $3x + 1$ Tuples

We are indebted to a computer scientist for the statement and proof of the following Lemma. We have edited the proof slightly, so any errors are entirely our fault.

### Lemma 14.8

*The tuple  $\langle Cx, Cy, Cy', \dots, Cz \rangle$  is a  $3x + C$  tuple iff the tuple  $\langle x, y, y', \dots, z \rangle$  is a  $3x + 1$  tuple.*

#### Proof (only if part):

Assume  $Cu$  is an element of a  $3x + C$  tuple. Then

$$\frac{3(Cu) + C}{2^{\text{ord}_2(3(Cu) + C)}} = \frac{C(3u + 1)}{2^{\text{ord}_2(C(3u + 1))}} = \frac{C(3u + 1)}{2^{\text{ord}_2(3u + 1)}}$$

The denominator of the middle term equals the denominator of the right-hand term because  $C$  is an odd, positive integer, and thus does not contain 2 as a factor and therefore has no effect on the value of the  $\text{ord}_2$  function.

The right-hand term gives us our desired result.  $\square$

#### Proof (if part):

Let  $x$  be an element of a  $3x + 1$  tuple. Then:

$$\frac{3x + 1}{2^{\text{ord}_2(3x + 1)}} = y \rightarrow \frac{C(3x + 1)}{2^{\text{ord}_2(3x + 1)}} = Cy \rightarrow \frac{3Cx + C}{2^{\text{ord}_2(3x + 1)}} = Cy \rightarrow \frac{3Cx + C}{2^{\text{ord}_2(3Cx + C)}} = Cy$$

The right-most equation gives us our result.

The denominators in the last two fractions are equal — that is,  $\text{ord}_2(3Cx + C) = \text{ord}_2(3(3x + 1)) = \text{ord}_2(3x + 1)$  — because 3 does not contain 2 as a factor, and therefore has no effect on the value of the  $\text{ord}_2$  function.  $\square$

### Remark

Lemma 14.8 shows that, informally, the  $3x + 1$  function is embedded in each  $3x + 1$ -like function. This suggests a strategy for proving the  $3x + 1$  Conjecture: show that each tuple  $\langle Cx, Cy, \dots, Cz \rangle$  in a  $3x + 1$ -like function  $3x + C$  eventually yields  $C$ . Other strategies based on the “convergence” of  $3x + C$  tuples to  $3x + 1$  tuples might be possible.

### Definition of “ $3x + 1$ -like” Problem

Let  $F_C$  be a  $3x + C$  function. Then if  $F_C(1) = 1$  we say that  $F_C$  gives rise to a  $3x + 1$ -like Problem and that  $F_C$  is a  $3x + 1$ -like function. It is by no means the case that all  $3x + C$  functions are  $3x + 1$ -like functions: For example,  $F_7(1) = 5$ .

We now establish all positive  $C$  that do, in fact, give rise to  $3x + 1$ -like Problems.

## All Positive $C$ That Give Rise to $3x + 1$ -like Problems

### Lemma 15.0

Let  $C$  define a  $3x + C$  function  $F_C$ . Then  $F_C$  gives rise to a  $3x + 1$ -like Problem iff  $C = -1$  or  $C = -1 + 2^1 + 2^2 + 2^3 + \dots + 2^k$ .

#### Proof (if part):

Let  $C = -1$ . Then by direct calculation we confirm that  $F_{-1}(1) = 1$ .

Let  $C = -1 + 2^1 + 2^2 + 2^3 + \dots + 2^k$ . Then  
(1)

$$\frac{3(1) - 1 + 2^1 + 2^2 + \dots + 2^k}{2^{k+1}} = \frac{1 + 2^0 + 2^1 + 2^2 + \dots + 2^k}{2^{k+1}} = \frac{2^{k+1}}{2^{k+1}} = 1$$

□

#### Proof (only if part):

If  $F_C$  gives rise to a  $3x + 1$ -like Problem, then by definition there must exist a  $k + 1$  such that

$$\frac{3(1) + C}{2^{k+1}} = 1$$

We find that solutions to this equation are  $C = -1$  and  $C = -1 + 2^1 + 2^2 + 2^3 + \dots + 2^k$ . □

The following corollary was brought to our attention by a computer scientist.

### Corollary to Lemma 15.0

(a) For each  $C$  such that  $3x + C$  is a  $3x + 1$ -like function, the exponent sequence associated with the infinite tuple  $\langle 1, 1, 1, \dots \rangle$  is as follows:

- $3x - 1$ : exponent sequence  $\{1, 1, \dots\}$
- $3x + 1$ : exponent sequence  $\{2, 2, \dots\}$
- $3x + 5$ : exponent sequence  $\{3, 3, \dots\}$
- $3x + 13$ : exponent sequence  $\{4, 4, \dots\}$

etc.

The proof follows directly from the  $C$  that yield  $3x + 1$ -like functions, as established in Lemma 15.0. □

(b) For each  $C$  (except  $C = 1$  and  $C = -1$ ) such that  $3x + C$  is a  $3x + 1$ -like function,  $C$  is a counterexample to the  $3x + C$  Conjecture because it yields the infinite cycle,  $\langle C, C, C, \dots \rangle$ . Thus:

$$3x + 5: (3(5) + 5)/2^2 = 5, \text{ giving rise to the infinite tuple } \langle 5, 5, 5, \dots \rangle$$

$$3x + 13: (3(13) + 13)/2^2 = 13, \text{ giving rise to the infinite tuple } \langle 13, 13, 13, \dots \rangle$$

etc.

**Proof:**  $(3(C) + C)/2^2 = C(3 + 1)/2^2 = C$ .  $\square$

Part (b) was brought to our attention by a computer scientist.

### On Trivial Infinite Cycles in $3x + 1$ -like Functions

The definition of  $3x + 1$ -like functions, along with “Lemma 15.0” on page 48 and its Corollary, make clear that there are at least two trivial infinite cycles in each  $3x + 1$ -like function:  $\langle 1, 1, 1, \dots \rangle$  and  $\langle C, C, C, \dots \rangle$ . In the  $3x + 1$  case, and only in this case, these cycles are the same. A naive question arises: if we are going to identify, for each  $3x + 1$ -like function, one of these infinite cycles as “the fundamental (trivial) cycle”, which one should it be? So far, researchers have regarded  $\langle 1, 1, 1, \dots \rangle$  in the  $3x + 1$  case as being “fundamental”, not least because it is part of the definition of the  $3x + 1$  Problem. If we do the same for all  $3x + 1$ -like functions, then the  $\langle C, C, C, \dots \rangle$  cycles are counterexamples to the  $3x + C$  Conjecture. On the other hand, if we make the  $\langle C, C, C, \dots \rangle$  cycles fundamental, then each  $\langle 1, 1, 1, \dots \rangle$  is a counterexample! In the  $3x + 1$  case, this wrecks the definition of the  $3x + 1$  Problem.

A computer scientist has suggested that we define, for each  $3x + 1$ -like functions except the  $3x + 1$  function, *both*  $\langle 1, 1, 1, \dots \rangle$  *and*  $\langle C, C, C, \dots \rangle$  as fundamental (trivial) cycles, call each odd, positive integer that maps to *either* 1 or  $C$ , a non-counterexample to the  $3x + C$  Conjecture, and call all other odd, positive integers, counterexamples. This convention has the advantage that each of the two fundamental (trivial) cycles has an obvious relationship to  $\langle 1, 1, 1, \dots \rangle$  in the  $3x + 1$  function. On the other hand, it seems a little strange to have non-counterexamples partitioned into two disjoint sets.

The fact that there are the two trivial infinite cycles,  $\langle 1, 1, 1, \dots \rangle$  and  $\langle C, C, C, \dots \rangle$ , for each  $3x + C$  function that is a  $3x + 1$ -like function — two such cycles *except* when  $C = 1$  (our familiar  $3x + 1$  function) suggests a possible “convergence” strategy for proving the  $3x + 1$  Conjecture: show that, as  $C$  decreases, a certain crucial property converges to a value such that counterexamples cannot exist for the  $3x + 1$  function.

### Conjectures Concerning $3x + 1$ -like Functions

**Conjecture C1.** *The tuple-set structure holds for all these functions. In particular, the distance functions established by parts (a) and (b) of Lemma 1.0 are the same for all these functions.*

Lemma 1.0 holds in tests of the  $3x - 1$ ,  $3x + 5$ , and  $3x + 13$  functions. It is proved to be true for the  $3x + 1$  function in “Lemma 1.0: Statement and Proof” on page 25. However, “Lemma 12.0: Statement and Proof” on page 37 does not hold in any  $3x + C$  function that is a  $3x + 1$ -like function if  $C \geq 5$ . This Lemma states that each range element is mapped to by all exponents of one parity only. The reason the Lemma doesn’t hold for these  $C$  is that, as  $C$  increases, the smallest

power of 2 that maps to 1 increases from  $2^3$ , to  $2^4$ , to  $2^5$ , etc., as shown in “Corollary to Lemma 15.0” on page 48.

**Conjecture C2.** *The recursive “spiral”’s structure holds for all these functions. In particular, the distance between successive elements  $x, x'$  in a “spiral” is given by  $x' = 4x + C$ .*

The Conjecture holds in tests of the  $3x - 1$ ,  $3x + 5$ , and  $3x + 13$  functions. It is proved to be true for the  $3x + 1$  function in the proof of “Lemma 12.0: Statement and Proof” on page 37.

It goes without saying that the precise applicability of the results in this paper to other  $3x + 1$ -like functions needs to be investigated.

## Definition of $3x - 1$ Function

The definition of the  $3x - 1$  function is similar to that of the  $3x + 1$  function. That is, for  $x$  an odd, positive integer, the  $3x - 1$  function  $C'$  is defined as:

$$C'(x) = \frac{3x - 1}{2^{\text{ord}_2(3x - 1)}}$$

where  $\text{ord}_2(3x - 1)$  is the largest exponent of 2 such that the denominator divides the numerator. The importance of the  $3x - 1$  function for our purposes is that it is the negative of the  $3x + 1$  function on the odd, negative integers. Thus, for example,

$$-\left(\frac{3(7) - 1}{2^2}\right) = -5 = \left(\frac{3(-7) + 1}{2^2}\right)$$

The  $3x - 1$  Conjecture asserts that for each odd, positive integer  $x$  (or for each odd, negative integer  $x'$  for the  $3x + 1$  function over the odd, negative integers) repeated iterations of the  $3x - 1$  eventually terminate in 1 (or  $-1$  in the negative case).

However, the  $3x - 1$  Conjecture is known to be false. Among the counterexamples are 5 and 17 ( $-5$  and  $-17$  in the negative case).

A useful test for the correctness of a proof of the  $3x + 1$  Conjecture is to see if it also proves that the  $3x - 1$  Conjecture is true. If it does, then we know that the proof is invalid.

We now prove several elementary facts about the  $3x - 1$  function. From here on we will use its negative version, denoting it as  $C'$ , although we will refer to this negative version as the  $3x - 1$  function.

## Elementary Facts About the $3x - 1$ Function

### Lemma 9.0

*For no odd, negative integer  $-u$  is it the case that  $C'(-u)$  is positive.*

**Proof:**

$(3(-u) + 1)/(ord_2(3x - 1))$  is negative because  $(3(-u) + 1)$  is negative and  $ord_2(3x - 1)$  is positive.  $\square$

**Lemma 9.05**

(a) *The negative of the  $3x - 1$  function over the odd, positive integers = the  $3x - 1$  function over the odd, negative integers. That is, for all odd, non-zero integers  $u$ ,*

$$-\left(\frac{3(u) - 1}{2^2}\right) = -w = \left(\frac{3(-u) + 1}{2^2}\right)$$

(b) *The negative of the  $3x + 1$  function is embedded in the  $3x - 1$  function. That is,  $\langle x, y \rangle$  is a tuple in the  $3x + 1$  function iff  $\langle -x, -y \rangle$  is a tuple in the  $3x - 1$  function.*

**Proof of part (a):**

Follows directly from algebra on the equation in the statement of part (a).  $\square$

**Proof of part (b):**

Follows directly from “Lemma 14.8” on page 47.  $\square$

**Remark**

Part (b) is the only case that we are aware of in which a  $3x + 1$ -like function’s domain includes both the positive and negative integers. The reader must clearly understand that the negative tuples in this case are *not* the negative tuples in the case of Part (a).

**Lemma 9.1**

*If  $y$  is an anchor for the  $3x + 1$  function at level  $i$ , then  $y - 2 \cdot 3^{(i-1)}$  is an  $i$ -level anchor for the  $3x - 1$  function.*

**Proof:**

It is reasonable to define an anchor for the  $3x - 1$  function analogously to an anchor for the  $3x + 1$  function, that is, to define an anchor for the  $3x - 1$  function as an odd, negative integer  $y'$  that is relatively prime to  $2 \cdot 3^{(i-1)}$  and greater than  $-2 \cdot 3^{(i-1)}$ . It is clear that  $y - 2 \cdot 3^{(i-1)}$  is such an integer. In fact, since  $y$  is a minimum positive residue of the integers mod  $2 \cdot 3^{(i-1)}$  that is relatively prime to  $2 \cdot 3^{(i-1)}$ , it follows that  $y - 2 \cdot 3^{(i-1)}$  is a maximum negative residue of the integers mod  $2 \cdot 3^{(i-1)}$  that is relatively prime to  $2 \cdot 3^{(i-1)}$ .  $\square$

**Lemma 9.2**

*For each  $i \geq 2$ , the set of all  $i$ -level anchor tuples for the  $3x - 1$  function is complete.*

**Proof:**

Follows directly from (1) the fact that the set of all  $i$ -level anchor tuples for the  $3x + 1$  function is complete, and (2) “Lemma 9.1” on page 51.  $\square$

**Lemma 9.3**

*“Lemma 1.0” on page 10 and “Lemma 5.0” on page 14 apply to the  $3x - 1$  function.*

**Proof:**

Each lemma applies to the  $3x - 1$  function because in our negative definition, the  $3x - 1$  function is simply the  $3x + 1$  function extended into the odd, negative integers, and this does not affect the proof of either Lemma, or of referenced lemmas..  $\square$

**Lemma 9.4**

*Let  $u$  be an odd, negative integer, and let  $\bar{t}_u = \langle u, u', \dots \rangle$  be the infinite tuple it generates. Let  $A(\bar{t}_u)$  be the infinite exponent sequence associated with  $\bar{t}_u$ . Let  $x$  be an odd, positive integer, and let  $t_x = \langle x, x', \dots \rangle$  be the infinite tuple it generates. Let  $A(\bar{t}_x)$  be the infinite exponent sequence associated with  $\bar{t}_x$ .*

*Then  $A(\bar{t}_u) \neq A(\bar{t}_x)$ .*

**Proof:**

Assume the contrary. Then  $u, x$  are at a distance  $d = x - u$  apart. But as the length of their respective tuples increases, they nevertheless remain in the same succession of tuple-set extensions, by our hypothesis. However, by “Lemma 1.0” on page 10, a level  $i$  must eventually be reached such that  $d$  is less than the minimum distance  $d(1, i)$  between first elements of successive  $i$ -level tuples, where

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3}) \dots (2^{a_i})$$

and the exponent sequence  $A$  for the  $i$ -level tuple-set  $T_A$  is  $\{a_2, a_3, \dots, a_i\}$ . This impossibility gives us our proof.  $\square$

**Remark:** Lemma 9.4 implies that there does not exist a counterexample  $x$  to the  $3x + 1$  Conjecture such that  $A(\bar{t}_x) = A(\bar{t}_u)$  for any counterexample  $u$  to the  $3x - 1$  Conjecture. In passing we point out that by “Lemma 5.0” on page 14 applied to the  $3x - 1$  function, we know that for each  $i$ , the set  $\{\bar{t}_u(i)\}$  of all  $i$ -level prefixes of all  $3x - 1$  counterexample infinite tuples  $\bar{t}_u$  is complete. The next lemma is another way of expressing this fact. It shows how a certain class of  $3x + 1$  counterexample tuples are “pushed away” to infinity, and hence to non-existence.

**Lemma 9.5**

*Let  $u$  be a counterexample to the  $3x - 1$  Conjecture, and let  $\bar{t}_u = \langle u, u', \dots \rangle$  be the infinite tuple it generates. Let  $A(\bar{t}_u(j))$  be the exponent sequence associated with the prefix  $\bar{t}_u(j)$ . And similarly for counterexamples  $x$  to the  $3x + 1$  Conjecture. Then for all counterexamples  $x$  to the  $3x + 1$  Conjecture:*

*If  $A(\bar{t}_x(2)) = A(\bar{t}_u(2))$  then  $x - u$  must be  $\geq 2 \cdot 2^{a_2}$ ; and*

*If  $A(\bar{t}_x(3)) = A(\bar{t}_u(3))$  then  $x - u$  must be  $\geq 2 \cdot 2^{a_2} 2^{a_3}$ ; and*

*If  $A(\bar{t}_x(4)) = A(\bar{t}_u(4))$  then  $x - u$  must be  $\geq 2 \cdot 2^{a_2} 2^{a_3} 2^{a_4}$ ; and*

...

**Proof:**

Same argument as in the proof of “Lemma 9.4” on page 52, plus part (b) of the distance function lemma, namely, “Lemma 1.0” on page 10□

### **How Is the Interleaving of Counterexample and Non-Counterexample Anchor Tuples Possible?**

“Lemma 5.0” on page 14, which also applies to the  $3x - 1$  function, states that if counterexamples exist, then each  $i$ -level tuple-set, where  $i \geq 2$ , contains an infinity of counterexample tuples and an infinity of non-counterexample tuples. Since for each level  $i \geq 2$ , there are both non-counterexample and counterexample anchor tuples in the case of the  $3x - 1$  function, this would seem to imply that, for each level  $i > 2$ , the set of non-counterexample anchor tuples is complete, and similarly for the set of counterexample anchor tuples. (At level 2, this does not hold.) This in turn would imply that each  $i$ -level tuple-set has two anchor tuples, one non-counterexample and the other counterexample, which is impossible.

But in fact the existence of non-counterexample and counterexample anchor tuples is possible if the following is always the case, namely, that for each level  $i > 2$ , there is a maximum  $j < i$  such that the set of  $j$ -level non-counterexample anchor tuple *prefixes* is complete, and similarly for the set of  $j$ -level counterexample anchor tuple *prefixes*. For longer prefixes, the corresponding sets are not complete, although the set of both non-counterexample and counterexample anchor tuples is always complete.

### **$3x - 1$ Anchor Tuples and a Failed Proof of the $3x + 1$ Conjecture**

The reader will naturally wonder how “Lemma 9.5” on page 52 can hold, given the fact that for each level  $i$  there is a complete set of positive anchor tuples. How is it possible that the distance function is not violated by the fact that (1) each negative counterexample  $u$  is eventually a negative anchor tuple, and remains so for an infinity of successive levels, and that (2) for each level  $i$ , there must be a positive anchor tuple  $\bar{t}_x(i)$  such that  $A(\bar{t}_u(i)) = A(\bar{t}_x(i))$ ?

The answer is clear from the lemma statement itself. It is *different* positive anchor tuples that fulfill the role of providing successive matching exponent sequences for the prefixes of  $\bar{t}_u$ . A single positive  $x$  does not give rise to all these positive anchor tuples.

If the reader asks what happens to the prefixes of infinite tuples  $\bar{t}_x$  once their exponent sequences no longer match those of prefixes of  $\bar{t}_u$ , the answer is that they are associated with the exponent sequences of different negative anchor tuples.

We see, therefore, that the  $3x - 1$  anchor tuples limit the possible exponent sequences for positive counterexample tuples. But not sufficiently for a proof of the  $3x + 1$  Conjecture, because, for example, it is possible that all negative counterexamples give rise to infinite cycles. If so, then it is still possible that positive counterexamples exist that do not give rise to infinite cycles.

### **Why Are There Counterexamples to the $3x - 1$ Conjecture?**

Two known counterexamples to the  $3x - 1$  Conjecture are 5 and 17, or, in our alternate version of the function,  $-5$  and  $-17$ . Both counterexamples give rise to infinite loops. For  $-5$  we have the infinite tuple  $\langle -5, -7, -5, \dots \rangle$  and for  $-17$  we have the infinite tuple  $\langle -17, -25, -37, -55, -41, -61, -91, -17, \dots \rangle$ . As in the case of the  $3x + 1$  function,  $-5$  (or 5) is the base element of an

infinite set of recursive “spiral”s in the  $3x - 1$  function, and similarly for  $-17$  (or  $17$ ). We conjecture that these infinite sets are disjoint, and in fact that the three infinite sets of recursive “spiral”s with base elements  $-5$  (or  $5$ ),  $-17$  (or  $17$ ), and  $-1$  (or  $1$ ) are disjoint.

If we sharpen our question to, “Why does the infinite cycle  $\langle -5, -7, -5, \dots \rangle$  (or  $\langle 5, 7, 5, \dots \rangle$ ), which consists of the counterexamples  $-5$  (or  $5$ ), and  $-7$  (or  $7$ ), exist?”, then the answer is simple: the cycle is a consequence of the distance functions  $d(1, 3)$ ,  $d(2, 3)$  and  $d(3, 3)$  in “Lemma 1.0” on page 10 operating on the 3-level tuple  $\langle 11, 17, 13 \rangle$  in the 3-level tuple-set  $T_A$ , where  $A = \{1, 2\}$ .

Specifically, subtracting  $d(1, 3) = 2 \bullet 2^1 \bullet 2^2 = 16$  from 11 gives us  $-5$ . Subtracting  $d(2, 3) = 2 \bullet 3 \bullet 2^2$  (see “Distances between elements of tuples  $t(r)$ ,  $t(s)$  consecutive at level  $i$ ” on page 11)  $= 24$  from 17 gives us  $-7$ . Finally, subtracting  $d(3, 3) = 2 \bullet 3^{3-1} = 18$  from 13 gives us  $-5$ .

Thus the distance functions give us the tuple  $\langle -5, -7, -5 \rangle$  in the odd, negative integers, which is the negative of the tuple  $\langle 5, 7, 5 \rangle$  for the  $3x - 1$  function. Informally, the infinite cycle  $\langle 5, 7, 5, \dots \rangle$  simply “falls out of the arithmetic”.