# A Solution to the 3*x* + 1 Problem

by

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Oct. 20. 2020

"Very often in mathematics the crucial problem is to recognize and to discover what are the relevant concepts; once this is accomplished the job may be more than half done."[1]

"One of the greatest contributions a mathematician can make is to spot something so simple and powerful that everybody else has missed it."[2]

***Note* 1**: The reader who is sorely pressed for time should begin by reading our shortest proof of the 3*x* + 1 Conjecture. (A proof of the Conjecture solves the 3*x* + 1 Problem.) See "Third Proof of the 3x + 1 Conjecture" on page 21.

***Note* 2**: Letters designating appendices have been recently changed as a result of a reorganization of this paper.

***Note* 3:** The reader can safely assume, *initially*, that all referenced lemmas in this paper are true, since their proofs have been checked and deemed correct by several mathematicians.

***Note* 4**: *We will offer shared-authorship* to any mathematician who creates a proof of the 3*x* +1 Conjecture that differs from those in this paper, but that makes use of materials in this paper.

***Note* 5:** The author is seeking a professional mathematician to help prepare this paper for publication. The author will pay any reasonable consulting fee, give generous credit in the Acknowledgments (but only with the mathematician's prior written approval), and offer shared-authorship for significant contribution to content.

*Key words*: 3*x* + 1 Problem, 3*n* + 1 Problem, Syracuse Problem, Ulam's Problem, Collatz Conjecture, computational number theory, proof of termination of programs, recursive function theory
American Mathematical Society Classification Numbers: 11Y16, 11Z05, 03D20, 68Q60

1. Herstein, I. N., *Topics in Algebra*, John Wiley & Sons, N.Y., 1975, p. 50.
2. Stewart, Ian, *The Problems of Mathematics*, Oxford University Press, N.Y., 1992, pp. 279-280.

## Abstract

The $3x + 1$ Problem asks if repeated iterations of the function $C(x) = (3x + 1)/(2^a)$ always terminate in 1 Here $x$ is an odd, positive integer, and $a = ord_2(3x + 1)$, the largest positive integer such that the denominator divides the numerator. The conjecture that the function always eventually terminates in 1 is the *3x + 1 Conjecture*. An odd, positive integer that maps to 1 is called a *non-counterexample*; an odd, positive integer that doesn't map to 1 is called a *counterexample* (to the Conjecture).

Our first proof (given under:"First Proof of the 3x + 1 Conjecture" on page 11) is based on a structure called *tuple-sets* that represents the $3x + 1$ function in the "forward" direction. In our proof, we show that the 35-level elements of all 35-level tuples in all 35-level tuple-sets are the same, regardless if counterexamples to the Conjecture exist or not[1]. From this fact, a simple inductive argument allows us to conclude that all tuple-sets are the same, whether counterexamples exist or not, and hence that counterexamples do not exist.

Our second proof (given in "Second Proof of the 3x + 1 Conjecture" on page 12), like the first, is based on tuple-sets. In this proof, we define *anchor*, which is the *i*-level element of the first *i*-level tuple in an *i*-level tuple-set. We then show that there is one and only one set of anchors for all *i*, regardless if counterexamples exist or not. We then show that this implies that there is one and only one set of infinite tuples, regardless if counterexamples exist or not, and from this we deduce that, if counterexamples exist, then some infinite tuples must be both counterexample and non-counterexample tuples, which is absurd, hence counterexamples do not exist and the Conjecture is true.

Our third proof (given in "Third Proof of the 3x + 1 Conjecture" on page 21) has three versions. The first is based on a structure called the 1-tree. This tree is a $y$-tree, where the root $y$ is a range element of the $3x + 1$ function; $y$-trees represent the $3x + 1$ function in the "inverse" direction. The second version is based on the remarkable fact that each finite sequence of iterations of the $3x + 1$ function can in principle be traced on a certain spiral diagram. The third version is also based on the 1-tree.

As far as we have been able to determine, our approaches to a solution of the Problem are original.


1. A phrase of the form "$q$ regardless if $p$" is equivalent to "(if $p$ then $q$) and (if *not-p* then $q$)". It is meaningful and in fact true as long as $q$ is true, which it always is in this paper. Instances of the phrase occur in everyday speech, for example, "Fermat's Last Theorem is true regardless if the Riemann Conjecture is true".,

# Introduction

## Statement of Problem

For $x$ an odd, positive integer, set

$$C(x) = \frac{3x+1}{2^{ord_2(3x+1)}}$$

where $ord_2(3x + 1)$ is the largest exponent of 2 such that the denominator divides the numerator. Thus, for example, C(17) = 13 ($ord_2(3(17) + 1) = 2$) , C(13) = 5 ($ord_2(3(13) + 1) = 3$), C(5) = 1 ($ord_2(3(5) + 1) = 4$). Each of these constitutes one iteration of the $3x + 1$ function. The $3x + 1$ Problem, also known as the $3n + 1$ Problem, the Syracuse Problem, Ulam's Problem, the Collatz Conjecture, Kakutani's Problem, and Hasse's Algorithm, asks if repeated iterations of $C$ always terminate at 1. The conjecture that they do is hereafter called the *3x + 1 Conjecture*, or sometimes, in this paper, just *the Conjecture*. We call $C$ the $3x + 1$ function; note that $C(x)$ is by definition odd.

An odd, positive integer such that repeated iterations of $C$ terminate at 1, we call a *non-counterexample*. An odd, positive integer such that repeated iterations of $C$ never terminate at 1, we call a *counterexample*.

Other equivalent formulations of the $3x + 1$ Problem are given in the literature; we base our formulation on the $C$ function (following Crandall) because, as we shall see, it brings out certain structures that are not otherwise evident.

## Summary of Research on the Problem

As stated in (Lagarias 1985), "The exact origin of the $3x + 1$ problem is obscure. It has circulated by word of mouth in the mathematical community for many years. The problem is traditionally credited to Lothar Collatz, at the University of Hamburg. In his student days in the 1930's, stimulated by the lectures of Edmund Landau, Oskar Petron, and Issai Schur, he became interested in number-theoretic functions. His interest in graph theory led him to the idea of representing such number-theoretic functions as directed graphs, and questions about the structure of such graphs are tied to the behavior of iterates of such functions. In the last ten years [that is, 1975-1985] the problem has forsaken its underground existence by appearing in various forms as a problem in books and journals..."

Lagarias has performed an invaluable service to the $3x + 1$ research community by publishing several annotated bibliographies relating to the Problem. These are accessible on the Internet.

## On the Structure of This Paper

To enhance readability, we have placed proofs of all lemmas in "Appendix A — Statement and Proof of Each Lemma" on page 26.

## In Memoriam

Several of the most important lemmas in this paper were originally conjectured by the author and then proved by the late Michael O'Neill. He made a major contribution to this research, and is sorely missed.

# Tuple-sets: The Structure of the 3*x* + 1 Function in the “Forward” Direction

## A Comment

The structure, *tuple-sets*, that we are about to describe, is one of two remarkably simple structures[1] that we have discovered underlying the $3x + 1$ function, a function that is still referred to, at least informally, as “chaotic”. The reader can get an idea of the alternative structures that, at the time of this writing, are used throughout the $3x + 1$ literature, by browsing papers in Google that come up in response to the search string, “Collatz graphs”. For example, see:

https://www.fq.math.ca/Scanned/40-1/andaloro.pdf,
http://go.helms-net.de/math/collatz/aboutloop/collatzgraphs.htm

## Brief Description of Tuple-sets

The following should be sufficient for the reader to understand our proofs of the $3x + 1$ Conjecture that are based on tuple-sets, namely those in “First Proof of the 3x + 1 Conjecture” on page 11 and “Second Proof of the 3x + 1 Conjecture” on page 12.

1. We use the definition of the $3x + 1$ function in which all successive divisions by 2 are collapsed into a single exponent of 2 (see “Statement of Problem” on page 3). Thus, for example, the tuple <9, 7, 11> represents the fact that

9 maps to 7 in one iteration of the function, via the exponent 2, because $(3(9) + 1)/2^2 = 7$ ;
7 maps to 11 in one iteration of the function, via the exponent 1, because $(3(7) + 1)/2^1 = 11$.

2. We see that the sequence of exponents associated with the tuple <9, 7, 11> is {2, 1}.

3. A tuple-set $T_A$ is the set of all finite tuples that are associated with the exponent sequence $A$ (and “approximations” to $A$, but this is not important for our proofs of the 3x + 1 Conjecture). In our example, $A = \{2, 1\}$.

In addition to the tuple <9, 7, 11> , the tuple-set $T_A = T_{\{2, 1\}}$ contains the tuples <25, 19, 29> , <41, 31, 47>, and an infinity of others, each associated with the exponent sequence {2, 1}. (See “Lemma 1.0: the “Distance” Functions d(i, i) and d(1, i)” on page 9.)

4. Facts about tuple-sets:

An *i*-level tuple-set $T_A$, $i \geq 2$, contains (among other tuples, see previous step) all $(i + 1)$-element tuples that are associated with the exponent sequence $A$.

There is an infinity of tuples in each tuple-set.

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1. The other is y-trees, as described under “y-Trees: The Structure of the 3x + 1 Function in the “Backward”, or Inverse, Direction” on page 18.

The set of all tuple-sets contains tuples representing all finite iterations of the $3x + 1$ function.

## Full Description of Tuple-sets

### Definitions

**Iteration**

An *iteration* takes an odd, positive integer, $x$, to an odd, positive integer, $y$, via one application of the $3x + 1$ function, $C$. Thus, in one iteration $C$ takes 17 to 13 because $C(17) = 13$.

**Tuple**

A (finite) *tuple* is a finite sequence of zero or more successive iterations of $C$, that is, <$x$, $C(x)$, $C^2(x)$, ..., $C^k(x)$>, where $k \geq 0$.

A finite tuple is the prefix of an infinite tuple. If $x$ is a non-counterexample, then $x$ is the first element of an infinite tuple <$x$, $y$, ..., 1, 1, 1, ... >. Of course, if $x$ is a range element of $C$, then $x$ can be an element other than the first in another non-counterexample tuple.

In the literature, a tuple (finite or infinite) is usually called a *trajectory* or an *orbit*.

If $x$ is a counterexample, then $x$ is the first element of an infinite tuple <$x$, $y$, ... > which does not contain 1. Of course, if $x$ is a range element of C, then $x$ can be an element other than the first in another counterexample tuple.

A counterexample tuple must be one of two types: either there is an infinitely-repeated finite cycle of elements (none of which is 1) in the infinite tuple having the counterexample $x$ as first element, or else there is no such cycle, but there is no 1 in the infinite tuple having the counterexample $x$ as first element — in other words, there is no upper bound to the elements of the infinite tuple.

**Exponent, Exponent Sequence**

If $C(x) = y$, with $y = (3x + 1)/2^a$, we say that *a is the exponent associated with* $x$. In more formal language, this can be expressed as $ord_2(3x + 1) = a$. Sometimes we simply write $e(x) = a$. The sequence $A = \{a_2, a_3, ..., a_i\}$, where $a_2, a_3, ..., a_i$ are the exponents associated with $x$, $C(x)$, ..., $C^{(i-1)}(x)$ respectively, is called an *exponent sequence*. We number exponents beginning with $a_2$ in order that the subscript corresponds to a level number in the corresponding tuple-set. See "Levels in Tuples and Tuple-sets" on page 7. For all $i \geq 2$, there are always $i - 1$ exponents in the exponent sequence associated with an $i$-level tuple-set

We say that *$x$ maps to $y$ via $a_i$* if $C(x) = y$ and $ord_2(3x + 1) = a_i$. By extension, we say that $x$ *maps to* $z$ if $z$ is the result of a finite sequence of iterations of $C$ beginning with $x$, that is if the tuple <$x$, $y$, ..., $z$> exists.

**Tuple-set[1]**

Let $A = \{a_2, a_3, ..., a_i\}$ be a finite sequence of exponents, where $i \geq 2$. The *tuple-set* $T_A$ consists of all and only the tuples that are associated with all successive approximations to $A$. Thus

$T_A$ consists of all and only the following tuples. (*Note*: First elements $x$ in different tuples are different odd, positive integers. No two tuples in a tuple-set have the same first element.)

all tuples $<x>$ such that $x$ does not map to an odd, positive integer via $a_2$;

all tuples $<x, y>$ such that $x$ maps to $y$ via $a_2$ but $y$ does not map to an odd, positive integer via $a_3$;

all tuples $<x, y, y'>$ such that $x$ maps to $y$ via $a_2$ and $y$ maps to $y'$ via $a_3$, but $y'$ does not map to an odd, positive integer via $a_4$;

...

all tuples $<x, y, y', ..., y^{(i-3)}, y^{(i-2)}>$ such that $x$ maps to $y$ via $a_2$ and $y$ maps to $y'$ via $a_3$ and ... and $y^{(i-3)}$ maps to $y^{(i-2)}$ via the exponent $a_i$. (The longest tuple in an $i$-level tuple-set has $i$ elements.)

In other words, for each $i$-level exponent sequence $A$:

there are tuples $<x>$ whose associated exponent sequence is a prefix of $A$ for no exponent of $A$, and
there are other tuples $<x, y>$ whose associated exponent sequence is a prefix of $A$ for the first exponent of $A$, and
there are other tuples $<x, y, y'>$ whose associated exponent sequence is a prefix of $A$ for the first two exponents of $A$, and
...
there are other tuples $<x, y, z, ..., y^{(i-2)}>$ whose associated exponent sequence is a prefix of $A$ for all $i - 1$ exponents of $A$.

Tuples are ordered in the natural way by their first elements.

The set of first elements of all tuples in a tuple-set is the set of odd, positive integers (see proof under "The Structure of Tuple-sets" on page 9). Thus, there is a countable infinity of tuples in each tuple-set.

For each $i \geq 2$, tuple-sets are a *partition* of the set of all $i$-level tuples.

## Levels in Tuples and Tuple-sets

Let $A$ be an $i$-level exponent sequence, $\{a_2, a_3, ..., a_i\}$. The reason subscripts of exponents begin with 2, rather than with 0 or 1, is so that they correspond to levels in each tuple-set. (No tuple-set has only one level, because that would mean it is associated with no exponent sequence.) Let $T_A$ be the tuple-set determined by $A$. Then, by definition of tuple-set, there exist $j$-level tuples in $T_A$, where $1 \leq j \leq i$, that is, tuples $t = <x, y, ..., z>$, where $x$ is the 1-level element of $t$, $y$ is the 2-

1. The literature contains several results that establish properties of the $3x + 1$ function that are equivalent to some of those for tuple-sets. However, the language is very different, and the definition of the $3x + 1$ function that is used is not ours, but the original one, in which each division by 2 is a separate node in the tree graph representing the function.

level element of $t$, ..., and $z$ is the $j$-level element of $t$. We say that $T_A$ is an $i$-level tuple-set, and we sometimes speak of the set of $j$-level tuple-elements in $T_A$, where $1 \leq j \leq i$.

For $2 \leq j \leq i$, two tuples are said to be *consecutive at level j* if no $j$-level or higher-level tuple exists between them.

**Example of Tuple-set**

As an example of (part of) a tuple-set: in Fig. 1, where $A = \{a_2, a_3, a_4\} = \{1, 1, 2\}$ and where we adopt the convention of orienting tuples vertically on the page, the tuple-set $T_A$ includes:

the tuple <1>, because $e(1) = 2 \neq (a_2 = 1)$;
the tuple <3, 5> , because $e(3) = (a_2 = 1)$, but $e(5) = 4 \neq (a_3 = 1)$;
the tuple <5>, because $e(5) = 4 \neq (a_2 = 1)$;
the tuple <7, 11, 17, 13> because $e(7) = 1$ $(a_2 = 1)$ and $e(11) = 1$ $(a_3 = 1)$ and $e(17) = 2$ $(a_4 = 2)$;
etc.

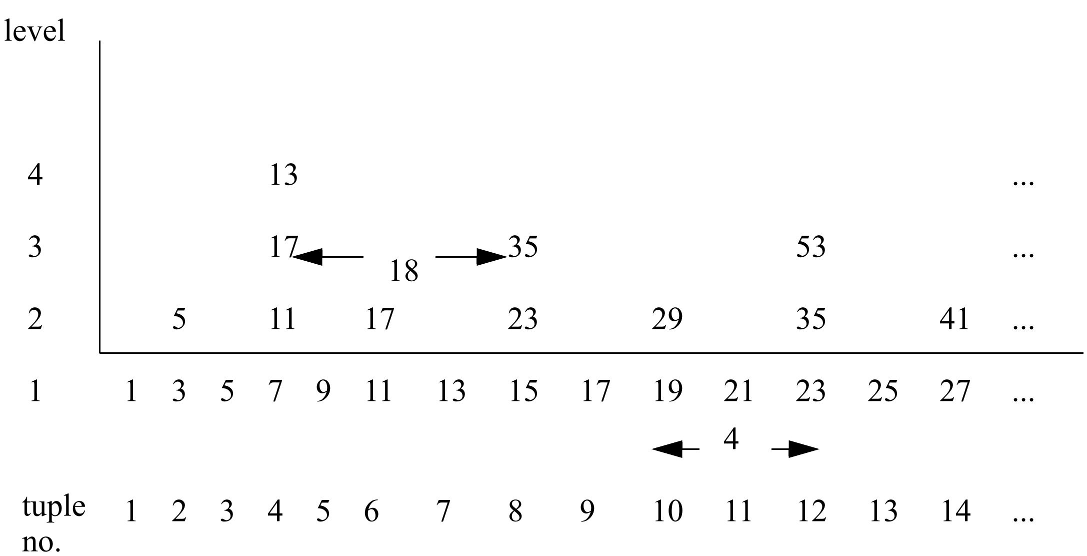


**Fig. 1. Part of the tuple-set $T_A$ associated with the sequence $A$ = {1, 1, 2}**

The number 18 between the arrows at level 3 and the number 4 between the arrows at level 1 are the values of the level 3 and level 1 distance functions, respectively, established by Lemma 1.0 (see "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 9).

In each $i$-level tuple-set $T_A$, where $i \geq 2$, for each odd, positive integer $x$ there exists a tuple whose first element is $x$. The tuple may be one-level (<$x$>), or 2-level (<$x$, $y$>), or ... or $i$-level (<$x, y, y', ..., y^{(i-3)}, y^{(i-2)}$>). Thus each tuple-set is non-empty.

**Graphical Representation of the Set of All Tuple-sets**

It is clear from the definition of *tuple-set* that the set of all tuple-sets can be represented by an infinitary tree in which each node is a tuple-set. We can imagine the tuple-set (which contains an infinity of tuples) extending into the page.

**The Structure of Tuple-sets**

It is important for the reader to understand that the structure of each tuple-set is unchanged by the presence or absence of counterexample tuples. Regardless if counterexample tuples exist or not, the set of first elements of all tuples in each tuple-set is always the same, namely, the set of odd, positive integers. *Proof:* Let $x$ be any odd, positive integer and let $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, be any exponent sequence. Then there are exactly two possibilities:

(1) $x$ maps to a $y$ in a single iteration of the $3x + 1$ function, $C$, via the exponent $a_2$, or
(2) $x$ does not map to a $y$ in a single iteration of $C$ via the exponent $a_2$.

But if (1) is true, then a tuple containing at least two elements, with $x$ as the first, is in $T_A$; if (2) is true, then the tuple $<x>$ is in $T_A$. There is no third possibility. □

For each tuple-set, the first element of the first tuple is 1, the first element of the second tuple is 3, the first element of the third tuple is 5, etc.

It can never be the case that, if counterexample tuples exist, then somehow there are "more" tuples in a tuple-set than if there are no counterexample tuples[1].

Furthermore, the distance functions defined in "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 9 are the same regardless if counterexample tuples exist or not.

**Extensions of Tuple-sets**

Since there is a tuple-set for each finite sequence $A$ of exponents, it follows that each tuple-set $T_A$ has an extension via the exponent 1, and an extension via the exponent 2, and an extension via the exponent 3, ... In other words, if $A = \{a_2, a_3, ..., a_i\}$, then there is a tuple-set $T_{A'}$, where $A' = \{a_2, a_3, ..., a_i, 1\}$, and a tuple-set $T_{A''}$, where $A'' = \{a_2, a_3, ..., a_i, 2\}$, and a tuple-set $T_{A'''}$, where $A''' = \{a_2, a_3, ..., a_i, 3\}$, ...

All this is true whether or not the tuple-set $T_A$ and/or any of its extensions contains counterexample tuples or not.

For further details on extensions of tuple-sets, see "How Tuple-sets 'Work'" and the proof that there exists an extension for each tuple-set ("Lemma 3.0 Statement and Proof") in our paper, "Are We Near a Solution to the 3x + 1 Problem?" on occampress.com.

**Lemma 1.0: the "Distance" Functions *d*(*i*, *i*) and *d*(1, *i*)**

(a) *Let* $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, *be a sequence of exponents, and let* $t_{(r)}$, $t_{(s)}$ *be tuples consecutive at level*[2] *i in* $T_A$. *Then d(i, i) is given by*:

1. To make this statement more precise: in no tuple-set does there ever exist a first element of a tuple, regardless how large that first element is, such that there are more tuples in that tuple-set having smaller first elements if counterexamples exist, than if counterexamples do not exist.
2. For $2 \leq j \leq i$, two tuples are consecutive at level $j$ if no $j$-level or higher-level tuple exists between them (see "Levels in Tuples and Tuple-sets" on page 7.

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) *Let $t_{(r)}$, $t_{(s)}$ be tuples consecutive at level i in $T_A$. Then d(1, i) is given by:*

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3})\ldots(2^{a_i})$$

**Proof:** see "Lemma 1.0: Statement and Proof" on page 26

It follows from part (a) of the Lemma that the set of all $i$-level elements of all $i$-level first tuples in all $i$-level tuple-sets is $\{z \mid 1 \le z < 2 \bullet 3^{i-1}\}$, where $z$ is an odd, positive integer not divisible by 3.

**Remark**: Relationships similar to those described in parts (a) and (b) of the Lemma hold for successive $j$-level tuples, where $2 \le j < i$. The following table shows these relationships for $(i - j)$-level elements of tuples consecutive at level $(i - j)$ in an $i$-level tuple-set, where $0 \le j \le (i - 1)$. The distances are easily proved using Lemma 1.0.

**Table 1: Distances between elements of tuples consecutive at level *i***

| Level | Distance between $(i - j)$-level elements of tuples consecutive at level $(i - j)$, where $0 \le j \le (i - 1)$ |
|---|---|
| $i$ | $2 \cdot 3^{i-1}$ |
| $i - 1$ | $2 \cdot 3^{i-2} \cdot 2^{a_i}$ |
| $i - 2$ | $2 \cdot 3^{i-3} \cdot 2^{a_{i-1}}2^{a_i}$ |
| $i - 3$ | $2 \cdot 3^{i-4} \cdot 2^{a_{i-2}}2^{a_{i-1}}2^{a_i}$ |
| ... | ... |
| 2 | $2 \cdot 3 \cdot 2^{a_3}\ldots2^{a_{i-1}}2^{a_i}$ |
| 1 | $2 \cdot 2^{a_2}2^{a_3}\ldots2^{a_{i-1}}\ 2^{a_i}$ |

Further details can be found in the section, "Remarks About the Distance Functions" in our paper, "Are We Near a Solution to the 3$x$ + 1 Problem?", on occampress.com.

**Lemma 2.0 Counterexample tuples in tuple-sets if counterexamples exist**

*Assume a counterexample exists. Then for all i ≥ 2, each i-level tuple-set contains an infinity of i-level counterexample tuples and an infinity of i-level non-counterexample tuples.*

**Proof**: "Lemma 2.0: Statement and Proof" on page 31

## First Proof of the 3x + 1 Conjecture

The $3x + 1$ Problem asks if repeated iterations of the function $C(x) = (3x + 1)/(2^a)$ always terminate in 1  Here $x$ is an odd, positive integer, and $a = ord_2(3x + 1)$, the largest positive integer such that the denominator divides the numerator.  The conjecture that the function always eventually terminates in 1 is the *3x + 1 Conjecture*.

(*Note*: the reader is asked to inform us of the first sentence that the reader believes contains an error, and what that error is.)

1. *Definitions*: an *anchor tuple* is the first $i$-level tuple in an $i$-level tuple-set, where $i \geq 2$.  An *anchor* is the $i$-level element of the anchor tuple in an $i$-level tuple-set.  It is easily shown that, for each $i \geq 2$, the number of anchors in *all* $i$-level tuple-sets is $2 \cdot 3^{((i-1)-1)}$ (see "Second Proof of the 3x + 1 Conjecture" on page 12).

We know, by computer test[1], that for all $i$, $2 \leq i \leq 35$, the anchor in each $i$-level tuple-set is a non-counterexample.  Thus, the following argument does not apply to the $3x - 1$ function, where already at $i = 2$, there is an $i$-level tuple that contains a counterexample.  (The tuple is <7, 5>, which is the start of the infinite cycle <7, 5, 7, 5, ... >, hence 5 is a counterexample  (as is 7).)

2.  For *each* 35-level tuple-set $T_A$, the sequence $S$ of 35-level elements in the sequence of 35-level tuples  is given by $y + n(2 \cdot 3^{(35-1)})$, where $n \geq 0$ and $y$ is the anchor tuple in $T_A$. The sequence $S$  is the  sequence if counterexamples do not exist.  It is also the sequence if counterexamples exist. (A similar statement occurs in Version 1 of "Third Proof of the 3x + 1 Conjecture" on page 21.)

*Proof*: Follows from part (a) of "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 9.  The Distance Functions are not themselves functions of the truth or falsity of the $3x + 1$ Conjecture.□

*Note*: the fact that the set of anchors in all 35-level tuple-sets are non-counterexamples is emphatically *not* the case for the $3x - 1$ function, where one of the elements, 5, of a two-level tuple-set is already a counterexample. Thus there exists a first 2-level tuple, namely <7, 5>, in a 2-level tuple-set that is a counterexample tuple.  The anchors for each greater $i$ include counterexamples.  Each of the anchor tuples is therefore a counterexample tuple.  *So our proof  cannot be used to prove the false 3x – 1 Conjecture*[2].

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1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$, which is greater than $3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35-1)}$, have been tested and found to be non-counterexamples. These include the set of anchors in the set of all 35-level tuple-sets .
2. See "Appendix B — On The 3x – 1 Test" on page 41.

3. Since each 35-level tuple-set has an extension via each possible exponent, namely, via 1, 2, 3, ..., we can use an inductive argument beginning with step 2 to arrive at the conclusion that the set of all non-counterexample tuples if counterexamples do not exist, is the same as the set of all non-counterexample tuples if counterexamples exist.

4.We must now ask if counterexamples can exist in $T_A$ in $j$-level tuples, where $j < 35$. The answer is No, because each $j$-level tuple in $T_A$ extends to a 35-level tuple in some other 35-level tuple-set, and $T_A$ is any 35-level tuple-set. From step 2 we know that there is one and only one sequence $S$ of 35-level elements in the sequence of 35-level tuples in any 35-level tuple-set. Thus if any $j$-level tuples are counterexample tuples, they always behave the same as non-counterexample tuples.

6. So we must conclude from step 3 that counterexamples behave the same as non-counterexamples, which is absurd, or else that the set of counterexamples is the empty set. In either case, our conclusion must be that the $3x + 1$ Conjecture is true.. □

. . . . .

**Remark 1**

The reader might enjoy answering — or attempting to answer — question (I), below, which arises from the following facts:

Let the $i$-level tuple $t = <x, ..., 1>$, which is clearly a non-counterexample tuple ($x$ is a non-counterexample). Let $A$ be the $i$-level exponent sequence associated with $t$. Then $t$ is the first $i$-level tuple in the tuple-set $T_A$. The level-1 (first) element of $t$ is $x$, the evel-i element of $t$ is 1.

The level-1 (first) element of the $n$th $i$-level tuple in $T_A$ is given by

$x + (n - 1)(2 \cdot (2^{a_2})(2^{a_3})\ldots(2^{a_i}))$, and

the level-i element of the $n$th $i$-level tuple in $T_A$ is given by $1 + (n - 1)(2 \bullet 3^{i-1})$, where $n \geq 1$ (by parts ($b$) and ($a$) of "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 9).

(I)

How does $T_A$ differ if (1) counterexamples exist, and (2) counterexamples do not exist?

**Remark 2**

The reader might enjoy reading at least some of the possible strategies in "Possible Strategies for Other Proofs of the 3x + 1 Conjecture Using Tuple-sets" on page 15.

**Remark 3**

A wealth of additional results and possible strategies is available in our paper, "Are We Near a Solution to the $3x + 1$ Problem?" on occampress.com.

## Second Proof of the 3*x* + 1 Conjecture

The following proof of the $3x + 1$ Conjecture is based on the idea underlying the proof of ("First Proof of the 3x + 1 Conjecture" on page 11). Like that one, it shows that the set of all tuple-sets (structure and contents) is the same, whether or not counterexamples exist. This implies that, if counterexamples exist, there is a contradiction, hence counterexamples do not exist.

(*Note*: we ask the reader to inform us of the first sentence that the reader believes contains an error, and what that error is.)

1. *Definitions*: an *anchor tuple* is the first $i$-level tuple in an $i$-level tuple-set, where $i \geq 2$. An *anchor* is the $i$-level element of the anchor tuple in an $i$-level tuple-set.

(a) If $x$ is a range element of the $3x + 1$ function, then $x$ is eventually — for some $i \geq 2$ — an anchor

*Proof*: If $x$ exists, then for some $i \geq 2$, $x < 2 \cdot 3^{(i-1)}$. Therefore, $x$ is an anchor (part (a) of "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 9). □

It follows trivially that $x$ is also an anchor for all greater $i$. ("Once an anchor, always an anchor.")

(b) If $x$ is a non-counterexample anchor, then it is a non-counterexample anchor whether or not counterexamples exist.

*Proof*: The arithmetic defining the $3x + 1$ function is not itself a function of the truth or falsity of the $3x + 1$ Conjecture. □

Thus, for example, 13 is a non-counterexample (maps to 1) today, and if the Conjecture is proved true tomorrow, it will be a non-counterexample tomorrow, and if the Conjecture is proved false tomorrow it will *still* be a non-counterexample.

(Actually, the statement (b) holds for non-counterexamples in general, not just non-counterexample anchors.)

2. At this point, it is reasonable to assume that there are two possible sets of anchors: one containing counterexamples if counterexamples exist, and one not containing counterexamples, if counterexamples do not exist.

However this assumption is false.

(1) There is one and only one set of anchors, regardless if counterexamples exist or not.

*Proof*:

(a) The "distance" between consecutive $i$-level elements of an $i$-level tuple-set is $2 \cdot 3^{(i-1)}$ (follows from part (a) in "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 9).

Thus, for example, the distance between the first and second 2-level elements of any 2-level tuple-set having 1 as first element, namely, between the elements 1 and 7, is $2 \cdot 3^{(2-1)} = 2 \cdot 3^1 = 6$. The distance between the second and third elements, that is, between the elements 7 and 13, is likewise 6. Etc.

(b) Each $i$-level anchor is less than $2 \cdot 3^{(i-1)}$ (follows from part (a) in "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 9). Of course, each anchor is greater than 0, by definition of the domain of the $3x + 1$ function.

(c) For each $i \geq 2$, the number of anchors in all $i$-level tuple-sets is $2 \cdot 3^{((i-1)-1)}$ (follows from part (a) in "Lemma 1.0: the "Distance" Functions d(i, i) and d(1, i)" on page 9).

Thus, for example, the number of anchors in all 2-level tuple-sets is $2 \cdot 3^{((2-1)-1))} = 2 \cdot 3^0 = 2$. These anchors are 1 and 5. It is easy to show that 1 is mapped to by all even exponents, and 5 is mapped to by all odd exponents. Those are the only two possibilities for the anchors of 2-level tuple-sets.

The number of anchors in all 3-level tuple-sets is $2 \cdot 3^{((3-1)-1))} = 2 \cdot 3^1 = 6$. These anchors are 1, 5, 7, 11, 13, 17.

(d) The set of $(i + 1)$-level anchors comes into being as follows:

If $a$ is an $i$-level anchor then $a$ is an $(i + 1)$-level anchor, because if $a$ is less than $2 \cdot 3^{(i-1)}$, as it must be if $a$ is an $i$-level anchor, then $a$ is certainly less than $2 \cdot 3^{((i+1)-1)}$.

Since the $i$-level tuple-set element $a + 1 \cdot (2 \cdot 3^{(i-1)})$ is less than $2 \cdot 3^{((i+1)-1)}$, the element is an $(i + 1)$-level anchor.

Since the $i$-level tuple-set element $a + 2 \cdot (2 \cdot 3^{(i-1)})$ is less than $2 \cdot 3^{((i+1)-1)}$, the element is an $(i + 1)$-level anchor.

No other element of an $i$-level tuple-set is less than $2 \cdot 3^{((i+1)-1)}$, and therefore no other element of an $i$-level tuple-set is an $(i + 1)$-level anchor.

The reader can see an example of this increase in anchors from level 2 to level 3 in step 2 (c).

(e) The process we have described is unique. It yields all $i$-level anchors for all $i \geq 2$. There is thus one and only one set of anchors. In other words:

If counterexamples do not exist, then the set of all anchors is exactly the set that results from the process we have described. Call that set *S*.

If counterexamples exist, then the set of all anchors is exactly the set that results from the process we have described. In other words, if counterexamples exist, then the set of all anchors is the same set *S*. □

*3.* Computer tests[1] have shown the Conjecture to be valid for all consecutive odd positive integers up to at least $10^{18} + 1$, which includes all the anchors (each of which is a non-counterexample anchor) from level 2 through at least level 35.

(This is not true in the case of the $3x - 1$ function since the first counterexample in the case of that function is 5.)

Furthermore, there is an infinity of non-counterexample anchors at levels greater than 35. *Proof*: each range element, hence each non-counterexample anchor at levels 2 through 35 is mapped to by an infinity of odd, positive integers (see Lemma 13.0 in our paper, "Are We Near a Solution to the 3x + 1 Problem?", on occampress.com), and range element there is an infinity of

1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$, which is greater than $3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35-1)}$, have been tested and found to be non-counterexamples. These include the set of all 35-level anchors.

them in that infinity of odd, positive integers is also mapped to by an infinity of odd, positive integers, etc.. The fact that there is an infinity of these range elements in each case means that an infinity of them are greater than the largest anchor at level 35. □

The unique process for generating anchors (step 2) then continues to generate anchors for all levels beyond 35. The set of anchors so generated for each level is the same whether or not counterexamples exist (the process is unique). So, in particular, we can regard the process as generating the set of all non-counterexample anchors.

If counterexamples exist, the set of anchors so generated is the same as the set of anchors if counterexamples do not exist. Each anchor is an element of an infinite tuple. Non-counterexample infinite tuples are, by definition, of the form <*x*, ..., 1, 1, 1, ... >, whereas counterexample infinite tuples are of the form <*y*, ... >, with no element equal to 1.

And so if counterexamples exist, then some counterexample anchors are the same as non-counterexample anchors, which is absurd. Therefore the $3x + 1$ Conjecture is true.

Another way of stating our argument here is:

The set of all tuple-sets (structure and contents) is the same, whether or not counterexamples exist. Therefore there is no difference between the set of all counterexamples and the set of all non-counterexamples. Therefore, counterexample tuples behave exactly the same as non-counterexample tuples, which is absurd. Therefore counterexamples do not exist, and the Conjecture is true. □

**Remark 1**

Suppose that the anchors were all and only those odd, positive integers that map to 1. Suppose, further, that if counterexamples exist, they never become anchors. Then there would be no difficulty: the set of anchors would be fixed, whether or not counterexamples existed, and they would all map to 1.

However, the simple argument in step 1 (a) shows that if counterexamples exist, they must eventually be anchors. And so there is, in reality, a difficulty: how to reconcile this fact with the fact that the set of anchors is fixed whether or not counterexamples exist. Our proof, above, shows one way to reconcile this fact.

**Remark 2**

The reader might enjoy reading at least some of the possible strategies in "Possible Strategies for Other Proofs of the 3x + 1 Conjecture Using Tuple-sets" on page 15.

**Remark 3**

A wealth of additional results and possible strategies is available in our paper, "Are We Near a Solution to the $3x + 1$ Problem?" on occampress.com.

## Possible Strategies for Other Proofs of the 3x + 1 Conjecture Using Tuple-sets

**First Possible Strategy**

1. Each range element of the $3x + 1$ function is mapped to by all finite exponent sequences,

plus a possible final "buffer" exponent ("Lemma 7.0: Statement and Proof" on page 37).

2. *Definition*: an *anchor tuple* is the first $i$-level tuple in an $i$-level tuple-set, where $i \geq 2$.

If an odd, positive integer exists, then for some smallest $i$, it must be an element of an anchor tuple. It is, furthermore, an element of all extensions of that anchor tuple, that is, it is an element of an anchor tuple for all greater $i$. (Easy proofs.)

Thus, if counterexamples exist, each counterexample must eventually — for some $i$ — be the first element of an anchor tuple.

3. For each finite exponent sequence $A$, there is one and only one tuple-set $T_A$ associated with $A$.

Thus for each finite exponent sequence $A$, there is one and only one anchor tuple associated with $A$.

4. Assume counterexamples exist. (By computer tests, we know that the minimum counterexample is greater than $10^{18}$, hence our argument here does not also apply to the $3x - 1$ function, where the smallest counterexample is known to be 5.)

Then eventually, that is, for some $i$, there must be anchor tuples that are counterexample tuples. However, since there is one and only one anchor tuple for each tuple-set, and one and only tuple-set associated with each exponent sequence, each of these counterexample anchor tuples takes the place of a non-counterexample anchor tuple.

But then there are exponent sequences that do not map to 1, contradicting Lemma 7.0 (see step 1). Hence counterexamples do not exist, and the $3x + 1$ Conjecture is true.

**Second Possible Strategy**

1. *Definition*: an *anchor tuple* is the first $i$-level tuple in an $i$-level tuple-set, where $i \geq 2$.

.

If an odd, positive integer exists, then for some smallest $i$, it must be an element of an anchor tuple. It is, furthermore, an element of all extensions of that anchor tuple, that is, it is an element of an anchor tuple for all greater $i$. (Easy proofs.)

2. If a counterexample exists, then there is a minimum counterexample, $y_c$. It must be an element of an infinite tuple no element of which is less than $y_c$ (otherwise, $y_c$ would not be the minimum counterexample).

3. Call any tuple the last element of which is less than the first, a *downward-slope* tuple. Clearly, $y_c$ can never be the first element of a downward slope tuple. We will say, $y_c$ must always be the first element of a *non-downward-slope* tuple. In particular, if $y_c$ is an element of an anchor tuple, then in all extensions of the anchor tuple, any sub-tuple of which $y_c$ is the first element, must be a non-downward-slope sub-tuple.

4. But there is one and only one set of tuple-sets. Each finite exponent sequence is associated with one and only one tuple-set. So every downward-slope tuple is associated with a downward-slope exponent sequence; and similarly for every non-downward-slope tuple.

5. But this is true whether or not counterexamples exist. Thus, in particular, if counterexamples do *not* exist, there is nevertheless the same set of non-downward-slope exponent sequences as there is if counterexamples exist.

It appears, then, that there is no difference (in structure and content) between the set of all tuple-sets if counterexamples exist, and the set of all tuple-sets if counterexamples do not exist.

We conclude that counterexamples must be the same as non-counterexamples, which is absurd. Therefore counterexamples do not exist.

**A Failed Strategy**

The following strategy has failed repeated attempts to make it yield a proof of the $3x + 1$ Conjecture.

- Show that for all $i \geq 2$, each $i$-level counterexample tuple is always the second, or third, or fourth, or ..., but never the first $i$-level tuple in any tuple-set. That would mean that no counterexample tuple exists, for if an odd positive integer exists (for example, a counterexample), it must eventually, for some $i$, and for all greater $i$, be an element of a first $i$-level tuple in an $i$-level tuple-set. (Such tuples are called *anchor* tuples.)

The problem is that this assumes that non-counterexample tuples having the same $i$-level exponent sequence as counterexample tuples, are always anchor tuples, and therefore that counterexample tuples are always the second, or third, or fourth, or ... tuple in the tuple-set.. This assumes the truth of what we are trying to prove, and hence is an invalid argument.

## y-Trees: The Structure of the 3x + 1 Function in the "Backward", or Inverse, Direction

### Definition of the *y*-Tree

Let $y$ be a range element of the $3x + 1$ function. Then $y$ and the set of all odd, positive integers that map to $y$ in one or more iterations of the $3x + 1$ function, is a tree called the *y-tree*.

### Properties of the y-Tree

• Let the set of all odd, positive integers that map to a range element $y$ in one iteration of the $3x + 1$ function be called a "spiral".

Then if $x$ is an element of a "spiral", $4x + 1$ is the next larger element.

*Proof*:

1. Assume the root $y$ is mapped to by an even exponent. Then there exists an $x$ such that:

$$\frac{3x+1}{2^{2k}} = y$$

2. Multiply numerator and denominator by $2^2$. Then we have

$$\frac{2^2}{2^2}\frac{(3x+1)}{2^{2k}} = y$$

or

$$\frac{(3(4x+1)+1)}{2^{2k+2}} = y$$

The reader can check that the result is not $y$ if we divide numerator and denominator by $2^1$.

A similar argument applies if $y$ is mapped to by an odd exponent. □

• If $x$ is an element of a "spiral", then the next *smaller* element is $(x-1)/4$, so that $4((x-1)/4) + 1 = x$. In other words, the $4x + 1$ rule applies to all successive elements of a "spiral".

*Proof*:

$$\frac{3\left(\frac{x-1}{4}\right)+1}{2^{a-2}} = \frac{3(x-1)+4}{2^{a-2+2}} = \frac{3x+1}{2^a} = y$$

So the next smaller element maps to $y$ in one iteration of the $3x + 1$ function, and hence is in the "spiral". □

• Each range element of a $y$-tree is mapped to, in one iteration of the $3x$+1 function, either by all even or by all odd exponents. Thus a "spiral" contains an infinity of odd, positive integers.

*Proof*:

In the previous proof, our exponent increases by 2, yielding the next exponent of the same parity. □

(*Note*: another explanation for the above properties of a "spiral" will be found in the section, "Some Properties of the Spiral Drawing" on page 23.)

• The successive elements of a "spiral" are mapped to in accordance with a rule that can be expressed as ... 2, 1, 3, 2, 1, 3, ..., where "2" means "is mapped to by all even exponents", "1" means "is mapped to by all odd exponents", and "3" means "is not mapped to because element is a multiple-of-3, hence not a range element". The proof in brief is the following:

$$\frac{3w+1}{2^2} = x \rightarrow \frac{3(2w+1)+1}{2^1} = 4x+1$$

The reader can substitute the left-hand side of the left-hand equation for $x$ in the right-hand side of the right-hand equation, and work through the algebra to see that the two equations in fact hold. □

The repetition of a multiple-of-3 every third successive element of a "spiral" can be seen from the following. Let $3m$ be a multiple-of-3. Then, by the $4x + 1$ rule described above in this list of properties, we have, for the third successive element after the $3m$ element:

$$4(4(4(3m)+1)+1)+1 = (64)(3)(m)+21$$

Each of the two terms on the right-hand side of the equation are multiples of 3, and so we have our result. □

Finally, we must prove that the next successive element of a "spiral" following a multiple-of-3 is an element that is mapped to by all even exponents.

Let $3m$ be a multiple-of-3. The next successive "spiral" element is $4(3m) + 1$. We ask if there exists a $w$ such that

$$\frac{3w+1}{2^2} = 4(3m)+1$$

Multiplying through by $2^2$ we get

$$3w = 2^2(4)(3m)+2^2-1 = 3U+3$$

Hence $w$ exists. It is equal to $U+1$. □

- Because each "spiral" contains an infinity of range elements, each $y$-tree is infinitely deep.

- There is one and only one possible $y$-tree for each $y$.

*Proof*: Otherwise, a range element could be mapped to, via the same exponent, by two or more odd, positive integers, contrary to the definition of the $3x + 1$ function. □

- Level 1 of a $y$-tree is the set of all odd, positive integers that map to $y$ in one iteration of the $3x + 1$ function;

  Level 2 of a $y$-tree is the set of all odd, positive integers that map to all elements of Level 1 in one iteration of the $3x + 1$ function;

  Level 3 of a $y$-tree is the set of all odd, positive integers that map to all elements of Level 2 in one iteration of the 3x + 1 function;

  etc.

## Can y-Trees and Tuple-sets Be Merged?

The answer is yes. Pick any node $x$ in a $y$-tree. Then the sequence of nodes mapped to $y$ from $x$ is a tuple in a tuple-set.

## Third Proof of the 3x + 1 Conjecture

### Version 1 of Proof

1. If the reader has come here directly from *Note* 1 on the first page of this paper, he or she should read "Statement of Problem" on page 3.

2. The First Level of the 1-tree is the set S = {1, 5, 21, 85, 341, ... }. It contains all odd, positive integers that map to 1 in one iteration of the 3***x*** + 1 function.

All odd, positive integers from 1 to at least $10^{20} + 1$ have been confirmed, by computer test, to be non-counterexamples.

3. Let *T* denote all and only the *y*-trees in the 1-tree such that $1 \le y \le 10^{20} + 1$. (The root *y* of each *y*-tree is a range element of the 3*x* + 1 function. Each *y*-tree contains an infinity of *y*-trees, and so, obviously, not all of these *y*-trees are such that $1 \le y \le 10^{20} + 1$. )

4. Clearly, the portion of the 1-tree that is *T* is what this portion would be if counterexamples did not exist.

But it is also what this portion would be if counterexamples did exist.

5. Consider the remainder of the 1-tree, namely, *not-T.* This is the portion of the 1-tree that is descended from all range elements of the set *S* such that the smallest is greater than $10^{20} + 1$.

It is clear that *not-T* must be the same whether or not counterexamples exist, given that, by "Lemma 3.0: Statement and Proof" on page 31, there is one and only one 1-tree, whether or not counterexamples exist.

6. Therefore if counterexamples exist, they must be the same as non-counterexamples, which is absurd. Hence counterexamples do not exist, and the 3*x* + 1 Conjecture is true.

**Remark 1**

This version passes the 3*x* – 1 Test for the following reason.

In the 3*x* – 1 function, make a list of all consecutive range elements that are known, by computer test, to be non-counterexamples. But this list contains only 1, because the next consecutive range element is 5, and 5 is a counterexample. And so, clearly *T* in the 3*x* – 1 case has no *y*-trees at all besides the redundant 1-tree, contrary to *T* in the 3*x* + 1 case.

### Version 2 of Proof

1. If the reader has come here directly from *Note* 1 on the first page of this paper, he or she should read "Statement of Problem" on page 3.

2. This Version utilizes a spiral drawing on which, remarkably, *any finite sequence of iterations of the 3x + 1 function can be traced*. The spiral is not the same type of "spiral" as associated with *y*-trees, although one of its properties (see"Some Properties of the Spiral Drawing" on page 23) in fact explains why that "spiral" is as it is. Our proposed proof is the same as for "Ver-

sion 1 of Proof" on page 21.

A drawing of the spiral is in preparation. However, it is simple enough that the reader can make his or her own drawing for the time being. Here are the instructions:

**Spiral Drawing**

2.1 Place the pencil point in the center of a white sheet of paper, make a dot, and label it 1.

2.2. Extend a line about 1/4-inch below the 1, make a dot, and label it 2.

2.3. Curve the line to the left 90 degrees, then curve it around until a point is reached that is about 1/4-inch directly above the original 1. Make a point, and label it 3. Continue curving the line around until you reach a point about 1/4-inch below the 2, and label it 4.

2.4. Continue curving the line clockwise, about 1/4-inch from the previous line, until you reach a point 90 degrees clockwise from the 4. Make a point, and label it 5.

2.5. Continue curving the line around until you reach a point directly over the 3, and about 1/4-inch above it. Make a point and label it 6.

Continue curving the line around until you reach a point 90 degrees clockwise from the 6, make a point, and label it 7.

Continue curving the line around until you reach a point about 1/4-inch below the 4, and label it 8.

3. Now go back and draw straight lines starting at each *odd* number, and perpendicular to the spiral at that number, and extending toward the edge of the page.

Do this for all odd numbers from now on.

4. Continue the line around until you reach a point 45 degrees clockwise from 8, and label it 9, and extend a straight line from it, as described in step 3.

Continue curving the line around you reach the straight line extending from 5. Make a point and label it 10

Proceed in this manner until you have labeled, say, 20 numbers.

5. Here is how to trace one iteration of the $3x + 1$ function. We will use the odd, number 3 as our starting number.

5.1 Place the pencil point on 3 and move it out to the point 6. This represents multiplying 3 by 2.

Now move the pencil point clockwise 3 + 1 points on the portion of the spiral you are on. That will bring you to 10, on the straight line extending from 5. You have, in effect, arrived at $3(3) + 1 = 2(3) + 3 + 1$.

Move the pencil point along the straight line to where the line starts. This is the odd number 5, and you have completed one iteration of the $3x + 1$ function.

6. We know by computer test that all consecutive odd, positive integers from 1 through at least $10^{20} + 1$ are non-counterexamples.

Thus the computation of each of these integers terminates on the vertical straight line in your diagram, namely, the one with points labeled 1, 2, 4, 8, ...

7. Now in principle you can place your pencil point on the smallest odd, positive integer that has not been computer-tested, and carry out as many iterations of the function as you wish.

The key point is that, *whether or not counterexamples exist, there is one and only one set of iterations that can be carried out on the spiral*. At no point in your movements of the pencil, is there a possibility of two different next movements. There is always only one. (Certainly the fact is obvious for the part of the spiral containing all consecutive points from 1 through $10^{20} + 1$.)

This is the equivalent of "Lemma 3.0: Statement and Proof" on page 31.

Therefore if counterexamples exist, they are the same as non-counterexamples, which is absurd. Hence the Conjecture is true.

**Some Properties of the Spiral Drawing**

• *Each odd, positive integer is in the drawing*. Thus there is no chance that the drawing could contain only non-counterexamples, with the counterexamples being "elsewhere".

• The following properties are obvious, yet we feel that the reader should keep them in mind:
For each $k \geq 1$, there are $2^k$ odd, positive integers between $2^k$ and $2^{k+1}$.
Half of these are consecutive odd numbers. They alternate with even numbers.

• The drawing explains why the "spiral"s that we introduced in the section "Properties of the y-Tree" on page 18, are as they are:

(1) Each odd, positive integer $n$ in the spiral is the start of a straight line that is in principle infinitely long.

(2) Points on the line are labeled $n$, $2n$, $4n$, $8n$, $16n$, ... Thus, for example, points on the line starting with $n = 7$ are 7, 14, 28, 56, 112, ...

(3) There is a first point $k$ on the line such that there exists an odd, positive integer $j$ such that $3j + 1 = k$. Thus, in the case of the line starting with $n = 7$, that first point $k$ is 28, because $3(9) + 1 = 28$ ($j$ in this case is 9).

(4) When $k$ is divided by one of $2^1$ or $2^2$, the result is our original $n$. Thus here, 28 divided by $2^2$ gives us our starting number 7.

(5) The number $4j + 1$ gives us another number, namely, $j'$, such that $3j' + 1$ is another number, namely, $k'$ on the straight line. In our case, $4(9) + 1 = 37$ (which is $j'$) and $3(37) + 1 = 112$ (which is $k'$).

(6) When $k'$ is divided by $2^2$ times the previous power of 2, or, in our case, by $2^2 2^2 = 16$, we get our starting number 7.

(7) In this manner, we generate all the successive numbers $j$, $j'$, $j''$, ... that constitute the elements of a "spiral". Each number maps to our original number $n$ by $2^2$ times the previous power that mapped to $n$.

**A Possible Trigonometric Proof of the 3*x* + 1 Conjecture**

The spiral drawing presents us with a tantalizing challenge, namely, that of obtaining a geometric — a trigonometric — proof of the Conjecture, based on the movement of an infinitely-long rod, one end fastened at the origin, and moving from odd number to odd number in accordance with successive iterations of the function.

Each position of the rod at an odd number, is at a unique angle. So in effect we are summing unique angles. Or summing unique fractions of a circle (which is how an angle can be regarded – here we measure fractions of a circle in the clockwise direction, starting with the rod pointing straight down). We ask if each such sum always terminates with the rod pointing straight down (which is equivalent to the function having yielded 1).

Some examples follow. The rod points straight down at the conclusion of each sum.

Starting at 3, we have 1/2 + 3/4 + 3/4 = 2; or, in terms of 1/4's: 2/4 + 3/4 + 3/4 = 2;

Starting at 5, we have 1/4 + 6/8 = 1; or, in terms of 1/8's: 2/8 + 6/8 = 1;

Starting at 17, we have 1/16 + 9/16 + 10/16 + 3/4 = 2; or, in terms of 1/16's: 1/16 + 9/16 + 10/16 + 12/16 = 2.

We want to prove that *each sufficiently long sequence of successive partial rotations of the rod — each sufficiently long sequence of successive fractions of a circle — will result in the rod pointing straight down*.

## Version 3 of Proof

1. If the reader has come here directly from *Note* 1 on the first page of this paper, he or she should read "Statement of Problem" on page 3.

2. If counterexamples do not exist, then the 1-tree contains all and only the odd, positive integers.

3. If counterexamples exist, then the 1-tree contains only a proper subset of the odd, positive integers.

4. Therefore there are two possible 1-trees.

5. However, "Lemma 3.0: Statement and Proof" on page 31 states that there is one and only one possible 1-tree, whether or not counterexamples exist. In other words, Lemma 3.0 states:

5.1 If counterexamples do not exist, then the 1-tree contains the elements of a set *J*.
5.2 If counterexamples exist, then the 1-tree contains the elements of the same set *J*.

5.3 Therefore there is one and only one possible 1-tree, whether or not counterexamples exist..

6. Steps 4 and 5.3 are a contradiction. It is brought about by the possibility that counterexamples can exist. Therefore they don't exist, and we have a proof of the Conjecture.

**Remark**

Our proof passes the $3x - 1$ Test, because step 2 is not relevant in the $3x - 1$ case, since counterexamples are known to exist (the smallest is 5). Therefore, when applied to the $3x - 1$ case, step 4 would have to read "There is only one possible 1-tree." Hence there is no contradiction between this modified step 4 and step 5. So our proof does not also prove the (false) $3x - 1$ Conjecture.

.

## References

Lagarias, J., (1985), “The 3*x* + 1 Problem and Its Generalizations”, *American Mathematical Monthly*, **93**, 3-23.

Wirsching, Günther J.. *The Dynamical System Generated by the 3n + 1 Function*, Springer-Verlag, Berlin, Germany, 1998.

# Appendix A — Statement and Proof of Each Lemma

## Lemma 1.0: Statement and Proof

*Definition*: let $T_A$ be an $i$-level tuple-set, where $i \geq 2$. Let $t(r)$, $t(s)$ denote tuples consecutive at level $i$, with $r < s$ in the natural ordering of tuples by first elements. Let $t(r)(h)$, $t(s)(h)$ denote the elements of $t(r)$, $t(s)$ at level $h$, where $1 \leq h \leq i$. Then we call $|t_{(s)(h)} - t_{(r)(h)}|$ the *distance* between $t(r)$ and $t(s)$ at level $h$. We denote this distance by $d(h, i)$ and call $d$ the *distance functions* (one function for each $h$).

### Lemma 1.0

(a) Let $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, *be a sequence of exponents, and let* $t_{(r)}$, $t_{(s)}$ *be tuples consecutive at level i in* $T_A$. *Then d(i, i) is given by*:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) *Let* $t_{(r)}$, $t_{(s)}$ *be tuples consecutive at level i in* $T_A$. *Then d(1, i) is given by:*

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3})\ldots(2^{a_i})$$

Thus, in “Fig. 1. Part of the tuple-set TA associated with the sequence A = {1, 1, 2}” on page 8, the distance $d(3, 3)$ between $t_{8(3)} = 35$ and $t_{4(3)} = 17$ is $2 \cdot 3^{(3-1)} = 18$. The distance $d(1, 2)$ between $t_{12(1)} = 23$ and $t_{10(1)} = 19$ is $2 \cdot 2^1 = 4$.

**Proof:**

The proof is by induction.

**Proof of Basis Step for Parts (a) and (b) of Lemma 1.0:**

Let $t_{(r)}$ and $t_{(s)}$ be the first and second 2-level tuples, in the standard linear ordering of tuples based on their first elements, that are consecutive at level $i = 2$ in the 2-level tuple-set $T_A$, where $A = \{a_2\}$. (See Fig. 2 (1).)

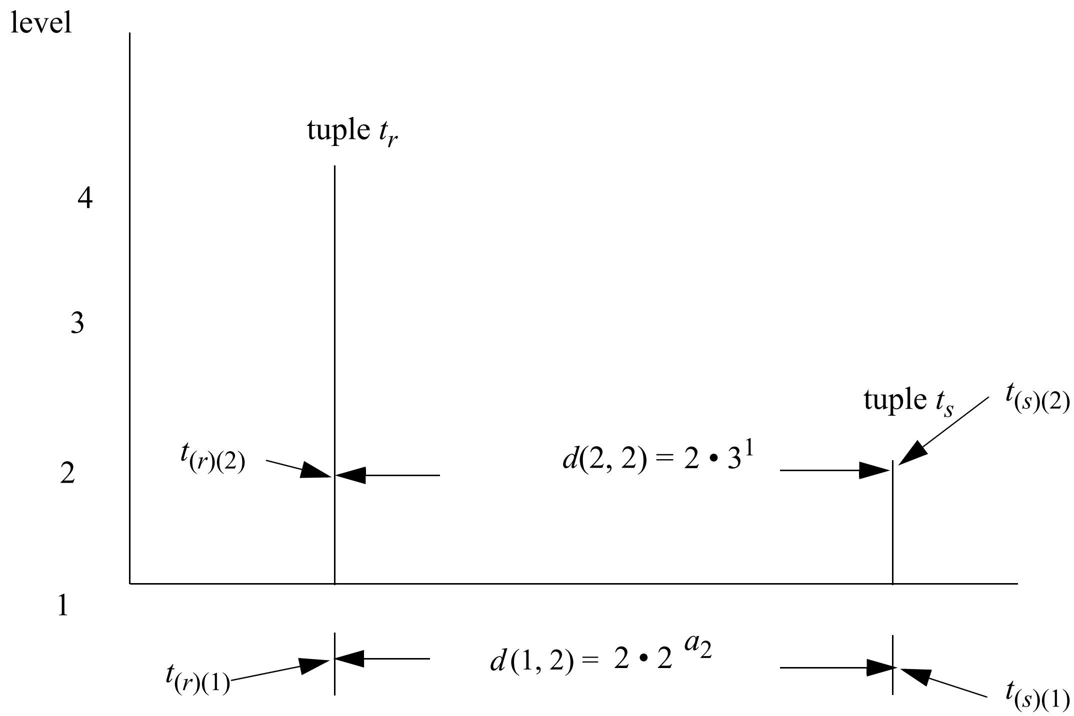


Fig. 2 (1). Illustration for proof of Basis Step of Lemma 1.0.

Then we have:

$$\frac{3t_{(r)(1)} + 1}{2^{a_2}} = t_{(r)(2)} \tag{1.1}$$

and since, by definition of $d(1, 2)$,

$$t_{(s)(1)} = t_{(r)(1)} + d(1, 2)$$

we have:

$$\frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}} = t_{(s)(2)} \tag{1.2}$$

Therefore, since, by definition of $d(i, i)$,

$$t_{(r)(2)} + d(2, 2) = t_{(s)(2)}$$

we can write, from (1.1) and (1.2):

$$\frac{3t_{(r)(1)} + 1}{2^{a_2}} + d(2, 2) = \frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}}$$

By elementary algebra, this yields:

$$2^{a_2}d(2, 2) = 3 \cdot d(1, 2)$$

Now $d(2, 2)$ must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level $i$, it must be the smallest such even number, whence it follows that $d(2, 2)$ must = 3 • 2, and necessarily

$$d(1, 2) = 2 \bullet 2^{a_2}$$

A similar argument establishes that $d(2, 2)$ and $d(1,2)$ have the above values for every other pair of tuples consecutive at level 2.

Thus we have our proof of the Basis Step for parts (a) and (b) of Lemma 1.0.

**Proof of Induction Step for Parts (a) and (b) of Lemma 1.0**

Assume the Lemma is true for all levels $j$, $2 \leq j \leq i$ and that $T_A$ is an $i$-level tuple-set, where A = $\{a_2, a_3, ..., a_i\}$.

Let $t_{(r)}$ and $t_{(s)}$ be tuples consecutive at level $i$, and let $t_{(r)}$ and $t_{(f)}$ be tuples consecutive at level $i$ +1. (See Fig. 2 (2).)

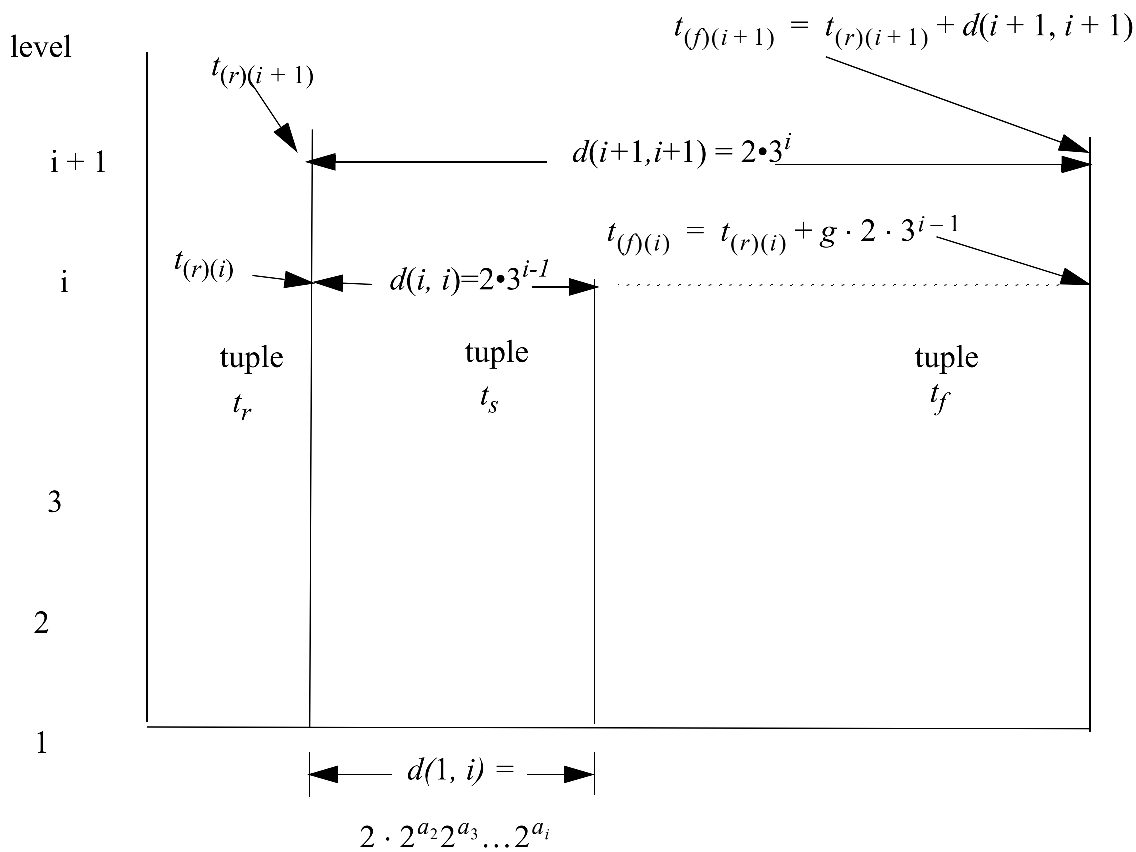


Fig. 2 (2). Illustration for proof of Induction Step of Lemma 1.0.

Then we have:

$$\frac{3t_{(r)(i)} + 1}{2^{a_{i+1}}} = t_{(r)(i+1)}$$

and since, by definition of $d(i, i)$,

$$t_{(f)(i)} = t_{(r)(i)} + g \cdot d(i, i)$$

for some $g \geq 1$, we have:

$$\frac{3(t_{(r)(i)} + g \cdot d(i, i)) + 1}{2^{a_{i+1}}} = t_{(f)(i+1)}$$

Thus, since

$$t_{(r)(i+1)} + d(i+1, i+1) = t_{(f)(i+1)}$$

we can write:

$$\frac{3t_{t_{(r)(i)}} + 1}{2^{a_{i+1}}} + d(i+1, i+1) = \frac{3(t_{t_{(r)(i)}} + gd(i, i)) + 1}{2^{a_{i+1}}}$$

This yields, by elementary algebra:

$$2^{a_{(i+1)}}d(i+1, i+1) = 3 \cdot gd(i, i)$$

As in the proof of the Basis Step, $d(i+1, i+1)$ must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level $i+1$, it must be the smallest such even number. Thus $d(i+1, i+1) = 3 \bullet d(i, i)$, and

$$g \cdot d(i, i) = 2^{a_{i+1}}d(i, i) \quad .$$

Hence

$$g = 2^{a_{i+1}}$$

Now $g$ is the number of tuples consecutive at level $i$ that must be "traversed" to get from $t_{(r)}$ to $t_{(f)}$. By inductive hypothesis, d(1, $i$) for *each* pair of these tuples is:

$$d(1, i) = 2 \cdot 2^{a_2} \cdot 2^{a_3} \cdot \quad \ldots \cdot 2^{a_i}$$

hence, since

$$g = 2^{a_{i+1}}$$

we have

$$d(1, i+1) = d(1, i) \cdot 2^{a_{i+1}} .$$

A similar argument establishes that $d(i+1, i+1)$ and $d(1, i+1)$ have the above values for every pair of tuples consecutive at level $i+1$.

Thus we have our proof of the Induction Step for parts (a) and (b) of Lemma 1.0. The proof of Lemma 1.0 is completed. □

## Lemma 2.0: Statement and Proof

*Assume a counterexample exists. Then for all $i \geq 2$, each i-level tuple-set contains an infinity of i-level counterexample tuples and an infinity of i-level non-counterexample tuples.*

**Proof:**

1. Assume a counterexample exists. Then:

There is a countable infinity of non-counterexample range elements.

*Proof*: Each non-counterexample maps to a range element, by definition of *range element*.

Each range element is mapped to by an infinity of elements ( "Lemma 6.0: Statement and Proof" on page 35). A countable infinity of these are range elements (proof of "Lemma 7.0: Statement and Proof" on page 37).

There is a countable infinity of counterexample range elements.

*Proof*: same as for non-counterexample case.

2. For each finite exponent sequence $A$, and for each range element $y$, non-counterexample or counterexample, there is an $x$ that maps to $y$ via $A$ possibly followed by a buffer exponent ("Lemma 7.0: Statement and Proof" on page 37). The presence of the buffer exponent does not change the fact that $x$ is the first element of a tuple associated with the exponent sequence $A$. □

## Lemma 3.0: Statement and Proof

*There is one and only one possib1e 1-tree, whether or not counterexamples exist.*
*(In other words,*
*If counterexample do not exist, then the 1-tree contains all and only the elements of a set J;*
*if counterexamples exist, then the 1-tree contains all and only the elements of the same set J.)*

**Short Proof:**

1. "Once a non-counterexample, always a non-counterexample." *Proof*: the proof is a generalization of our canonical example, 13: "13 is a non-counterexample today; if the $3x + 1$ Conjecture is proved true tomorrow, it will be a non-counterexample; and if the Conjecture is proved *false* tomorrow it will *still* be a non-counterexample". Similar facts follow for all non-counterexamples by definition of the $3x + 1$ function (no odd, positive integer can map to two or more different values in one iteration of the function.) □

2. "Once a non-counterexample, always a non-counterexample" can be expressed as the statement of Lemma 3.0. □

**Longer Proof:**

**Proof of "There is one and only one possible 1-tree..."**

The 1-tree =

(Level 1 = {odd, positive integers $y$ | $y$ maps to 1 in one iteration of the $3x + 1$ function}[1]) ∪
(Level 2 = {odd, positive integers $y$ | $y$ maps to an element of Level 1 in one iteration of the $3x + 1$ function}) ∪
(Level 3 = ({odd, positive integers $y$ | $y$ maps to an element of Level 2 in one iteration of the $3x + 1$ function} ∪
...

Since 1 is a range element of the $3x + 1$ function, it is the root of a $y$-tree (in this case, the 1-tree). Each $y$-tree has several basic, well-defined properties. (For full details, and references to the elementary proofs, see "Properties of the y-Tree" on page 18 and "y-Trees: The Structure of the 3x + 1 Function in the "Backward", or Inverse, Direction" on page 18):

Each $y$ is mapped to by an infinity of odd, positive integers in one iteration of the $3x + 1$ function. We call this infinity of odd, positive integers, a "spiral".

If $x$ is an element of a "spiral", then $4x + 1$ is the next larger element.

Each "spiral" contains an infinity of range elements, and an infinity of multiples of 3, which are not range elements because they are not mapped to by any odd, positive integer.

Each "spiral" element maps to $y$ (in one iteration of the $3x + 1$ function), by either all odd exponents, or by all even exponents.

The sequence of these types of "spiral" elements is given by a rule that can be expressed as ... 2, 1, 3, 2, 1, 3, ..., where "2" means "is mapped to by all even exponents", "1" means "is mapped to by all odd exponents", and "3" means "is not mapped to because element is a multiple-of-3, hence not a range element".

Because of the infinity of range elements in each "spiral", it is clear that the structure of each $y$-tree is the result of an infinitely recursive process. Thus each $y$-tree is infinitely deep.

**Proof of "...whether or not counterexamples exist**

If an odd, positive integer $x$ maps to 1 (that is, if $x$ is a non-counterexample, hence an element of the 1-tree), then it maps to 1 regardless if counterexamples exist or not. Informally, we say, "Once a non-counterexample, always a non-counterexample." Thus, for example,

13 maps to 1 today;
if the $3x + 1$ Conjecture is proved true tomorrow it will still map to 1;
if the $3x + 1$ Conjecture is proved *false* tomorrow it will *still* map to 1.

If it were not the case that "Once a non-counterexample, always a non-counterexample", some odd, positive integers could map to two different odd, positive integers, contrary to the definition of the $3x + 1$ function. □

## Remark 1

Some readers claim that the Lemma is trivial, "unnecessary". But this claim is based on a false assumption. These readers assume (correctly) that 1 is mapped to by all exponents of only

1. This set is $S = \{1, 5, 21, 85, 341, ... \}$.

one parity, but they assume (incorrectly) that there are nevertheless two possibilities: (1) the 1-tree contains all odd, positive integers, or (2) the 1-tree contains only a proper subset of the odd, positive integers.

However, that implies that a given range element, although it is mapped to by one and only one exponent (in one iteration of the $3x + 1$ function), nevertheless can be mapped to by two different odd, positive integers via that one exponent! But that is impossible, given the definition of the $3x + 1$ function.

In actuality, we know that 1 is mapped to by all even exponents. There is no possibility that it might be mapped to by any odd exponents. Furthermore, for each range element $y$ (and 1 is a range element) and each (even) exponent, $y$ is mapped to by one and only odd, positive integer in one iteration of the $3x + 1$ function.

Hence there is one and only one possible 1-tree, whether or not counterexamples exist.

### Remark 2

The Lemma passes the $3x - 1$ Test. The reason is that the Lemma asserts that there is one and only one possible 1-tree, whether or not counterexamples exist. At the time of this writing, no counterexample to the 3x + 1 Conjecture is known, even though all consecutive odd, positive integers between 1 and at least $10^{18} - 1$ have been found, by computer test[1], to be non-counterexamples. But a counterexample to the $3x - 1$ Conjecture is known (the smallest is 5), and so it is emphatically not true that there is one and only one possible 1-tree for the $3x - 1$ function, whether or not counterexamples exist. If no counterexamples to the $3x - 1$ Conjecture existed, the 1-tree for the $3x - 1$ function would certainly be different than the existing one.

### Remark 3

The Lemma statement is, of course, very counter-intuitive. Even we who first stated it, and then proved it, found ourselves spending time trying to understand how it could be true.

But it is true, as the reader can check by going over the proof.

## Lemma 4.0: Statement and Proof

*No multiple of 3 is a range element.*

**Proof :**

If

$$\frac{3x+1}{2^a} = 3m$$

then $1 \equiv 0 \bmod 3$, which is false. □

1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers less than $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$, which is greater than $3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35-1)}$, have been tested and found to be non-counterexamples.

## Lemma 5.0: Statement and Proof

*Each odd, positive integer (except a multiple of 3) is mapped to by a multiple of 3 in one iteration of the 3x + 1 function.*

**Proof:**

Since the domain of the $3x + 1$ function is the odd, positive integers, the only relevant generators are $3(2k + 1)$, $k \geq 0$. We show that, for each odd, positive integer $y$ not a multiple of 3, there exists a $k$ and an $a$ such that

$$y = \frac{(3(3(2k+1))+1)}{2^a} \quad , \tag{11.1}$$

where $a$ is necessarily the largest such $a$, since $y$ is assumed odd.

Rewriting (11.1), we have:

$$y2^{a-1} - 5 = 9k \ . \tag{11.2}$$

Without loss of generality, we can let $y \equiv r \bmod 18$, where $r$ is one of 1, 5, 7, 11, 13, or 17 (since $y$ is odd and not a multiple of 3, these values of $r$ cover all possibilities mod 18). Or, in other words, for some $q$, $r$, $y = 18q + r$. Then, from (11.2) we can write:

$$18(2^{a-1})q + (2^{a-1})r - 5 = 9k \ . \tag{11.3}$$

Since the first term on the left-hand side is a multiple of 9, $(2^{a-1})r - 5$ must also be if the equation is to hold. We can thus construct the following table. (Certain larger $a$ also serve equally well, but those given suffice for purposes of this proof.)

**Table 2: Values of *r*, *a*, for Proof of Lemma**

| $r$ | $a$ | $(2^{a-1})r - 5$ |
|---|---|---|
| 1 | 6 | 27 |
| 5 | 1 | 0 |
| 7 | 2 | 9 |
| 11 | 5 | 171 |
| 13 | 4 | 99 |
| 17 | 3 | 63 |

Given $q$ and $r$ (hence $y$), we can use $r$ to look up $a$ in the table, and then solve (11.3) for integral $k$, thus producing the multiple of 3 that maps to $y$ in one iteration of the $3x + 1$ function. □

## Lemma 6.0: Statement and Proof

*(a) Each range element y is mapped to, in one iteration of the 3x + 1 function, by every exponent of one parity only. Furthermore,*

*(b) For each of the two parities, there exists a range element that is mapped to by every exponent of that parity.*

**Proof of part (a)**:

Steps 1 and 2 are slightly edited versions of proofs by Jonathan Kilgallin and Alex Godofsky. Any errors are entirely ours. Step 3 is a slightly edited version of a proof by Michael Klipper. Any errors are entirely ours.

1. We first show that if $y$ is mapped to by the exponent $a$, then $y$ is mapped to by every exponent greater than $a$ that is of the same parity as $a$.

Let $y$ be a range element, and let $x$ map to $y$ via the exponent $a$. Then

$$y = \frac{3x+1}{2^a}$$

We wish to show that there exists an $x'$ such that $x'$ maps to $y$ via the exponent $2^{a+2}$. That is, we wish to show that there exists an $x'$ such that

$$y = \frac{3x'+1}{2^{a+2}}$$

Rewriting, this gives

$$x' = \frac{2^{a+2}y-1}{3}$$

Substituting for $y$ yields

$$x' = \frac{2^{a+2}\left(\frac{3x+1}{2^a}\right)-1}{3}$$

Simplifying, this gives $x' = 4x + 1$. Since $x$ is an odd, positive integer, clearly $x'$ is as well.

Thus, by induction, if $y$ is mapped to via the exponent $a$ , it is mapped to by every exponent greater than $a$ of the same parity. □

2. Next we show that if $y$ is mapped to by the exponent $a$ which is greater than 2, then it is mapped to by every exponent less than $a$ that is of the same parity as $a$.

Let $y$ be a range element, and let $x$ map to $y$ via the exponent $a$ where $a > 2$. Then

$$y = \frac{3x + 1}{2^a}$$

We wish to show that there exists an $x'$ such that $x'$ maps to $y$ via the exponent $2^{a-2}$. That is, we wish to show that there exists an $x'$ such that

$$y = \frac{3x' + 1}{2^{a-2}}$$

Rewriting, this gives

$$x' = \frac{2^{a-2}y - 1}{3}$$

Substituting for $y$ yields

$$x' = \frac{2^{a-2}\left(\frac{3x + 1}{2^a}\right) - 1}{3}$$

Simplifying yields

$$x' = \frac{x - 1}{4}$$

3. We must now show that $x' = (x - 1)/4$ is an odd, positive integer. This means we must show that $(x - 1) = 4(2k + 1)$ for some $k \geq 0$, or that $(x - 1) = 8k + 4$, hence that $x = 8k + 5$. Thus, we must prove $x \equiv 5 \bmod 8$.

We know that $x$ maps to $y$ via $a$, where $a \geq 3$. Thus, $y = (3x + 1)/2^a$, so $2^a y = 3x + 1$. Because $a \geq 3$, $2^a y$ is a multiple of 8. Thus, $(3x + 1) \equiv 0 \bmod 8$, and $3x \equiv 7 \bmod 8$. This readily implies $x \equiv 5 \bmod 8$.

4. Thus, by induction, if $y$ is mapped to via the exponent $a$, where $a > 2$, then it is mapped to by every exponent less than $a$ of the same parity. □

**Proof of part (b):**

We now show that for each of the two parities there exists a range element that is mapped to by every exponent of that parity.

1. Fix a range element $y$, and suppose that $x$ maps to $y$ via the exponent $a$. Now $a$ is either even or odd, hence $a = 2n + h$, where $h$ is either 0 or 1. Since $y = (3x + 1)/2^a$, it follows that $(2^a)y = 3x+1$. Reduce the equation mod 3, and we get $(2^h)y \equiv 1 \bmod 3$, by the following reasoning: $(2^a)y \equiv 1 \bmod 3$ implies $(2^{2n+h})y \equiv 1 \bmod 3$ implies $2^{2n}\, 2^h y \equiv 1 \bmod 3$ implies $2^h y \equiv 1 \pmod 3$ because $2^{2n} = 4^n \equiv 1 \bmod 3$.

2. Since $y$ is fixed, either $y \equiv 1$ or $y \equiv 2 \bmod 3$. (We know that $y$, a range element, is not a multiple of 3 by "Lemma 4.0: Statement and Proof" on page 33). If $y \equiv 1 \bmod 3$, then we have $2^h(1) \equiv 1 \bmod 3$, which implies that $h$ must be 0. If $y \equiv 2 \bmod 3$, then we have $(2^h)(2) \equiv 1 \bmod 3$, implying that $h$ must be 1. □

## Lemma 7.0: Statement and Proof

*Let y be a range element of the 3x + 1 function. Then for each finite exponent sequence A, there exists an x that maps to y via A possibly followed by a "buffer" exponent. (For example, if y is mapped to by even exponents, and our exponent sequence A ends with an odd exponent, then there must be an even "buffer" exponent following A, and similarly if y is mapped to by odd exponents and A ends with an even exponent. However, there are other cases in which a "buffer" exponent is required.)*

**Proof:**

1. Each range element $y$ is mapped to by all exponents of one parity ("Lemma 6.0: Statement and Proof" on page 35).

2. Each range element $y$ is mapped to by a multiple of 3 ("Lemma 5.0: Statement and Proof" on page 34).

Each range element is mapped to by an infinity of range elements ("Lemma 5.0: Statement and Proof" on page 34).

3. Let $y$ be a range element and let $S = \{s_1, s_2, s_3, ... \}$ be the set of all odd, positive integers that map to $y$ in one iteration of the $3x + 1$ function. In other words, $S$ is the set of all elements in a "spiral". Furthermore, let the $s_i$ be in increasing order of magnitude. It is easily shown that $s_{i+1} = 4s_i + 1$.

(In Fig. 18, $y = 13$, $S = \{17, 69, 277, 1109, ... \}$

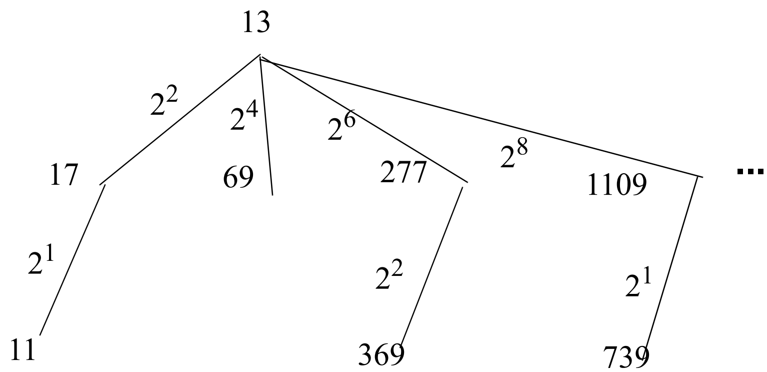


Fig. 18

(*Note*: for a graphical representation of part of the tree having 1 as its root instead of 13, see "Recursive "Spiral"s: The Structure of the 3x + 1 Function in the "Backward", or Inverse, Direction" in our paper, "Are We Near a Solution to the 3x + 1 Problem?", on occampress.com.)

4. If $s_i$ is a multiple of 3, then $4s_i + 1$ is mapped to, in one iteration of the $3x + 1$ function, by all exponents of even parity.

To prove this, we need only show that $x$ is an integer in the equation

$$4(3u) + 1 = \frac{3x + 1}{2^2}$$

Multiplying through by $2^2$ and collecting terms we get

$$(48u) + 4 = 3x + 1$$

and clearly $x$ is an integer.

5. If $s_j$ is mapped to by all even exponents, then $4s_j + 1$ is mapped to, in one iteration of the $3x + 1$ function, by all exponents of odd parity.
(The proof is by an algebraic argument similar to that in step 4.)

6. If $s_k$ is mapped to by all odd exponents, then $4s_k + 1$ is a multiple of 3.
(The proof is by an algebraic argument similar to that in step 4.)

7. The Lemma follows by an inductive argument that we now describe.

Let $y$ be a range element. It is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. As a consequence of steps 1 through 6, we can represent an infinite sub-sequence of the sequence by

...3, 2, 1, 3, 2, 1, ...

where
"3" means "this odd, positive integer is a multiple of 3 and therefore is not mapped to by any odd, positive integer";
"2" means "this odd, positive integer is mapped to by all even exponents";
"1" means "this odd, positive integer is mapped to by all odd exponents".

Each type "2" and type "1" odd, positive integer is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. We can represent an infinite sub-sequence of the sequence by

...3, 2, 1, 3, 2, 1, ...

where each integer has the same meaning as above.

Temporarily ignoring the case in which a buffer exponent is needed, it should now be clear that, for each range element $y$, and for each finite sequence of exponents $B$, we can find a finite path down through the infinitary tree we have just established, starting at the root $y$. The path will end in an odd, positive integer $x$. Let $A$ denote the path $B$ taken in reverse order. Then we have our result for the non-buffer-exponent case. The buffer-exponent case follows from the fact that the buffer exponent is one among an infinity of exponents of one parity. Thus $y$ is mapped to by an infinite sequence of odd, positive integers. We then simply apply the above argument.. □

# Appendix B — On The 3*x* – 1 Test

There is no question but that what we have called the “3*x* – 1 Test” has helped us to recognize errors in our proposed proofs of the Conjecture. (In this Test, we ask if our proof also proves the 3*x* – 1 Conjecture. If it does, then it is claimed that our proof is in error, because there are known counterexamples to the 3*x* – 1 Conjecture, the smallest of which is 5.)

However, we make the following counterargument to this claim, and hence to the validity of the 3*x* – 1 Test:

The question we ask those who use the 3*x* – 1 Test to claim a proof of ours has an error, is: “Suppose you didn’t know about the 3*x* – 1 function, and hence about the 3*x* – 1 Test. What would your criticism be then?”

*A proof must stand or fall on its own terms.* Mathematical logic, and, in particular, computerized proof-checking, would face an insurmountable obstacle if each proof of a conjecture required that the author, or the proof-checker, find all and only the related conjectures (whatever “related” means) that are known to be false, and then prove that the proof in question did not also prove one of those conjectures.

Furthermore, even if our proof of the Conjecture proves none of the related conjectures, each of which is known to be false, *that in no way confirms the validity of our proof*! There may be other infinite cycles in one or more of the 3*x* + 1 -like functions, and/or there may be computations that never yield 1, although they are not infinite cycles.

Our proofs in this paper apply to a function having the property that all consecutive odd, positive integers between 1 and $10^{18} - 1$ are known, by computer test, to be non-counterexamples. No counterexample is known.

The 3*x* – 1 function, on the other hand, has the property that only the first two consecutive odd, positive integers, starting at 1, are known, by computer test, to be non-counterexamples. The first counterexample is 5. (In passing, we remark that it might be significant that the first two consecutive odd, positive integers that are non-counterexamples, namely, 1 and 3, are both integers that map directly to 1: $(3(1) - 1)/2^1 = 1$, and $(3(3) - 1)/2^3 = 1$. The counterexample 7 is the first element of the tuple <7, 5>, which is the first tuple in the tuple-set $T_A$, where $A = \{2\}$.)

It would be of considerable importance if it could be shown that a counterexample to the Conjecture for a 3*x* + 1-like function, must always be small, for example, an element of the first tuple in a one-exponent tuple-set. That in itself would give us a proof of the 3*x* + 1 Conjecture.

So, it seems to us entirely possible that a proof can be valid when applied to the 3*x* + 1 function, and invalid when applied to the 3*x* – 1 function.

# Appendix C — For Professional Mathematicians Only

## Understandable Reluctance of Mathematicians to Read This Paper

There has been an understandable reluctance on the part of professional mathematicians to give serious attention to this paper, or its predecessors  It seems clear to us that the main reason is mathematicians' difficulty in believing that such an extraordinarily difficult problem can have been solved by a non-mathematician (our degree is in computer science, and we have spent most of our working life doing research in the computer industry).  This skepticism is reinforced by the fact that there have been many false claims of solutions to the $3x + 1$ Problem, the overwhelming majority of which having been made by non-mathematicians.

But we must point out that the occasionally-heard remark, "Nothing of importance in mathematics has ever come from outside the university", is, in fact, false, considering that some of the best of the best worked entirely outside the university — Descartes, Pascal, Fermat, Leibniz, and Galois, to name only the most famous.

We must also not fail to mention another reason for mathematicians' reluctance to read this paper, and that is the online presence of obsolete criticisms of the paper. For example, Stack Overflow has a website  containing criticisms of a proof in a 2015 version of the paper. Not only were the criticisms false, but the proof that was criticized has long since been removed from the paper. This website appears next to the website containing this paper, and thus unquestionably discourages potential readers — especially mathematicians — from reading this paper.  Yet despite many pleas on our part, the managers of the website have refused to delete the criticisms or to add a note to the website stating that the criticisms do not apply to the current version of the paper.  Nor have they explained to us the reason for their refusals.  Apparently we have no recourse in this matter, except to encourage others to boycott the Stack Overflow websites, and to write to the organization explaining the reason for the boycott.  The email address is team@stackoverflow.com, the item no. is 201708202111462820.

It appears that the managers of the above website have no experience of actually doing research.  They believe that if a paper is published online and contains an error, that means that the author is incapable of correcting the error, and that his underlying ideas do not deserve any attention.  But errors are almost inevitable in the course of attempting to solve very difficult  problems.  We remind the reader that Wiles' first proposed proof of the Taniyama–Shimura–Weil Conjecture in the early 90s, which implied a proof of Fermat's Last Theorem, contained an error that took Wiles, with the help of the mathematician Richard Taylor,  more than a year to repair.  The important question obviously was, Do the underlying ideas in this paper offer hope for correcting the error?  And the answer was yes.

We have been struck by the eagerness with which readers of this paper look for anything they can regard as an error, and the indifference they display to understanding, and thinking about, the underlying ideas.

## If You Do Not Accept Our Proofs of the 3x + 1 Conjecture...

If you do not accept our proofs of the Conjecture, or any of the possible strategies for a proof that are set forth in the above appendices, we urge you to at least peruse our paper, "Are We Near a Solution to the $3x + 1$ Conjecture?" on occampress.com.  This paper contains a wealth of results, insights, possible strategies for a proof, plus a section on what we have called "$3x + 1$-like functions".  We will welcome comments.

We are confident that at least two publishable, significant papers can be produced from the material in our $3x + 1$ papers, and that this is true *even if* the proof of the Conjecture in the present paper and all the possible strategies in the appendices, are faulty and cannot be repaired.

We feel that the two structures we have discovered that underlie the $3x + 1$ function, namely, tuple-sets and recursive "spiral"s, are of fundamental importance, and should be brought to the attention of the entire $3x + 1$ research community.

## Difficulty, So Far, In Getting This Paper Published

Not surprisingly, so far, no journal that we know of is willing to even consider this paper for possible publication. The reason seems to be that editors cannot believe that such a difficult problem might have been solved by a non-mathematician.

## Incentives for Mathematicians to Take This Paper Seriously and to Spread the Word About It

Unquestionably, if this paper contains a solution to the $3x + 1$ Problem, or can easily be modified to contain a solution, considerable prestige will be gained by the first mathematicians who promote the paper. Of course, mathematicians who believe that the proof of the Conjecture is correct, and/or that at least one of the possible strategies in the appendices look promising, but do not want to risk their reputations by saying so, especially given that the author of the paper is not an academic mathematician, will not be recipients of that prestige. We are offering three incentives:

(1) Any reasonable consulting fee;

(2) Generous mention in the Acknowledgments when the paper is published. (But no name will be mentioned without the prior written approval of the mathematician concerned.)

(3) An offer of shared authorship to the first mathematician who makes a significant contribution to the paper prior to publication.

In any case, all communications we receive about this paper will be kept strictly confidential.