

# **A Solution to the $3x + 1$ Problem**

by

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“Very often in mathematics the crucial problem is to recognize and to discover what are the relevant concepts; once this is accomplished the job may be more than half done.”<sup>1</sup>

“One of the greatest contributions a mathematician can make is to spot something so simple and powerful that everybody else has missed it.”<sup>2</sup>

**Note 1:** The reader may wish to begin by reading “Appendix E —A Brief Summary of Our Solution to the Problem” on page 39.

**Note 2:** The reader can safely assume, *initially*, that all referenced lemmas are true, since their proofs have been checked and deemed correct by several mathematicians. Thus it is only necessary to read pages 1 - 9 in order to understand in detail our proof of the  $3x + 1$  Conjecture.

Key words:  $3x + 1$  Problem,  $3n + 1$  Problem, Syracuse Problem, Ulam’s Problem, Collatz Conjecture, computational number theory, proof of termination of programs, recursive function theory

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1. Herstein, I. N., *Topics in Algebra*, John Wiley & Sons, N.Y., 1975, p. 50.

2. Stewart, Ian, *The Problems of Mathematics*, Oxford University Press, N.Y., 1992, pp. 279-280.

## **Abstract**

We present a proof of the  $3x + 1$  Conjecture, which asserts that repeated iterations of the function  $C(x) = (3x + 1)/(2^a)$  always terminate in 1. Here  $x$  is an odd, positive integer, and  $a = \text{ord}_2(3x + 1)$ , the largest positive integer such that the denominator divides the numerator. Our proof is based on a structure called “tuple-sets” that represents the  $3x + 1$  function in the “forward” (as opposed to the inverse) direction. In our proof, we show that the 35-level elements of all 35-level tuples in all 35-level tuple-sets are the same, regardless if counterexamples to the Conjecture exist or not. From this fact, a simple inductive argument allows us to conclude that all tuple-sets are the same, whether counterexamples exist or not, and hence that counterexamples do not exist.

As far as we have been able to determine, our approach to a solution of the Problem is original.

## Introduction

### Statement of Problem

For  $x$  an odd, positive integer, set

$$C(x) = \frac{3x + 1}{2^{\text{ord}_2(3x + 1)}}$$

where  $\text{ord}_2(3x + 1)$  is the largest exponent of 2 such that the denominator divides the numerator. Thus, for example,  $C(17) = 13$ ,  $C(13) = 5$ ,  $C(5) = 1$ . The  $3x + 1$  Problem, also known as the  $3n + 1$  Problem, the Syracuse Problem, Ulam's Problem, the Collatz Conjecture, Kakutani's Problem, and Hasse's Algorithm, asks if repeated iterations of  $C$  always terminate at 1. The conjecture that they do is hereafter called the  $3x + 1$  Conjecture, or sometimes, in this paper, just *the Conjecture*. We call  $C$  the  $3x + 1$  function; note that  $C(x)$  is by definition odd.

An odd, positive integer such that repeated iterations of  $C$  terminate at 1, we call a *non-counterexample*. An odd, positive integer such that repeated iterations of  $C$  never terminate at 1, we call a *counterexample*.

Other equivalent formulations of the  $3x + 1$  Problem are given in the literature; we base our formulation on the  $C$  function (following Crandall) because, as we shall see, it brings out certain structures that are not otherwise evident.

### Summary of Research on the Problem

As stated in (Lagarias 1985), "The exact origin of the  $3x + 1$  problem is obscure. It has circulated by word of mouth in the mathematical community for many years. The problem is traditionally credited to Lothar Collatz, at the University of Hamburg. In his student days in the 1930's, stimulated by the lectures of Edmund Landau, Oskar Petron, and Issai Schur, he became interested in number-theoretic functions. His interest in graph theory led him to the idea of representing such number-theoretic functions as directed graphs, and questions about the structure of such graphs are tied to the behavior of iterates of such functions. In the last ten years [that is, 1975-1985] the problem has forsaken its underground existence by appearing in various forms as a problem in books and journals..."

Lagarias has performed an invaluable service to the  $3x + 1$  research community by publishing several annotated bibliographies relating to the Problem. These are accessible on the Internet.

### On the Structure of This Paper

To enhance readability, we have placed proofs of all lemmas in "Appendix A — Statement and Proof of Each Lemma" on page 17.

## Tuple-sets: The Structure of the $3x + 1$ Function in the “Forward” Direction

### Definitions

#### Iteration

An *iteration* takes an odd, positive integer,  $x$ , to an odd, positive integer,  $y$ , via one application of the  $3x + 1$  function,  $C$ . Thus, in one iteration  $C$  takes 17 to 13 because  $C(17) = 13$ .

#### Tuple

A (finite) *tuple* is a finite sequence of zero or more successive iterations of  $C$ , that is,  $\langle x, C(x), C^2(x), \dots, C^k(x) \rangle$ , where  $k \geq 0$ .

A finite tuple is the prefix of an infinite tuple. If  $x$  is a non-counterexample, then  $x$  is the first element of an infinite tuple  $\langle x, y, \dots, 1, 1, 1, \dots \rangle$ . Of course, if  $x$  is a range element of  $C$ , then  $x$  can be an element other than the first in another non-counterexample tuple.

In the literature, a tuple (finite or infinite) is usually called a *trajectory* or an *orbit*.

If  $x$  is a counterexample, then  $x$  is the first element of an infinite tuple  $\langle x, y, \dots \rangle$  which does not contain 1. Of course, if  $x$  is a range element of  $C$ , then  $x$  can be an element other than the first in another counterexample tuple.

A counterexample tuple must be one of two types: either there is an infinitely-repeated finite cycle of elements (none of which is 1) in the infinite tuple having the counterexample  $x$  as first element, or else there is no such cycle, but there is no 1 in the infinite tuple having the counterexample  $x$  as first element — in other words, there is no upper bound to the elements of the infinite tuple.

### Exponent, Exponent Sequence

If  $C(x) = y$ , with  $y = (3x + 1)/2^a$ , we say that  $a$  is the *exponent associated with  $x$* . In more formal language, this can be expressed as  $ord_2(3x + 1) = a$ . Sometimes we simply write  $e(x) = a$ . The sequence  $\{a_2, a_3, \dots, a_i\}$ , where  $a_2, a_3, \dots, a_i$  are the exponents associated with  $x, C(x), \dots, C^{(i-1)}(x)$  respectively, is called an *exponent sequence*. We number exponents beginning with  $a_2$  in order that the subscript corresponds to a level number in the corresponding tuple-set. See “Levels in Tuples and Tuple-sets” on page 5

We say that  $x$  *maps to  $y$  via  $a_i$*  if  $C(x) = y$  and  $ord_2(3x + 1) = a_i$ . By extension, we say that  $x$  *maps to  $z$*  if  $z$  is the result of a finite sequence of iterations of  $C$  beginning with  $x$ , that is if the tuple  $\langle x, y, \dots, z \rangle$  exists.

### Tuple-set<sup>1</sup>

Let  $A = \{a_2, a_3, \dots, a_i\}$  be a finite sequence of exponents, where  $i \geq 2$ . The *tuple-set*  $T_A$  consists of all and only the tuples that are associated with all successive approximations to  $A$ . Thus  $T_A$  consists of all and only the following tuples. (*Note:* First elements  $x$  in different tuples are different odd, positive integers. No two tuples in a tuple-set have the same first element.)

all tuples  $\langle x \rangle$  such that  $x$  does not map to an odd, positive integer via  $a_2$ ;

all tuples  $\langle x, y \rangle$  such that  $x$  maps to  $y$  via  $a_2$  but  $y$  does not map to an odd, positive integer via  $a_3$ ;

all tuples  $\langle x, y, y' \rangle$  such that  $x$  maps to  $y$  via  $a_2$  and  $y$  maps to  $y'$  via  $a_3$ , but  $y'$  does not map to an odd, positive integer via  $a_4$ ;

...

all tuples  $\langle x, y, y', \dots, y^{(i-3)}, y^{(i-2)} \rangle$  such that  $x$  maps to  $y$  via  $a_2$  and  $y$  maps to  $y'$  via  $a_3$  and ... and  $y^{(i-3)}$  maps to  $y^{(i-2)}$  via the exponent  $a_i$ . (The longest tuple in an  $i$ -level tuple-set has  $i$  elements.)

In other words, for each  $i$ -level exponent sequence  $A$ :

there are tuples  $\langle x \rangle$  whose associated exponent sequence is a prefix of  $A$  for no exponent of  $A$ , and

there are other tuples  $\langle x, y \rangle$  whose associated exponent sequence is a prefix of  $A$  for the first exponent of  $A$ , and

there are other tuples  $\langle x, y, y' \rangle$  whose associated exponent sequence is a prefix of  $A$  for the first two exponents of  $A$ , and

...

there are other tuples  $\langle x, y, z, \dots, y^{(i-2)} \rangle$  whose associated exponent sequence is a prefix of  $A$  for all  $i - 1$  exponents of  $A$ .

Tuples are ordered in the natural way by their first elements.

The set of first elements of all tuples in a tuple-set is the set of odd, positive integers (see proof under “The Structure of Tuple-sets” on page 7). Thus, there is a countable infinity of tuples in each tuple-set.

For each  $i \geq 2$ , tuple-sets are a *partition* of the set of all  $i$ -level tuples.

### Levels in Tuples and Tuple-sets

Let  $A$  be an  $i$ -level exponent sequence,  $\{a_2, a_3, \dots, a_i\}$ . The reason subscripts of exponents begin with 2, rather than with 0 or 1, is so that they correspond to levels in each tuple-set. (No

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1. The literature contains several results that establish properties of the  $3x + 1$  function that are equivalent to some of those for tuple-sets. However, the language is very different, and the definition of the  $3x + 1$  function that is used is not ours, but the original one, in which each division by 2 is a separate node in the tree graph representing the function.

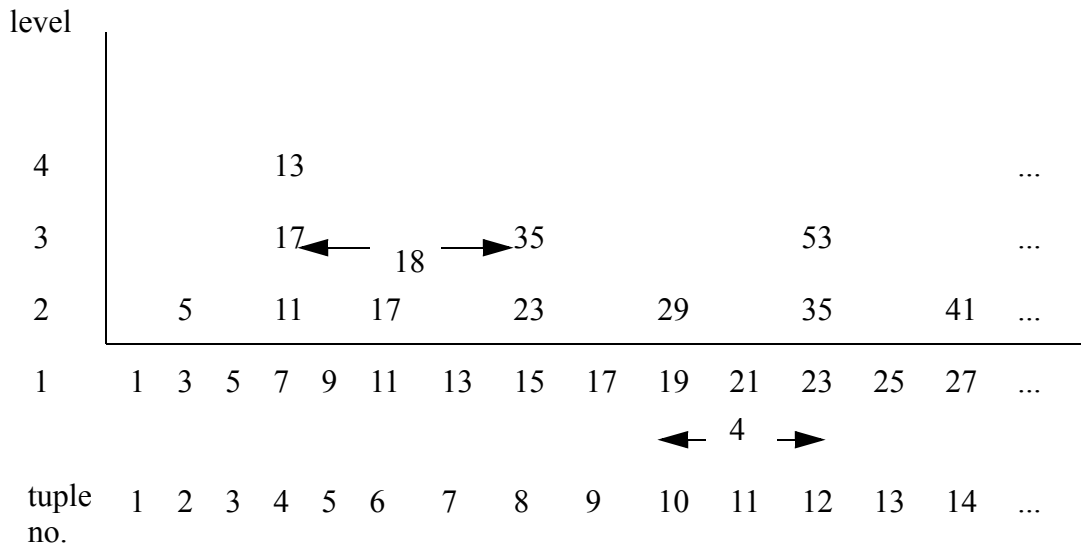
tuple-set has only one level, because that would mean it is associated with no exponent sequence.) Let  $T_A$  be the tuple-set determined by  $A$ . Then, by definition of tuple-set, there exist  $j$ -level tuples in  $T_A$ , where  $1 \leq j \leq i$ , that is, tuples  $t = \langle x, y, \dots, z \rangle$ , where  $x$  is the 1-level element of  $t$ ,  $y$  is the 2-level element of  $t$ , ..., and  $z$  is the  $j$ -level element of  $t$ . We say that  $T_A$  is an  $i$ -level tuple-set, and we sometimes speak of the set of  $j$ -level tuple-elements in  $T_A$ , where  $1 \leq j \leq i$ .

For  $2 \leq j \leq i$ , two tuples are said to be *consecutive at level  $j$*  if no  $j$ -level or higher-level tuple exists between them.

**Example of Tuple-set**

As an example of (part of) a tuple-set: in Fig. 1, where  $A = \{a_2, a_3, a_4\} = \{1, 1, 2\}$  and where we adopt the convention of orienting tuples vertically on the page, the tuple-set  $T_A$  includes:

- the tuple  $\langle 1 \rangle$ , because  $e(1) = 2 \neq (a_2 = 1)$ ;
- the tuple  $\langle 3, 5 \rangle$ , because  $e(3) = (a_2 = 1)$ , but  $e(5) = 4 \neq (a_3 = 1)$ ;
- the tuple  $\langle 5 \rangle$ , because  $e(5) = 4 \neq (a_2 = 1)$ ;
- the tuple  $\langle 7, 11, 17, 13 \rangle$  because  $e(7) = 1 (a_2 = 1)$  and  $e(11) = 1 (a_3 = 1)$  and  $e(17) = 2 (a_4 = 2)$ ;
- etc.



**Fig. 1. Part of the tuple-set  $T_A$  associated with the sequence  $A = \{1, 1, 2\}$**

The number 18 between the arrows at level 3 and the number 4 between the arrows at level 1 are the values of the level 3 and level 1 distance functions, respectively, established by Lemma 1.0 (see “Lemma 1.0: the “Distance” Functions  $d(i, i)$  and  $d(1, i)$ ” on page 7).

In each  $i$ -level tuple-set  $T_A$ , where  $i \geq 2$ , for each odd, positive integer  $x$  there exists a tuple whose first element is  $x$ . The tuple may be one-level ( $\langle x \rangle$ ), or 2-level ( $\langle x, y \rangle$ ), or ... or  $i$ -level ( $\langle x, y, y', \dots, y^{(i-3)}, y^{(i-2)} \rangle$ ). Thus each tuple-set is non-empty.

## Graphical Representation of the Set of All Tuple-sets

It is clear from the definition of *tuple-set* that the set of all tuple-sets can be represented by an infinitary tree in which each node is a tuple-set. We can imagine the tuple-set (which contains an infinity of tuples) extending into the page.

## The Structure of Tuple-sets

It is important for the reader to understand that the structure of each tuple-set is unchanged by the presence or absence of counterexample tuples. Regardless if counterexample tuples exist or not, the set of first elements of all tuples in each tuple-set is always the same, namely, the set of odd, positive integers. *Proof:* Let  $x$  be any odd, positive integer and let  $A = \{a_2, a_3, \dots, a_i\}$ , where  $i \geq 2$ , be any exponent sequence. Then there are exactly two possibilities:

- (1)  $x$  maps to a  $y$  in a single iteration of the  $3x + 1$  function,  $C$ , via the exponent  $a_2$ , or
- (2)  $x$  does not map to a  $y$  in a single iteration of  $C$  via the exponent  $a_2$ .

But if (1) is true, then a tuple containing at least two elements, with  $x$  as the first, is in  $T_A$ ; if (2) is true, then the tuple  $\langle x \rangle$  is in  $T_A$ . There is no third possibility.  $\square$

For each tuple-set, the first element of the first tuple is 1, the first element of the second tuple is 3, the first element of the third tuple is 5, etc.

It can never be the case that, if counterexample tuples exist, then somehow there are “more” tuples in a tuple-set than if there are no counterexample tuples<sup>1</sup>.

Furthermore, the distance functions defined in “Lemma 1.0: the “Distance” Functions  $d(i, i)$  and  $d(1, i)$ ” on page 7 are the same regardless if counterexample tuples exist or not.

## Extensions of Tuple-sets

Since there is a tuple-set for each finite sequence  $A$  of exponents, it follows that each tuple-set  $T_A$  has an extension via the exponent 1, and an extension via the exponent 2, and an extension via the exponent 3, ... In other words, if  $A = \{a_2, a_3, \dots, a_i\}$ , then there is a tuple-set  $T_{A'}$ , where  $A' = \{a_2, a_3, \dots, a_i, 1\}$ , and a tuple-set  $T_{A''}$ , where  $A'' = \{a_2, a_3, \dots, a_i, 2\}$ , and a tuple-set  $T_{A'''}$ , where  $A''' = \{a_2, a_3, \dots, a_i, 3\}$ , ...

All this is true whether or not the tuple-set  $T_A$  and/or any of its extensions contains counterexample tuples or not.

For further details on extensions of tuple-sets, see “How Tuple-sets ‘Work’” and the proof that there exists an extension for each tuple-set (“Lemma 3.0 Statement and Proof”) in our paper, “Are We Near a Solution to the  $3x + 1$  Problem?” on [occampress.com](http://occampress.com).

## Lemma 1.0: the “Distance” Functions $d(i, i)$ and $d(1, i)$

(a) Let  $A = \{a_2, a_3, \dots, a_i\}$ , where  $i \geq 2$ , be a sequence of exponents, and let  $t_{(r)}$ ,  $t_{(s)}$  be tuples consecutive at level<sup>2</sup>  $i$  in  $T_A$ . Then  $d(i, i)$  is given by:

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1. To make this statement more precise: in no tuple-set does there ever exist a first element of a tuple, regardless how large that first element is, such that there are more tuples in that tuple-set having smaller first elements if counterexamples exist, than if counterexamples do not exist.

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) Let  $t_{(r)}, t_{(s)}$  be tuples consecutive at level  $i$  in  $T_A$ . Then  $d(1, i)$  is given by:

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3}) \dots (2^{a_i})$$

**Proof:** see “Lemma 1.0: Statement and Proof” on page 17

It follows from part (a) of the Lemma that the set of all  $i$ -level elements of all  $i$ -level first tuples in all  $i$ -level tuple-sets is  $\{z \mid 1 \leq z < 2 \cdot 3^{i-1}\}$ , where  $z$  is an odd, positive integer not divisible by 3.

**Remark:** Relationships similar to those described in parts (a) and (b) of the Lemma hold for successive  $j$ -level tuples, where  $2 \leq j < i$ . The following table shows these relationships for  $(i-j)$ -level elements of tuples consecutive at level  $(i-j)$  in an  $i$ -level tuple-set, where  $0 \leq j \leq (i-1)$ . The distances are easily proved using Lemma 1.0.

**Table 1: Distances between elements of tuples consecutive at level  $i$**

Level	Distance between $(i-j)$ -level elements of tuples consecutive at level $(i-j)$ , where $0 \leq j \leq (i-1)$
$i$	$2 \cdot 3^{i-1}$
$i-1$	$2 \cdot 3^{i-2} \cdot 2^{a_i}$
$i-2$	$2 \cdot 3^{i-3} \cdot 2^{a_{i-1}} 2^{a_i}$
$i-3$	$2 \cdot 3^{i-4} \cdot 2^{a_{i-2}} 2^{a_{i-1}} 2^{a_i}$
...	...
2	$2 \cdot 3 \cdot 2^{a_3} \dots 2^{a_{i-1}} 2^{a_i}$
1	$2 \cdot 2^{a_2} 2^{a_3} \dots 2^{a_{i-1}} 2^{a_i}$

Further details can be found in the section, “Remarks About the Distance Functions” in our paper, “Are We Near a Solution to the  $3x + 1$  Problem?”, on [occampress.com](http://occampress.com).

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2. For  $2 \leq j \leq i$ , two tuples are consecutive at level  $j$  if no  $j$ -level or higher-level tuple exists between them (see “Levels in Tuples and Tuple-sets” on page 5).



**Lemma 2.0**

*Assume a counterexample exists. Then for all  $i \geq 2$ , each  $i$ -level tuple-set contains an infinity of  $i$ -level counterexample tuples and an infinity of  $i$ -level non-counterexample tuples.*

**Proof:** "Lemma 2.0: Statement and Proof" on page 22



The exponents of 2 are not even in all “spiral”s, of course. For example, the “spiral” of numbers (not shown) mapping to 341 has odd exponents.

The tree is not strictly an infinitary tree, because if a node is not a range element of the  $3x + 1$  function, that is, if a node is a multiple-of-3, then it has no descending nodes (no odd, positive integers map to a multiple-of-3).

We call the set of odd, positive integers that map to a range element  $y$  in one iteration of the function, a *recursive “spiral”*, or just a *“spiral”* for short. Thus, for example,  $\{1, 5, 21, 85, 341, \dots\}$  is a “spiral”. It maps to 1 in one iteration of the  $3x + 1$  function. It is easily shown that the elements of a “spiral” map to  $y$  either by all odd exponents, or by all even exponents (which is the case with  $\{1, 5, 21, 85, \dots\}$ ). It is likewise easily shown that if  $x$  is an element of a “spiral”, then the next “spiral” element is  $4x + 1$ .

Each node in the 1-tree is a “spiral” element. From now on, we will use the term *“spiral” element*.

Each “spiral” element  $y$  that is a range element of the  $3x + 1$  function is the root of a  $y$ -tree. The definition of the  $y$ -tree is the same as that of the 1-tree, with  $y$  replacing 1.

The odd, positive integers between successive “spiral” elements we call an *interval* in the “spiral”. Thus, for example, in the “spiral”  $\{1, 5, 21, 85, 341, \dots\}$ , 3 is the only element in the first interval; 7, 9, 11, 13, 15, 17, 19, are the elements of the second interval, etc.

If a counterexample exists, then it is an element of an interval, but it is never a “spiral” element. The tree or trees generated by counterexamples are of course separate (disjoint) from the 1-tree. There are “spiral”s, “spiral” elements, and intervals in counterexample trees just as there are in the 1-tree.

Let  $y$  denote a “spiral” element in  $\{1, 5, 21, 85, \dots\}$  that is a range element. Then if  $x$  is a non-counterexample, it must be an element of a  $y$ -tree.

So, to summarize: In the case of the 1-tree, all “spiral” elements are non-counterexamples. Some interval elements we know are non-counterexamples. If a counterexample exists, then it is an element of an interval, but never of a “spiral” in the 1-tree. In the case of a counterexample tree, all spiral elements are counterexamples. Some interval elements are counterexamples, others are non-counterexamples.

*There is exactly one 1-tree.* This means that an odd, positive integer maps to 1 regardless if counterexamples exist or not. Thus, for example, 13 (an element of the 5-tree) maps to 1 (in two iterations of the function) today. If a proposed proof of the  $3x + 1$  Conjecture is accepted as valid tomorrow, 13 will still map to 1. And if the Conjecture is proved *false* tomorrow, 13 will *still* map to 1. (A proof that there is only one 1-tree is the proof of Lemma 8.8 in our paper, “Are We Near a Solution to the  $3x + 1$  Problem?”, on occampress.com. That Lemma is accompanied by several aids to overcoming the counterintuitive aspect of the Lemma.)

If the  $3x + 1$  Conjecture is true, then the 1-tree contains all odd, positive integers. If the Conjecture is false, then the 1-tree contains only a proper subset of the odd, positive integers.

## Understanding the 1-Tree Relative to Successive Non-Counterexamples Beginning With 1

1. It is known by computer test<sup>1</sup> that all successive odd, positive integers less than approximately  $5.76 \cdot 10^{18}$  are non-counterexamples. Call this set  $W$ . So let us consider the “spiral”  $S = \{1, 5, 21, 85, \dots\}$ , all of whose elements map to 1 in one iteration of the  $3x + 1$  function, and proceed as follows.

2. Obviously, 1 maps to 1, and so we color 1 green in  $S$ .

Moving to the next odd, positive integer to the right of 1, which is 3, we can easily find that 3 maps to 1. So we color 3 green in the first interval of  $S$ .

Moving to the next odd, positive integer to the right of 3, which is 5, we know that 5 maps to 1, because all elements of  $S$  map to 1. So we color 5 green in  $S$ . Since 5 is a range-element of the  $3x + 1$  function, a “spiral” maps to it in one iteration of the function. That “spiral” is  $\{3, 13, 53, \dots\}$ . We color all elements of the “spiral” green in intervals of  $S$ . We also color all other elements in the 5-tree green. We then color green in intervals of  $S$ , all elements of the 5-tree.

Moving to the next odd, positive integer to the right of 5, which is 7, we proceed similarly. But from now on, we also color green all elements of upward extensions of 7 if they are not already colored green. and we color green the same interval elements and elements of  $S$ .

Etc.

If we come to an odd, positive integer that is already green (because it is a member of an earlier  $y$ -tree) we move to the next odd, positive integer to the right.

3. If all odd, positive integers (including those that are greater than the largest odd, positive integer in  $W$  (see step 1), are either already green, or are elements of  $S$ , then we have a proof of the  $3x + 1$  Conjecture.

The reader is urged to contemplate the fact that the portion of the 1-tree (including intervals) that is green is *exactly* what the portion of the 1-tree looks like if counterexamples do not exist — or if they do exist (see previous sub-section regarding fact that there is exactly one 1-tree).

One fact that seems promising for a proof of the  $3x + 1$  Conjecture is that each successive element of each “spiral” is an element of an infinity of successive intervals in the “spiral”  $S = \{1, 5, 21, 85, 341, \dots\}$ . Thus, consider the elements of the “spiral”  $\{3, 13, 53, 213, \dots\}$ .

Many relevant results and ideas will be found in our paper, “Are We Near a Solution to the  $3x + 1$  Problem?”, on occampress.com, in the sub-section, “Spiral’s, Intervals and Levels” of the section, “Strategy of ‘Filling-in’ of Intervals in the Base Sequence Relative to 1”.

In the case of the 1-tree for the  $3x - 1$  function, the first counterexample, 5, occurs in the first interval in the “spiral”  $\{1, 3, 11, 43, \dots\}$  that maps to 1 in one iteration of the  $3x - 1$  function —

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1. See results of tests performed by Tomás Oliveira e Silva, [www.ieeta.pt/~tos/3x+1/html](http://www.ieeta.pt/~tos/3x+1/html). All consecutive odd, positive integers to at least  $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$ , which is greater than  $3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35-1)} - 1$ , have been tested and found to be non-counterexamples.

namely, in the interval whose elements are 5, 7, 9. (The element 7 is also a counterexample, because  $\langle 5, 7, 5, \dots \rangle$  is an infinite cycle that never yields 1.)

...

**Theorem: The  $3x + 1$  Conjecture is true.**

**Proof:**

(Note: at the time of this writing, we believe that we may have a second proof of the Conjecture that is considerably shorter than the one presented here. See “Possible Proof of Conjecture Based on 1-Tree” on page 31.)

1. It is easily shown that, for each  $i \geq 2$ , the set  $E_i$  of  $i$ -level elements in *first*  $i$ -level tuples in all  $i$ -level tuple-sets is the set of odd, positive integers less than  $2 \cdot 3^{(i-1)}$  that are not divisible by 3 (by part (a) of “Lemma 1.0: the “Distance” Functions  $d(i, i)$  and  $d(1, i)$ ” on page 7). Thus, for example,  $E_2$  is the set of all the odd, positive integers less than  $2 \cdot 3^{(2-1)} = 6$ , that are not divisible by 3, and these integers are 1 and 5.

2. By computer test, it is known that  $E_2, E_3, E_4, \dots$ , up to at least  $E_{35}$  each consists solely of non-counterexamples<sup>1</sup>.

3.

(1) For *each* 35-level tuple-set  $T_A$ , the sequence  $S$  of 35-level elements in the sequence of 35-level tuples is given by  $y + n(2 \cdot 3^{(35-1)})$ , where  $n \geq 0$  and  $y$  is the 35-level element of the first 35-level tuple in the tuple-set  $T_A$ . The sequence  $S$  is the sequence if counterexamples do not exist, and it is also the sequence if counterexamples exist. As we stated in step 2,  $y$  is a non-counterexample element,

*Proof:* Follows from part (a) of “Lemma 1.0: the “Distance” Functions  $d(i, i)$  and  $d(1, i)$ ” on page 7. The Distance Functions are not themselves functions of the truth or falsity of the  $3x + 1$  Conjecture. □

*Note:* the fact that all elements of  $E_{35}$  are non-counterexamples is emphatically *not* the case for the  $3x - 1$  function, where one of the elements, 5, of  $E_2$  is already a counterexample. Thus there exists a first 2-level tuple, namely  $\langle 7, 5 \rangle$ , in a 2-level tuple-set that is a counterexample tuple. Each subsequent  $E_i$  contains counterexamples, each of which is the  $i$ -level element of the first  $i$ -level tuple in an  $i$ -level tuple-set. Each of these tuples is therefore a counterexample tuple. *So our proof cannot be used to prove the false  $3x - 1$  Conjecture.*

4. Since each 35-level tuple-set— and indeed each  $i$ -level tuple-set, where  $i \geq 2$  — has an extension via each possible exponent, namely, via 1, 2, 3, ..., we can use an inductive argument beginning with (1) to arrive at the conclusion that the set of all non-counterexample tuples if counterexamples do not exist, is the same as the set of all non-counterexample tuples if counterexamples exist.

5. We must now ask if counterexamples can exist in  $T_A$  in  $j$ -level tuples, where  $j < 35$ . The answer is No, because each  $j$ -level tuple in  $T_A$  is a 35-level tuple in some other 35-level tuple-set,

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1. See results of tests performed by Tomás Oliveira e Silva, [www.ieeta.pt/~tos/3x+1/html](http://www.ieeta.pt/~tos/3x+1/html). All consecutive odd, positive integers to at least  $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$ , which is greater than  $3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35-1)} - 1$ , have been tested and found to be non-counterexamples. These include the set of all 35-level elements of all 35-level first tuples in all 35-level tuple-sets .

and  $T_A$  is any 35-level tuple-set.

6. So we must conclude from step 4 that the set of counterexamples is the empty set, and therefore that the  $3x + 1$  Conjecture is true..□

### **Remark 1**

The reader might enjoy answering — or attempting to answer — question (A), below, which arises from the following facts:

Let the tuple  $t = \langle x, \dots, 1, 1, \dots, 1 \rangle$ , which is clearly a non-counterexample tuple ( $x$  is a non-counterexample). Let  $A$  be the  $i$ -level exponent sequence associated with  $t$ . Then  $t$  is the first  $i$ -level tuple in the tuple-set  $T_A$ . The 1-level (first) element of  $t$  is  $x$ , the  $i$ -level element of  $t$  is 1.

The 1-level (first) element of the  $n$ th  $i$ -level tuple in  $T_A$  is given by  $x + (n - 1)(2 \cdot (2^{a_2})(2^{a_3}) \dots (2^{a_i}))$ , and the  $i$ -level element of the  $n$ th  $i$ -level tuple in  $T_A$  is given by  $1 + (n - 1)(2 \cdot 3^{i-1})$ , where  $n \geq 1$  (by parts (b) and (a) of “Lemma 1.0: the “Distance” Functions  $d(i, i)$  and  $d(1, i)$ ” on page 7).

(A)

How does  $T_A$  differ if (1) counterexamples exist, and (2) counterexamples do not exist?

### **Remark 2**

The reader might enjoy reading at least some of the appendices below that give other possible proofs of the  $3x + 1$  Conjecture.

### **Remark 3**

A wealth of additional results and possible strategies is available in our paper, “Are We Near a Solution to the  $3x + 1$  Problem?” on [occampress.com](http://occampress.com).

## References

Lagarias, J., (1985), “The  $3x + 1$  Problem and Its Generalizations”, *American Mathematical Monthly*, **93**, 3-23.

Wirsching, Günther J.. *The Dynamical System Generated by the  $3n + 1$  Function*, Springer-Verlag, Berlin, Germany, 1998.



## Appendix A — Statement and Proof of Each Lemma

### Lemma 1.0: Statement and Proof

*Definition:* let  $T_A$  be an  $i$ -level tuple-set, where  $i \geq 2$ . Let  $t(r), t(s)$  denote tuples consecutive at level  $i$ , with  $r < s$  in the natural ordering of tuples by first elements. Let  $t(r)(h), t(s)(h)$  denote the elements of  $t(r), t(s)$  at level  $h$ , where  $1 \leq h \leq i$ . Then we call  $|t(s)(h) - t(r)(h)|$  the *distance* between  $t(r)$  and  $t(s)$  at level  $h$ . We denote this distance by  $d(h, i)$  and call  $d$  the *distance functions* (one function for each  $h$ ).

### Lemma 1.0

(a) Let  $A = \{a_2, a_3, \dots, a_i\}$ , where  $i \geq 2$ , be a sequence of exponents, and let  $t_{(r)}, t_{(s)}$  be tuples consecutive at level  $i$  in  $T_A$ . Then  $d(i, i)$  is given by:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) Let  $t_{(r)}, t_{(s)}$  be tuples consecutive at level  $i$  in  $T_A$ . Then  $d(1, i)$  is given by:

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3}) \dots (2^{a_i})$$

Thus, in “Fig. 1. Part of the tuple-set  $T_A$  associated with the sequence  $A = \{1, 1, 2\}$ ” on page 6, the distance  $d(3, 3)$  between  $t_{8(3)} = 35$  and  $t_{4(3)} = 17$  is  $2 \cdot 3^{(3-1)} = 18$ . The distance  $d(1, 2)$  between  $t_{12(1)} = 23$  and  $t_{10(1)} = 19$  is  $2 \cdot 2^1 = 4$ .

### Proof:

The proof is by induction.

### Proof of Basis Step for Parts (a) and (b) of Lemma 1.0:

Let  $t_{(r)}$  and  $t_{(s)}$  be the first and second 2-level tuples, in the standard linear ordering of tuples based on their first elements, that are consecutive at level  $i = 2$  in the 2-level tuple-set  $T_A$ , where  $A = \{a_2\}$ . (See Fig. 2 (1).)

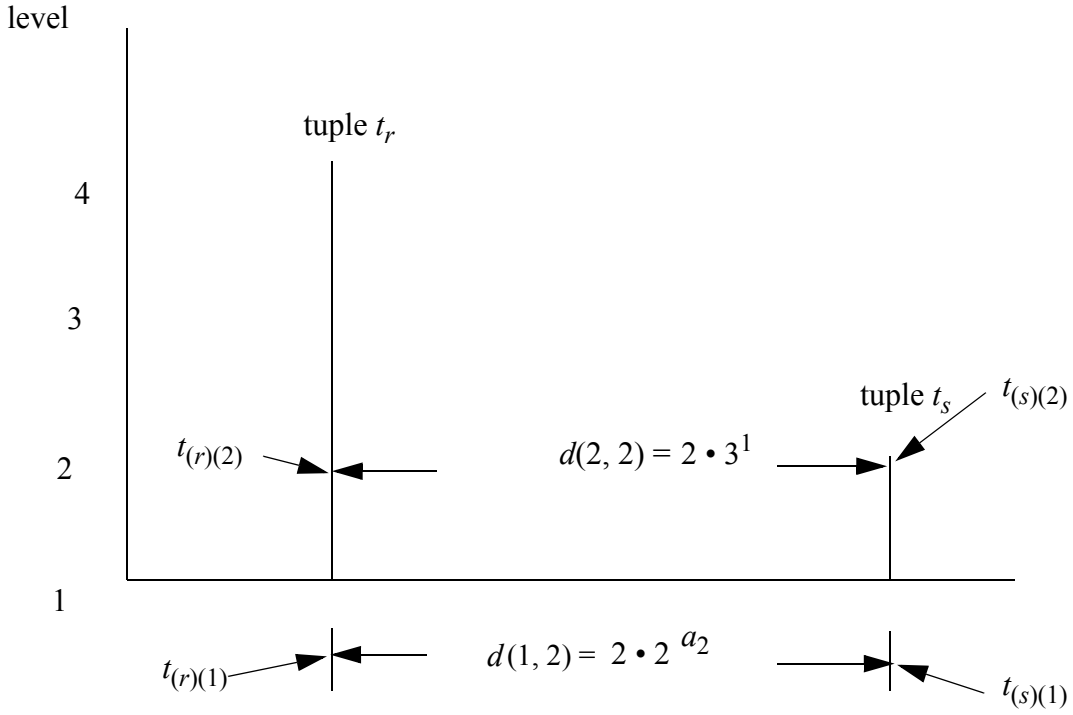


Fig. 2 (1). Illustration for proof of Basis Step of Lemma 1.0.

Then we have:

$$\frac{3t_{(r)(1)} + 1}{2^{a_2}} = t_{(r)(2)} \quad (1.1)$$

and since, by definition of  $d(1, 2)$ ,

$$t_{(s)(1)} = t_{(r)(1)} + d(1, 2)$$

we have:

$$\frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}} = t_{(s)(2)} \quad (1.2)$$

Therefore, since, by definition of  $d(i, i)$ ,

$$t_{(r)(2)} + d(2, 2) = t_{(s)(2)}$$

we can write, from (1.1) and (1.2):

$$\frac{3t_{(r)(1)} + 1}{2^{a_2}} + d(2, 2) = \frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}}$$

By elementary algebra, this yields:

$$2^{a_2}d(2, 2) = 3 \cdot d(1, 2)$$

Now  $d(2, 2)$  must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level  $i$ , it must be the smallest such even number, whence it follows that  $d(2, 2)$  must  $= 3 \cdot 2$ , and necessarily

$$d(1, 2) = 2 \cdot 2^{a_2}$$

A similar argument establishes that  $d(2, 2)$  and  $d(1, 2)$  have the above values for every other pair of tuples consecutive at level 2.

Thus we have our proof of the Basis Step for parts (a) and (b) of Lemma 1.0.

### **Proof of Induction Step for Parts (a) and (b) of Lemma 1.0**

Assume the Lemma is true for all levels  $j$ ,  $2 \leq j \leq i$  and that  $T_A$  is an  $i$ -level tuple-set, where  $A = \{a_2, a_3, \dots, a_i\}$ .

Let  $t_{(r)}$  and  $t_{(s)}$  be tuples consecutive at level  $i$ , and let  $t_{(r)}$  and  $t_{(f)}$  be tuples consecutive at level  $i + 1$ . (See Fig. 2 (2).)

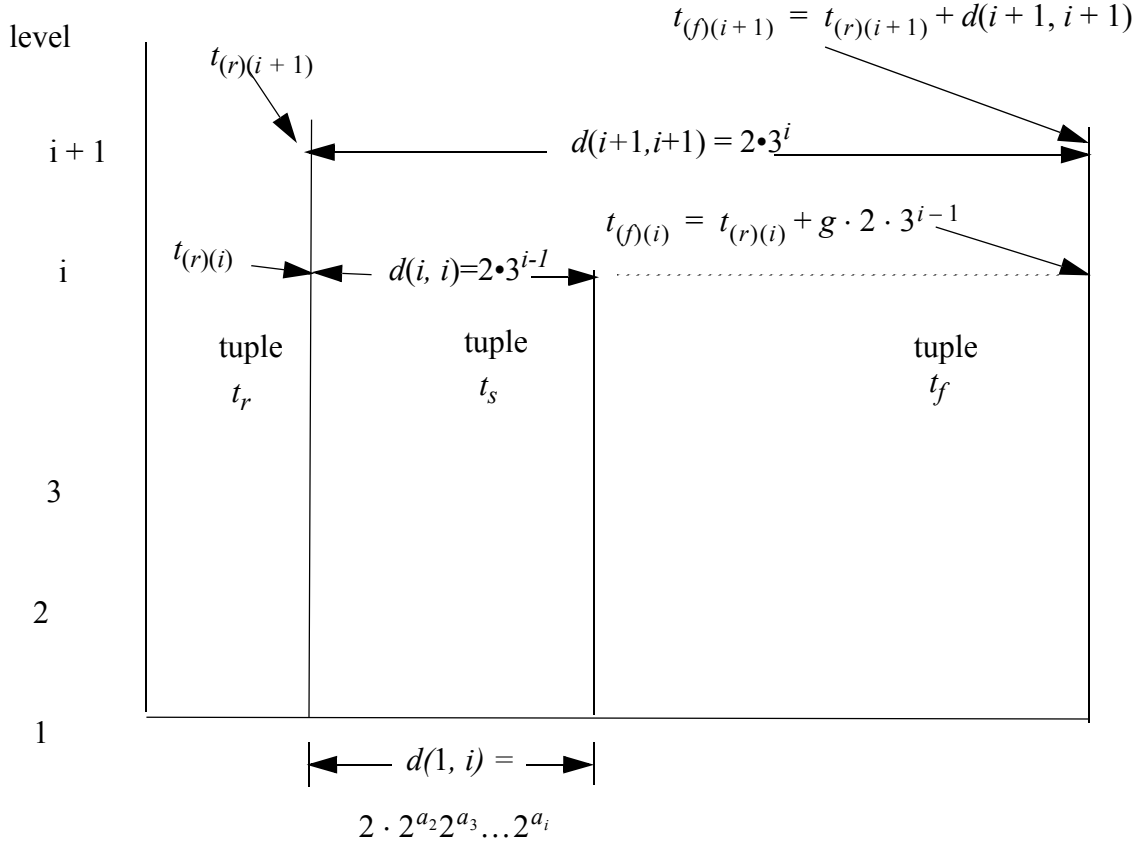


Fig. 2 (2). Illustration for proof of Induction Step of Lemma 1.0.

Then we have:

$$\frac{3t_{(r)(i)} + 1}{2^{a_{i+1}}} = t_{(r)(i+1)}$$

and since, by definition of  $d(i, i)$ ,

$$t_{(f)(i)} = t_{(r)(i)} + g \cdot d(i, i)$$

for some  $g \geq 1$ , we have:

$$\frac{3(t_{(r)(i)} + g \cdot d(i, i)) + 1}{2^{a_{i+1}}} = t_{(f)(i+1)}$$

Thus, since

$$t_{(r)(i+1)} + d(i+1, i+1) = t_{(f)(i+1)}$$

we can write:

$$\frac{3t_{(r)(i)} + 1}{2^{a_{i+1}}} + d(i+1, i+1) = \frac{3(t_{(r)(i)} + gd(i, i)) + 1}{2^{a_{i+1}}}$$

This yields, by elementary algebra:

$$2^{a_{i+1}}d(i+1, i+1) = 3 \cdot gd(i, i)$$

As in the proof of the Basis Step,  $d(i+1, i+1)$  must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level  $i+1$ , it must be the smallest such even number. Thus  $d(i+1, i+1) = 3 \cdot d(i, i)$ , and

$$g \cdot d(i, i) = 2^{a_{i+1}}d(i, i) \quad .$$

Hence

$$g = 2^{a_{i+1}}$$

Now  $g$  is the number of tuples consecutive at level  $i$  that must be “traversed” to get from  $t_{(r)}$  to  $t_{(f)}$ . By inductive hypothesis,  $d(1, i)$  for *each* pair of these tuples is:

$$d(1, i) = 2 \cdot 2^{a_2} \cdot 2^{a_3} \cdot \dots \cdot 2^{a_i}$$

hence, since

$$g = 2^{a_{i+1}}$$

we have

$$d(1, i+1) = d(1, i) \cdot 2^{a_{i+1}} \quad .$$

A similar argument establishes that  $d(i+1, i+1)$  and  $d(1, i+1)$  have the above values for every pair of tuples consecutive at level  $i+1$ .

Thus we have our proof of the Induction Step for parts (a) and (b) of Lemma 1.0. The proof of Lemma 1.0 is completed.  $\square$

## Lemma 2.0: Statement and Proof

Assume a counterexample exists. Then for all  $i \geq 2$ , each  $i$ -level tuple-set contains an infinity of  $i$ -level counterexample tuples and an infinity of  $i$ -level non-counterexample tuples.

### Proof:

1. Assume counterexamples exist. Then:

There is a countable infinity of non-counterexample range elements.

*Proof:* Each non-counterexample maps to a range element, by definition of *range element*.

Each range element is mapped to by an infinity of elements

(“Lemma 6.0: Statement and Proof” on page 24). A countable infinity of these are range elements (proof of “Lemma 7.0: Statement and Proof” on page 27).

There is a countable infinity of counterexample range elements.

*Proof:* same as for non-counterexample case.

2. For each finite exponent sequence  $A$ , and for each range element  $y$ , non-counterexample or counterexample, there is an  $x$  that maps to  $y$  via  $A$  possibly followed by a buffer exponent (“Lemma 7.0: Statement and Proof” on page 27). The presence of the buffer exponent does not change the fact that  $x$  is the first element of a tuple associated with the exponent  $A$ .  $\square$

## Lemma 3.0: Statement and Proof

Exactly one set  $J$  of odd, positive integers maps to 1, regardless if counterexamples exist or not.<sup>1</sup>

### Proof:

The set  $J =$

{odd, positive integers  $y$  |  $y$  maps to 1 in *one* iteration of the  $3x + 1$  function}  $\cup$   
 {odd, positive integers  $y$  |  $y$  maps to 1 in *two* iterations of the  $3x + 1$  function}  $\cup$   
 {odd, positive integers  $y$  |  $y$  maps to 1 in *three* iterations of the  $3x + 1$  function}  $\cup$   
 ...

If an odd, positive integer  $y$  maps to 1, then it maps to 1 regardless if counterexamples exist or not. For, certainly  $y$  will continue to map to 1 if counterexamples do not exist. If counterexamples exist, and  $y$  does not continue to map to 1, that implies that the proof that counterexamples

---

1. Some readers consider the phrase “regardless if counterexamples exist or not” as unnecessary. The reason we include it is that when it was not present, other readers assumed that the meaning of the Lemma was: exactly one set  $J$  of odd, positive integers maps to 1 (if counterexamples do not exist), and exactly one (different) set  $J$  of odd, positive integers maps to 1 if counterexamples exist. This meaning is incorrect. A second reason for including the phrase will become clear when the reader reads our proofs of the  $3x + 1$  Conjecture.

exist somehow changes the definition of the  $3x + 1$  function or the laws of arithmetic. But such changes are absurd. Thus the Lemma is true.  $\square$

### **Lemma 4.0: Statement and Proof**

*No multiple of 3 is a range element.*

**Proof :**

If

$$\frac{3x + 1}{2^a} = 3m$$

then  $1 \equiv 0 \pmod{3}$ , which is false.  $\square$

### **Lemma 5.0: Statement and Proof**

*Each odd, positive integer (except a multiple of 3) is mapped to by a multiple of 3 in one iteration of the  $3x + 1$  function.*

**Proof:**

Since the domain of the  $3x + 1$  function is the odd, positive integers, the only relevant generators are  $3(2k + 1)$ ,  $k \geq 0$ . We show that, for each odd, positive integer  $y$  not a multiple of 3, there exists a  $k$  and an  $a$  such that

$$y = \frac{(3(3(2k + 1)) + 1)}{2^a} , \tag{11.1}$$

where  $a$  is necessarily the largest such  $a$ , since  $y$  is assumed odd.

Rewriting (11.1), we have:

$$y2^{a-1} - 5 = 9k . \tag{11.2}$$

Without loss of generality, we can let  $y \equiv r \pmod{18}$ , where  $r$  is one of 1, 5, 7, 11, 13, or 17 (since  $y$  is odd and not a multiple of 3, these values of  $r$  cover all possibilities mod 18). Or, in other words, for some  $q$ ,  $r$ ,  $y = 18q + r$ . Then, from (11.2) we can write:

$$18(2^{a-1})q + (2^{a-1})r - 5 = 9k . \tag{11.3}$$

Since the first term on the left-hand side is a multiple of 9,  $(2^{a-1})r - 5$  must also be if the equation is to hold. We can thus construct the following table. (Certain larger  $a$  also serve equally well, but those given suffice for purposes of this proof.)

**Table 2: Values of  $r, a$ , for Proof of Lemma**

$r$	$a$	$(2^{a-1})r - 5$
1	6	27
5	1	0
7	2	9
11	5	171
1 3	4	99
1 7	3	63

Given  $q$  and  $r$  (hence  $y$ ), we can use  $r$  to look up  $a$  in the table, and then solve (11.3) for integral  $k$ , thus producing the multiple of 3 that maps to  $y$  in one iteration of the  $3x + 1$  function.  $\square$

**Lemma 6.0: Statement and Proof**

(a) Each range element  $y$  is mapped to, in one iteration of the  $3x + 1$  function, by every exponent of one parity only. Furthermore,

(b) For each of the two parities, there exists a range element that is mapped to by every exponent of that parity.

**Proof of part (a):**

Steps 1 and 2 are slightly edited versions of proofs by Jonathan Kilgallin and Alex Godofsky. Any errors are entirely ours. Step 3 is a slightly edited version of a proof by Michael Klipper. Any errors are entirely ours.

1. We first show that if  $y$  is mapped to by the exponent  $a$ , then  $y$  is mapped to by every exponent greater than  $a$  that is of the same parity as  $a$ .

Let  $y$  be a range element, and let  $x$  map to  $y$  via the exponent  $a$ . Then

$$y = \frac{3x + 1}{2^a}$$

We wish to show that there exists an  $x'$  such that  $x'$  maps to  $y$  via the exponent  $2^{a+2}$ . That is, we wish to show that there exists an  $x'$  such that



$$y = \frac{3x' + 1}{2^{a+2}}$$

Rewriting, this gives

$$x' = \frac{2^{a+2}y - 1}{3}$$

Substituting for  $y$  yields

$$x' = \frac{2^{a+2}\left(\frac{3x+1}{2^a}\right) - 1}{3}$$

Simplifying, this gives  $x' = 4x + 1$ . Since  $x$  is an odd, positive integer, clearly  $x'$  is as well.

Thus, by induction, if  $y$  is mapped to via the exponent  $a$ , it is mapped to by every exponent greater than  $a$  of the same parity.  $\square$

2. Next we show that if  $y$  is mapped to by the exponent  $a$  which is greater than 2, then it is mapped to by every exponent less than  $a$  that is of the same parity as  $a$ .

Let  $y$  be a range element, and let  $x$  map to  $y$  via the exponent  $a$  where  $a > 2$ . Then

$$y = \frac{3x + 1}{2^a}$$

We wish to show that there exists an  $x'$  such that  $x'$  maps to  $y$  via the exponent  $2^{a-2}$ . That is, we wish to show that there exists an  $x'$  such that

$$y = \frac{3x' + 1}{2^{a-2}}$$

Rewriting, this gives

$$x' = \frac{2^{a-2}y - 1}{3}$$

Substituting for  $y$  yields

$$x' = \frac{2^{a-2}\left(\frac{3x+1}{2^a}\right) - 1}{3}$$

Simplifying yields

$$x' = \frac{x-1}{4}$$

3. We must now show that  $x' = (x-1)/4$  is an odd, positive integer. This means we must show that  $(x-1) = 4(2k+1)$  for some  $k \geq 0$ , or that  $(x-1) = 8k+4$ , hence that  $x = 8k+5$ . Thus, we must prove  $x \equiv 5 \pmod{8}$ .

We know that  $x$  maps to  $y$  via  $a$ , where  $a \geq 3$ . Thus,  $y = (3x+1)/2^a$ , so  $2^a y = 3x+1$ . Because  $a \geq 3$ ,  $2^a y$  is a multiple of 8. Thus,  $(3x+1) \equiv 0 \pmod{8}$ , and  $3x \equiv 7 \pmod{8}$ . This readily implies  $x \equiv 5 \pmod{8}$ .

4. Thus, by induction, if  $y$  is mapped to via the exponent  $a$ , where  $a > 2$ , then it is mapped to by every exponent less than  $a$  of the same parity.  $\square$

**Proof of part (b):**

We now show that for each of the two parities there exists a range element that is mapped to by every exponent of that parity.

1. Fix a range element  $y$ , and suppose that  $x$  maps to  $y$  via the exponent  $a$ . Now  $a$  is either even or odd, hence  $a = 2n + h$ , where  $h$  is either 0 or 1. Since  $y = (3x+1)/2^a$ , it follows that  $(2^a)y = 3x+1$ . Reduce the equation mod 3, and we get  $(2^h)y \equiv 1 \pmod{3}$ , by the following reasoning:  $(2^a)y \equiv 1 \pmod{3}$  implies  $(2^{2n+h})y \equiv 1 \pmod{3}$  implies  $2^{2n} 2^h y \equiv 1 \pmod{3}$  implies  $2^h y \equiv 1 \pmod{3}$  because  $2^{2n} = 4^n \equiv 1 \pmod{3}$ .

2. Since  $y$  is fixed, either  $y \equiv 1$  or  $y \equiv 2 \pmod{3}$ . (We know that  $y$ , a range element, is not a multiple of 3 by "Lemma 4.0: Statement and Proof" on page 23). If  $y \equiv 1 \pmod{3}$ , then we have  $2^h(1) \equiv 1 \pmod{3}$ , which implies that  $h$  must be 0. If  $y \equiv 2 \pmod{3}$ , then we have  $(2^h)(2) \equiv 1 \pmod{3}$ , implying that  $h$  must be 1.  $\square$

### Lemma 7.0: Statement and Proof

Let  $y$  be a range element of the  $3x + 1$  function. Then for each finite exponent sequence  $A$ , there exists an  $x$  that maps to  $y$  via  $A$  possibly followed by a “buffer” exponent. (For example, if  $y$  is mapped to by even exponents, and our exponent sequence  $A$  ends with an odd exponent, then there must be a “buffer” exponent following  $A$ , and similarly if  $y$  is mapped to by odd exponents and  $A$  ends with an even exponent. However, there are other cases in which a “buffer” exponent is required.)

**Proof:**

1. Each range element  $y$  is mapped to by all exponents of one parity (“Lemma 6.0: Statement and Proof” on page 24).

2. Each range element  $y$  is mapped to by a multiple of 3 (“Lemma 5.0: Statement and Proof” on page 23).

Each range element is mapped to by an infinity of range elements (“Lemma 5.0: Statement and Proof” on page 23).

3. Let  $y$  be a range element and let  $S = \{s_1, s_2, s_3, \dots\}$  be the set of all odd, positive integers that map to  $y$  in one iteration of the  $3x + 1$  function. In other words,  $S$  is the set of all elements in a “spiral”. Furthermore, let the  $s_i$  be in increasing order of magnitude. It is easily shown that  $s_{i+1} = 4s_i + 1$ .

(In Fig. 18,  $y = 13$ ,  $S = \{17, 69, 277, 1109, \dots\}$ )

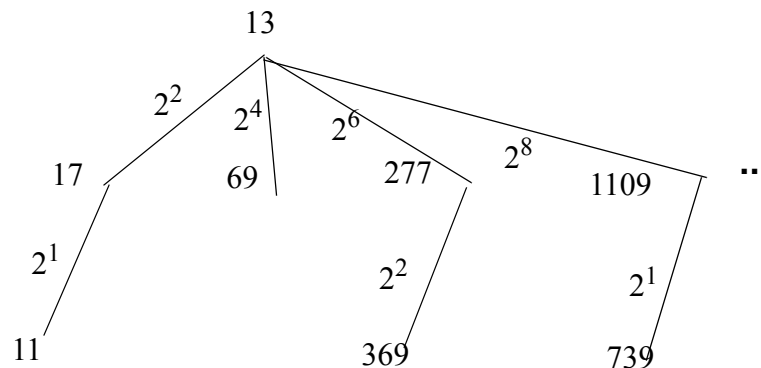


Fig. 18

(*Note*: for a graphical representation of part of the tree having 1 as its root instead of 13, see “Recursive “Spiral”s: The Structure of the  $3x + 1$  Function in the “Backward”, or Inverse, Direction” in our paper, “Are We Near a Solution to the  $3x + 1$  Problem?”, on [occampress.com](http://occampress.com).)

4. If  $s_i$  is a multiple of 3, then  $4s_i + 1$  is mapped to, in one iteration of the  $3x + 1$  function, by all exponents of even parity.

To prove this, we need only show that  $x$  is an integer in the equation

$$4(3u) + 1 = \frac{3x + 1}{2^2}$$

Multiplying through by  $2^2$  and collecting terms we get

$$(48u) + 4 = 3x + 1$$

and clearly  $x$  is an integer.

5. If  $s_j$  is mapped to by all even exponents, then  $4s_j + 1$  is mapped to, in one iteration of the  $3x + 1$  function, by all exponents of odd parity.

(The proof is by an algebraic argument similar to that in step 4.)

6. If  $s_k$  is mapped to by all odd exponents, then  $4s_k + 1$  is a multiple of 3.

(The proof is by an algebraic argument similar to that in step 4.)

7. The Lemma follows by an inductive argument that we now describe.

Let  $y$  be a range element. It is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. As a consequence of steps 1 through 6, we can represent an infinite sub-sequence of the sequence by

...3, 2, 1, 3, 2, 1, ...

where

“3” means “this odd, positive integer is a multiple of 3 and therefore is not mapped to by any odd, positive integer”;

“2” means “this odd, positive integer is mapped to by all even exponents”;

“1” means “this odd, positive integer is mapped to by all odd exponents”.

Each type “2” and type “1” odd, positive integer is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. We can represent an infinite sub-sequence of the sequence by

...3, 2, 1, 3, 2, 1, ...

where each integer has the same meaning as above.

Temporarily ignoring the case in which a buffer exponent is needed, it should now be clear that, for each range element  $y$ , and for each finite sequence of exponents  $B$ , we can find a finite path down through the infinitary tree we have just established, starting at the root  $y$ . The path will end in an odd, positive integer  $x$ . Let  $A$  denote the path  $B$  taken in reverse order. Then we have our result for the non-buffer-exponent case. The buffer-exponent case follows from the fact that the buffer exponent is one among an infinity of exponents of one parity. Thus  $y$  is mapped to by an infinite sequence of odd, positive integers. We then simply apply the above argument..  $\square$

*o*

## **Appendix B — A Possible Strategy for A Second Proof of the $3x + 1$ Conjecture**

In the above proof of the Conjecture, we argued in what we might call the “vertical” direction, that is, the direction of increasing  $i$ . We showed that the set of all 35-level elements of all 35-level tuples in all 35-level tuple-sets is the same regardless if counterexamples exist or not, and from that, via an elementary inductive argument, we showed that the contents of all tuple-sets are the same regardless if counterexamples exist or not, which implies that counterexamples do not exist.

In the following possible strategy, we argue in what we might call the “horizontal” direction.

1, Assume counterexamples exist, and consider the set of all first 35-level tuples in the set of all 35-level tuple-sets. By what we established in the above proof of the Conjecture, we know that all these tuples are non-counterexample tuples. By Lemma 8.8 in our paper, “Are We Near a Solution to the  $3x + 1$  Problem?”, on [occampress.com](http://occampress.com), we know that all these tuples are the same regardless if counterexamples exist or not.

2. Now, moving in the “horizontal” direction in any of these tuple-sets, that is, in the direction of increasing first elements of tuples, starting from the first 35-level tuple, we see from Table 1, “Distances between elements of tuples consecutive at level  $i$ ,” on page 8, that each successive tuple remains the same regardless if counterexamples exist or not, because the Table is not a function of the truth or falsity of the  $3x + 1$  Conjecture.

So we must conclude that all the tuples in the tuple-set remain the same regardless if counterexamples exist or not. But this implies that the contents of all tuple-sets remain the same, regardless if counterexamples exist or not, and therefore we must conclude, as we did in our proof of the Conjecture, that counterexamples do not exist.

## **Appendix C — Possible Proof, and Possible Strategies for a Proof, of the $3x + 1$ Conjecture Based on the 1-Tree**

For definitions and background on the 1-tree, see “The 1-Tree” on page 10.

### **Possible Proof of Conjecture Based on 1-Tree**

1. There is one and only one 1-tree, regardless if counterexamples exist or not (Lemma 8.8 in our paper, “Are We Near a Solution to the  $3x + 1$  Problem?”, on [occampress.com](http://occampress.com)). Thus, for example, 13 maps to 1 today, it will map to 1 if the  $3x + 1$  Conjecture is proved true tomorrow, and it will map to 1 if the Conjecture is proved *false* tomorrow.

2. Since each node (“spiral” element) in the 1-tree is a non-counterexample, then from step 1 we have that the set of non-counterexamples if counterexamples do not exist, is the same as the set of non-counterexamples if counterexamples exist.

3. Assume counterexamples exist. Then the counterexample  $x$  is not in the 1-tree. But if counterexamples do not exist, then  $x$  is in the 1-tree. But by step 1, the 1-tree is the same regardless if counterexamples exist or not. So our assumption that counterexamples exist has led to a contradiction. Therefore, counterexamples do not exist, and the  $3x + 1$  Conjecture is true.

### **Possible Strategy for a Proof Based on 1-Tree**

1. We must remember that there is exactly one 1-tree, whether or not counterexamples exist (see “The 1-Tree” on page 10). If the set of all “spiral” elements in the 1-tree is the set of all odd, positive integers, then counterexamples do not exist. Otherwise, counterexamples exist.

2. If a counterexample exists, it is the first element of a 2-tuple in a 2-level tuple-set. However, by what we have said regarding the table below (see “Partial List of Facts Pertaining to the 1-Tree” on page 34), this means that the counterexample is an element of a counterexample “spiral”. The “spiral” maps to a range-element  $y$  of the  $3x + 1$  function in a single iteration of the function, and therefore  $y$  is the root of a  $y$ -tree. The  $y$ -tree contains an infinity of counterexample “spiral”s.

If a counterexample exists, there is an infinity of counterexamples (“Lemma 2.0” on page 9, and the fact that, by the ... 2, 1, 3, 2, 1, 3, ..., pattern in “spiral” elements (see “Partial List of Facts Pertaining to the 1-Tree” on page 34)) we know that there is an infinity of counterexample range-elements. Hence there is an infinity of  $y$ -trees, where  $y$  is a counterexample range element).

3. There are no counterexample tuples in the set of all tuple-sets if the set contains only non-counterexample tuples.

It is easily shown that in order for there to be an infinity of tuples associated with each finite exponent sequence there must be an infinity of non-counterexample tuples in each tuple-set that are extended by the exponent 1, and an infinity that are extended by the exponent 2, etc. (See “Extensions of Tuple-sets” on page 7.) However, this property cannot survive the omission of a countable infinity of “spiral”s.

It seems, therefore, that we must conclude that the existence of a counterexample results in the failure of non-counterexample tuples to be associated with *each* finite exponent sequence, contradicting “Lemma 2.0” on page 9.

If this reasoning can be made precise and valid, we have a proof of the  $3x + 1$  Conjecture.

(Note: we must reconcile the omission of non-counterexample “spiral”s from tuple-sets, with the fact that the 1-tree is the same, whether or not counterexamples exist.)

## Second Possible Strategy for a Proof Based on 1-Tree

1. Let  $S$  denote the “spiral”  $\{1, 5, 21, 85, 341, \dots\}$  in the 1-tree. (For background on “spiral”s see “Recursive “Spiral”s: The Structure of the  $3x + 1$  Function in the “Backward”, or Inverse, Direction” on page 10.)

This “spiral” is the set of odd, positive integers that map to 1 in one iteration of the  $3x + 1$  function. The set of odd, positive integers lying between “spiral” elements is called an *interval*. Thus,  $\{3\}$  is the first interval in  $S$ ,  $\{7, 9, 11, 13, 15, 17, 19\}$  is the second interval, etc.

2. By computer test<sup>1</sup>, we know that all odd, positive integers less than  $5 \cdot 10^{18}$ , are non-counterexamples. These integers occupy more than the first 30 successive intervals in  $S$ .

Call the largest odd, positive integer less than  $5 \cdot 10^{18}$ ,  $w$ .

3. A basic fact governing “spiral”s is that the elements of each “spiral” other than  $S$ , occupy an infinity of successive intervals in  $S$ . Thus, consider the “spiral”  $\{3, 13, 53, 213, \dots\}$  and the intervals in the “spiral”  $S$ .

We remark in passing that each “spiral” element  $x$ , and the range element  $y$  it maps to in one iteration of the  $3x + 1$  function, is a tuple  $\langle x, y \rangle$  in a tuple-set. Since the set of elements of a “spiral” map to their range element either by all odd exponents or by all even exponents, it is easy to see that the set of all elements of all “spiral”s in the 1-tree is the set of all first elements of all non-counterexample 2-tuples in all 2-level tuple-sets.

The set of all (finite) upward paths (that is, paths in the direction of the root, 1, of the 1-tree, or away from 1 in the upward direction) is the set of all non-counterexample tuples in the set of all tuple-sets.

4. Therefore we can go through 1, 2, 3, ...,  $w$ . Call the set of these odd, positive integers,  $W$ . We can color green each interval element  $y$  in  $W$  that is a range element of the  $3x + 1$  function, and then color green each element of the  $y$ -tree (same structure as the 1-tree, except with  $y$  as its root — see the above-mentioned section on “spiral”s). We can then color green the elements of all *extensions* of each  $y$  in  $W$ , stopping at 1.

We can then color green each interval element  $v$  in  $W$  that is not a range-element (that is, which is a multiple-of-3), and all *extensions* of each such  $v$ , stopping at 1.

---

1. See results of tests performed by Tomás Oliveira e Silva, [www.ieeta.pt/~tos/3x+1/html](http://www.ieeta.pt/~tos/3x+1/html). All consecutive odd, positive integers to at least  $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$ , which is greater than  $3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35-1)} - 1$ , have been tested and found to be non-counterexamples.



We can then color green each *interval element* in  $S$  that equals an element that we have already colored green. Step 3 assures us that infinities of interval elements in  $S$  greater than  $w$  will be colored green.

5. Counterexamples occupy intervals only. They are never “spiral” elements.

If counterexamples exist, they can never themselves fill an interval in  $S$  as non-counterexamples less than  $w$  have done. The reason is that this would prevent non-counterexample “spiral” elements from having “room” in the interval, as these elements must, by step 3.

But after the first counterexample, *non-counterexamples* can never themselves fill an interval in  $S$  as non-counterexamples have done. The reason is that this would prevent *counterexample* elements from having room in the interval, as these elements must, by step 3.

So after the first counterexample, all subsequent intervals in  $S$  must be shared between non-counterexamples and counterexamples. There can never again be an interval containing solely non-counterexamples, and there can never be one containing solely counterexamples.

It seems hard to believe that the behavior of the  $3x + 1$  function would change so radically once the first counterexample has appeared — especially since a counterexample is simply another odd, positive integer that, if it and all other counterexamples were *non-counterexamples*, would result in an infinite succession of intervals containing solely non-counterexamples.

We might ask, informally, “How does the function ‘know’ what to do when it is given an odd, positive integer that may or may not be a counterexample?” Keeping in mind that, as far as the function is concerned, a counterexample is simply an odd, positive integer that is not in the 1-tree, one answer (invalid!) is the following.

For each odd, positive integer the function is given, it asks:

1. Is this integer in the 1-tree?

If the answer is yes, then the function asks,

Have I been given an integer *not* in the 1-tree before?

If the answer is yes, then the function simply proceeds with its mixed-intervals-only rule;

If the answer is no, then the function continues its practice of filling successive intervals entirely with non-counterexamples.

2. If the answer to 1. is no, then the function asks,

Have I been given an integer *not* in the 1-tree before?

If the answer is yes, then the function simply proceeds with its mixed-intervals-only rule;

If the answer is no, then the function begins the permanent application of its mixed-intervals-only rule.

But of course there is nothing like this question/answer sequence in the definition of the  $3x + 1$  function.

Whether all this can lead to a proof of the  $3x + 1$  Conjecture, remains to be seen.

### **Third Possible Strategy for a Proof Based on 1-Tree**

We can color green all “spiral” elements that must be in the 1-tree as a result of all consecutive odd, positive integers in the range 1 through  $w$  being non-counterexamples (this is what we did in

“Second Possible Strategy for a Proof Based on 1-Tree” on page 32).

These “spiral” elements are part of the 1-tree *if counterexamples do not exist*.

We now argue that the rest of the “spiral” elements of the 1-tree must likewise be part of the 1-tree if counterexamples do not exist, for there is only one such “rest of the elements”. Therefore, the 1-tree is the 1-tree if counterexamples do not exist.

But one could argue, no, the rest of the elements could be those if counterexamples do exist.

To which one could reply, but then the 1-tree is the same in both cases, and hence counterexamples if they exist are the same as non-counterexamples, which is a contradiction. Therefore counterexamples do not exist.

### Remark

A similar argument can be applied to the smallest counterexample  $y_c$  if counterexamples exist. This counterexample must occupy a position in an interval in the “spiral”  $S = \{1, 5, 21, 85, \dots\}$ . By assumption, all “spiral” elements and all interval elements less than  $y_c$  are non-counterexamples.

Now if  $y_c$  is a range element<sup>1</sup>, then  $y_c$  is the root of a  $y_c$ -tree. But if  $y_c$  were *not* a counterexample then the  $y_c$  tree would be the same as if  $y_c$  were a counterexample, because the definition of the  $3x + 1$  function is not subject to the truth or falsity of the  $3x + 1$  Conjecture.

The reader might reply that it is *extensions* of  $y_c$  that demonstrate counterexample behavior. But all these extensions must consist of range elements, and each range element occupies a position in an interval in the “spiral”  $S$ . And so what we said in the previous paragraph applies to each of these range elements.

We tentatively conclude that there is no difference between non-counterexample and counterexamples and hence that counterexamples do not exist.

### Partial List of Facts Pertaining to the 1-Tree

It is a major challenge to find a proof of the  $3x + 1$  Conjecture based on the 1-tree. In this section, we set forth a list of facts that may enable the creative mathematician to arrive at such a proof. The reader should not overlook “Possible Proof of Conjecture Based on 1-Tree” on page 31.

- There is exactly one 1-tree, whether or not counterexamples exist ( Lemma 8.8 in our paper, “Are We Near a Solution to the  $3x + 1$  Problem?”, on [occampress.com](http://occampress.com)). The question is, Does the 1-tree contain all odd, positive integers (in which case the Conjecture is true) or does it contain only a proper subset of the odd, positive integers (in which case the Conjecture is false).

- If  $x$  is an element of a “spiral”, then  $4x + 1$  is the next element.

Each “spiral” maps to a range element  $y$  in one iteration of the  $3x + 1$  function either by all odd exponents, or by all even exponents.

The successive elements of a “spiral” are mapped to in accordance with a rule that can be expressed as ... 2, 1, 3, 2, 1, 3, ..., where “2” means “is mapped to by all even exponents”, “1” means “is mapped to by all odd exponents”, and “3” means “is not mapped to because element is a multiple-of- $x$ , hence not a range element”.

---

1. We can make appropriate adjustments in our argument if  $y_c$  is not a range element.

Thus, for example, in the “spiral”  $S = \{1, 5, 21, 85, \dots\}$ , 1 is mapped to by all even exponents, 5 is mapped to by all odd exponents, 21 is not mapped to because it is a multiple-of-3, 85 is mapped to by all even exponents, ...

- The elements of each “spiral” other than  $S$ , occupy an infinity of successive intervals in  $S$ , one element per interval. Thus, consider the “spiral”  $\{3, 13, 53, 213, \dots\}$ . The same applies recursively down through the 1-tree, except that if a “spiral”  $s$  maps via *even* exponents to  $y$  in another “spiral”  $r$  in one iteration of the  $3x + 1$  function, then the elements of  $s$  skip one interval in  $r$  before occupying an infinity of successive intervals in  $r$ , one element per interval.

- Each “spiral” element  $x$ , and the range element  $y$  it maps to in one iteration of the  $3x + 1$  function, is a tuple  $\langle x, y \rangle$  in a tuple-set. Since the set of elements of a “spiral” map to their range element either by all odd exponents or by all even exponents, it is easy to see that the set of all elements of all “spirals” in the 1-tree is the set of all first elements of all non-counterexample 2-tuples in all 2-level tuple-sets.

The set of all (finite) upward paths (that is, paths in the direction of the root, 1, of the 1-tree, or away from 1 in the upward direction) is the set of all non-counterexample tuples in the set of all tuple-sets.

The following table shows the first three 2-tuples in the 2-level tuple-sets  $T_A$  for  $A = \{1\}$ ,  $\{3\}$ , and  $\{5\}$ . (The reader might find it more natural to imagine the exponent 1 tuple-set as being foremost, with the exponent 3 tuple-set parallel and directly behind the exponent 1 tuple-set, and the exponent 5 tuple-set as parallel and directly behind the exponent 3 tuple-set, etc.)

Observe the first three elements of the “spiral” that maps to 5 in one iteration of the  $3x + 1$  function, namely, the elements 3, 13, 53 running vertically.

Observe the first three elements of the “spiral” that maps to 11 in one iteration of the  $3x + 1$  function, namely, the elements 7, 29, 117 running vertically.

Observe the first three elements of the “spiral” that maps to 17 in one iteration of the  $3x + 1$  function, namely, the elements 11, 45, 181 running vertically.

In keeping with the rule governing successive elements of “spirals”, namely, that if  $x$  is a “spiral” element, then  $4x + 1$  is the next element, we observe that  $3(4) + 1 = 13$ , etc.;  $4(7) + 1 = 29$ , etc., and  $11(4) + 1 = 45$ , etc.

But observe also, in keeping with “Lemma 1.0: the “Distance” Functions  $d(i, i)$  and  $d(1, i)$ ” on page 7 :

the sequence of level-2 elements in each 2-level tuple-set is 5, 6, 11, ...

The difference between level-1 elements in the tuple-set  $T_A$  where  $A = \{1\}$  is  $2(2^1)$ ,

The difference between level-1 elements in the tuple-set  $T_A$  where  $A = \{3\}$  is  $2(2^3)$ ,

The difference between level-1 elements in the tuple-set  $T_A$  where  $A = \{5\}$  is  $2(2^5)$ ,

**Table 3: Partial View of Relationship Between 2-Level Tuple-sets and Odd-Exponent “Spiral”s**

Exponent				
5	5	11	17	...
	53	117	181	...
3	5	11	17	...
	13	29	45	...
1	5	11	17	...
	3	7	11	...

Similar 2-level tuple-sets exist for even exponents, that is, for 2-level tuple-sets  $T_A$  where  $A = \{2\}, \{4\}, \{6\}, \dots$ .

- If counterexamples exist, it is not because a sub-tree of the 1-tree has somehow been “broken off” and become a counterexample tree. The structure of the 1-tree is the same, whether or not counterexamples exist.

- If counterexamples exist, then the intervals in each “spiral” — not just the “spiral”  $\{1, 5, 21, 85, \dots\}$  — contain an infinity of counterexamples (“Lemma 2.0” on page 9).

- It is not possible to tell, from a single “spiral”, if counterexamples exist or not. For each range element  $y$  of the  $3x + 1$  function, there is exactly one “spiral”, whether or not counterexamples exist. The function is not itself a function of the existence or non-existence of counterexamples.

- Is there a contradiction in the fact that, in the tree(s) of counterexamples, the intervals of each “spiral” contain all the non-counterexamples?

- Is it possible to traverse from any point (“spiral” element)  $y$  in the 1-tree to any other point (“spiral” element)  $z$  in the 1-tree via the four movements: one element to the right or to the left in a “spiral” (the latter is possible only if  $y$  is not the first element of the “spiral”), up one element to the range-element  $u$  that the “spiral” maps to in one iteration of the  $3x + 1$  function, or down one element to the range element  $v$  that maps to  $y$  in one iteration of the function (there is no such  $v$  if  $y$  is a multiple-of-3, since then  $v$  is not a range element).

## **Appendix D — For Professional Mathematicians Only**

### **Understandable Reluctance of Mathematicians to Read This Paper**

There has been an understandable reluctance on the part of professional mathematicians to give serious attention to this paper, or its predecessors. It seems clear to us that the main reason is mathematicians' difficulty in believing that such an extraordinarily difficult problem can have been solved by a non-mathematician (our degree is in computer science, and we have spent most of our working life doing research in the computer industry). This skepticism is reinforced by the fact that there have been many false claims of solutions to the  $3x + 1$  Problem, the overwhelming majority of which having been made by non-mathematicians.

But we must point out that the occasionally-heard remark, "Nothing of importance in mathematics has ever come from outside the university", is, in fact, false, considering that some of the best of the best worked entirely outside the university — Descartes, Pascal, Fermat, Leibniz, and Galois, to name only the most famous.

We must also not fail to mention another reason for mathematicians' reluctance to read this paper, and that is the online presence of obsolete criticisms of the paper. We know of one website that contains criticisms of a 2015 version of the paper. Not only were the criticisms false, but the proof that was criticized has long since been removed from the paper. Yet despite repeated pleas on our part, the managers of the website have refused to add a note to the website stating that the criticisms do not apply to the current version of the paper. The result is that potential readers — especially mathematicians — are turned away. Apparently we have no recourse in this matter.

It appears that the managers of the above website have no experience of actually doing research. They believe that if a paper is published online and contains an error, that means that the author is incapable of correcting the error, and that his underlying ideas do not deserve any attention. But errors are almost inevitable in the course of attempting to solve very difficult problems. We remind the reader that Wiles' first proposed proof of the Taniyama–Shimura–Weil Conjecture in the early 90s, which implied a proof of Fermat's Last Theorem, contained an error that took him, with the help of the mathematician Richard Taylor, more than a year to repair. The important question obviously was, Do the underlying ideas in this paper offer hope for correcting the error? And the answer was yes.

We have been struck by the eagerness with which readers of this paper look for anything they can regard as an error, and the indifference they display to understanding, and thinking about, the underlying ideas.

### **If You Do Not Accept Our Proof of the $3x + 1$ Conjecture...**

If you do not accept our proof of the Conjecture, or any of the possible strategies for a proof that are set forth in the above appendices, we urge you to at least peruse our paper, "Are We Near a Solution to the  $3x + 1$  Conjecture?" on [occampress.com](http://occampress.com). This paper contains a wealth of results, insights, possible strategies for a proof, plus a section on what we have called " $3x + 1$ -like functions". We will welcome comments.

We are confident that at least two publishable, significant papers can be produced from the material in our  $3x + 1$  papers, and that this is true *even if* the proof of the Conjecture in the present paper and all the possible strategies in the appendices, faulty and cannot be repaired.

We feel that the two structures we have discovered that underlie the  $3x + 1$  function, namely, tuple-sets and recursive "spiral"s, are of fundamental importance, and should be brought to the

attention of the entire  $3x + 1$  research community.

### **Difficulty, So Far, In Getting This Paper Published**

Not surprisingly, so far, no journal that we know of is willing to even consider this paper for possible publication. The reason seems to be that editors cannot believe that such a difficult problem might have been solved by a non-mathematician.

### **Incentives for Mathematicians to Take This Paper Seriously and to Spread the Word About It**

Unquestionably, if this paper contains a solution to the  $3x + 1$  Problem, or can easily be modified to contain a solution, considerable prestige will be gained by the first mathematicians who promote the paper. Of course, mathematicians who believe that the proof of the Conjecture is correct, and/or that at least one of the possible strategies in the appendices look promising, but do not want to risk their reputations by saying so, especially given that the author of the paper is not an academic mathematician, will not be recipients of that prestige. We are offering three incentives:

- (1) Any reasonable consulting fee;
- (2) Generous mention in the Acknowledgments when the paper is published. (But no name will be mentioned without the prior written approval of the mathematician concerned.)
- (3) An offer of shared authorship to the first mathematician who makes a significant contribution to the paper prior to publication.

In any case, all communications we receive about this paper will be kept strictly confidential.

## **Appendix E —A Brief Summary of Our Solution to the Problem**

The following is a summary of our solution to the Problem — that is, of our proof of the  $3x + 1$  Conjecture.

1. Using Crandall's alternate, but equivalent, definition of the  $3x + 1$  function, we define an infinite set of infinite  $i$ -level tuple-sets, where  $i \geq 2$ , and where a tuple is the result of a finite sequence of iterations of the function. Thus, for example,  $\langle 17, 13, 5, 1 \rangle$  is a tuple in a tuple-set.

2. By computer test, we know that the 35-level elements in all first 35-level tuples in all 35-level tuple-sets are non-counterexamples. This fact holds regardless if counterexamples exist or not -- that is, no known non-counterexample can somehow become a counterexample, under any circumstances.

3. The distance function (part (a) of “Lemma 1.0: the “Distance” Functions  $d(i, i)$  and  $d(1, i)$ ” on page 7) gives the value of the  $i$ th element of the  $n$ th  $i$ -level tuple in an  $i$ -level tuple-set, where  $n \geq 2$ . This function is not a function of the truth or falsity of the  $3x + 1$  Conjecture.

4. Therefore, from step 2, the value of the 35th-level element of the  $n$ th 35-level tuple in any 35-level tuple-set is the same, whether or not counterexamples exist.

5. But since each  $(i + 1)$  - level tuple-set is an extension of  $i$ -level tuples in an  $i$ -level tuple-set, we see that the value of the 36th-level element in the  $n$ th 36-level tuple in any 36-level tuple-set is the same, whether or not counterexamples exist. And so on for all  $i$ -level tuple-sets, where  $i \geq 36$ .

6. But then we must arrive at the conclusion that the set of all non-counterexample tuples if counterexamples do not exist, is the same as the set of all non-counterexample tuples if counterexamples exist. Therefore the set of counterexamples is the empty set, and the  $3x + 1$  Conjecture is true.