The Structure of the 3*x* + 1 Function

by

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August 30, 1999

with Offers of Prizes for Proofs of Conjectures and for Answers to Questions

Important Note (Dec. 10, 2002)

This paper gives proofs of all theorems and lemmas in Schorer's "The Structure of the 3*x* + 1 Function: An Introduction" that are not given in that paper. (That paper, like this one, is accessible on the web site www.occampress.com.) All of these proofs have been checked and deemed correct by at least half a dozen mathematicians. However, this paper has not been revised since the indicated date, and therefore numbering of theorems, lemmas and conjectures is not necessarily the same as that in the Introduction. Furthermore, improved versions of definitions and of much of the exposition, plus extensive additional material on possible strategies and possible proofs of the 3*x* + 1 Conjecture, plus new lemmas, will be found in the Introduction.

The author will be glad to answer any questions concerning either paper.

Abstract

The $3x + 1$ Problem asks if iterations of the $3x + 1$ function

$$\frac{3x + 1}{2^{ord_2(3x + 1)}}$$

always terminate in 1, where x is an odd, positive integer, $ord_2(3x + 1) = a_j$ is the largest exponent such that the denominator evenly divides the numerator. (The result of each iteration is thus again an odd, positive integer.) In this paper, we describe two structures underlying this function. The "spacial", "geometric" nature of each structure suggests several strategies for a solution to the problem which otherwise are not at all apparent. The first structure is called "tuple-sets". Each tuple-set consists of all tuples whose elements are the successive odd, positive integers produced by a finite sequence A of exponents $a_j = ord_2(3x + 1)$. We define distance functions between the elements of the tuples in tuple-sets (Lemmas 1.0 and 1.1), then describe several possible strategies for solving the Problem using tuple-sets, supporting each strategy with various lemmas. Among these lemmas are that for each odd, positive integer y not a multiple of 3, and for each finite sequence A of exponents, there exists an x that maps to y via A with the possible concatenation of one additional exponent (Lemma 7.0). Also that the infinitary tree of all possible tuple-sets can be reduced to an equivalent finitary tree (Lemma 7.3).

The second structure is called "recursive 'spiral's". Each range element y of the $3x + 1$ function defines an infinite set of such "spiral"s. This set represents all odd, positive integers x that map to y in a finite number of iterations of the $3x + 1$ function. The set is self-similar in the sense defined by Mandelbrodt. We define distance functions between the elements of "spiral"s (Lemmas 11.0, 12.1, and 12.2), then describe several possible strategies for solving the Problem using recursive "spiral"s, supporting each strategy with various lemmas. Among these are Lemma 15.85, which establishes the sequence of congruence classes mod $2 \cdot 3^{i - 1}$ that each element of each possible "spiral" belongs to.

Finally, we show how the two structures can be merged into a single structure that suggests another strategy for a solution to the Problem.

Key words: $3x + 1$ Problem, $3n + 1$ Problem, Syracuse Problem, Ulam's Problem, computational number theory, proof of termination of programs, recursive function theory

American Mathematical Society Classification Numbers: 11Y16, 11Z05, 03D20, 68Q60

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Introduction

Statement of Problem

The $3x + 1$ Problem, also known as the $3n + 1$ Problem, the Syracuse Problem, the Collatz Problem, Ulam's Problem, Kakutani's Problem, and Hasse's Algorithm,asks if repeated iterations of the *3x + 1 function*

$$\frac{3x + 1}{2^{ord_2(3x + 1)}}$$

always terminate in 1, where x is an odd, positive integer, $ord_2(3x + 1) = a_j$ is the largest exponent of the prime *2* such that the denominator evenly divides the numerator. (The result of each iteration is again an odd, positive integer.) Thus, for example, beginning with $x = 5$, we have the result of one iteration, y, = 1, and we stop. Here $a_j = 4$. Beginning with $x = 13$, we have the result of one iteration, y, = 5, with $a_j = 3$. Then, as in the previous example, 5 yields 1, and we stop.

The conjecture that all repeated iterations eventually terminate in 1, that is, that no counterexamples exist, will hereafter be referred to as Conjecture 1.

Although many results are known — see, for example, [1], [2], and [3] — no proof or counterexample is known.

Summary of Research on the Problem

As stated in [1], "The exact origin of the $3x + 1$ problem is obscure. It has circulated by word of mouth in the mathematical community for many years. The problem is traditionally credited to Lothar Collatz, at the University of Hamburg. In his student days in the 1930's, stimulated by the lectures of Edmund Landau, Oskar Petron, and Issai Schur, he became interested in number-theoretic functions. His interest in graph theory led him to the idea of representing such number-theoretic functions as directed graphs, and questions about the structure of such graphs are tied to the behavior of iterates of such functions... In the last ten years [that is, 1975-1985] the problem has forsaken its underground existence by appearing in various forms as a problem in books and journals..."

[1] and [2] are the best summaries of research on the problem through Jan. 1996, as far as the author is aware. More than 150 papers are listed. [1] discusses some of the more important results in addition to giving an unannotated list of papers. [2] is an annotated bibliography. Because of the apparent lack of structure in the $3x + 1$ function, much of the research has been based on analytic (in the number-theoretic sense) and probabilistic approaches. p-adic number theory has been utilized.

[3], published in 1998, is an excellent, detailed presentation, with proofs, of many of the significant results connected with the analytical (in the number-theoretic sense) and probabilistic approaches to the Problem.

Note: The author discovered tuple-sets in the early eighties, but was unable to get any papers on them published because editors were not convinced they would lead to a solution of the Problem. Then, during the refereeing process of an earlier version of the present paper in 1998, the author was informed that results similar to some of those in the paper had recently appeared in chapter II of [3]. However, these results do not explicitly set forth the tuple-sets or recursive "spiral"s structures, nor do they include most of the results, or any of the possible strategies for a proof of Conjecture 1, that are contained in this paper. This paper points out correspondences, both in terminology and in results, with [3]. They are listed in the Index under “Wirsching...”.

Prizes for Proofs of Conjectures and for Answers to Questions

The offering of prizes for proofs of conjectures has a long history in mathematics. Prizes were offered for a proof of Fermat’s Last Theorem. Prizes have been offered for a proof of Conjecture 1 (see (Lagarias, 1985)). However, since some mathematicians are reluctant to accept money for doing proofs, winners are not required to accept the prize money offered in this paper.

Prizes are offered for proofs or disproofs of conjectures, and for answers to certain questions. However, since some of the questions are less precisely stated than the conjectures, prizes are not always offered for answers. Offers of prizes in this paper are given in the Index under “Conjecture...” and “Question...”.

Rules governing prizes are as follows:

(1) Conjectures: prize money will be awarded for the first valid proof *or disproof* that the author of this paper receives. *The proof or disproof need not be original.* Questions: prize money will be awarded for the first correct answer to the question. *The answer need not be original*.

(2) The winner implicitly agrees to allow the author of this paper to publish the winner’s proof or answer as part of future versions of this and other papers by the author. If the winner so requests, his or her name will not be mentioned in future versions of this and other papers by the author, until a paper by the author containing the proof or disproof or answer, is published in a refereed journal.

(3) The author of this paper and the prize winner will keep a record of the actual prize-winning proof or answer in case later versions of this paper result in accidental errors.

(4) If a prize winner’s proof or answer is found to contain an error, the prize winner will have up to one month to repair the error. If the prize winner is unable to do so, he or she will return the prize money to the author of this paper within one month following the end of the month allowed to repair the error.

(5) The prize winner has permission to publish the proof or answer on his or her own, provided due and appropriate credit is given to the author of this paper, specifically, as to the ideas and proofs existing in this paper at the time the winner won the prize.

(6) *It is possible that a given prize may already have been won at the time a reader decides to attempt a proof or disproof or answer a question. Also, amounts of prizes are subject to change without notice!* Therefore, prospective prize winners are advised to contact the author of this paper before beginning to work on any proof or disproof or answer to a question!

Consultants Sought

The author is seeking consultants to help bring this work to fruition. Terms are as follows.

Consultant must have a knowledge of elementary number theory at least equivalent to that of a senior undergraduate math major.

Consultant will be payed for hours worked, regardless of the ultimate outcome of this research. Hours worked can be as little as one per week.

Shared authorship will be offered if the consultant makes a significant contribution, where the meaning of “significant” is to be agreed upon beforehand between author and consultant.

About This Paper

First of all, the author encourages the reader to *use the extensive Index* to rapidly find definitions of terms, statements and proofs of lemmas, etc.

Second of all, the author is only too aware that the style and notation of this paper are much in need of improvement. On the other hand, the author believes that, in attempting to solve difficult problems, ideas must come first, even if the initial presentation of these ideas is awkward and long-winded.

This paper is a compendium of results from several other papers. In order to avoid confusion for persons who received these other papers, and who will receive this one, original numbering of lemmas has been retained, even when this numbering contradicts the order in which the lemma occurs in this paper. The reader will find it easy to locate the page containing any specific lemma by simply looking up the lemma in the Index under “Lemma...”.

Periods, commas, and similar punctuation following equations that occupy separate lines are absent due to a limitation of the word-processor used to write this paper. The author believes this will not be a source of confusion.

Equations are always numbered relative to a given proof.

All terms defined in this paper, as well as lemmas, remarks, questions, and conjectures are listed in the Index.

This paper is a work in progress. Hence there are almost certainly to be errors, for which the author apologizes.

Suggestions for a First Reading of (Parts of) This Paper

There is absolutely no reason for the reader to feel that this paper is an all-or-nothing proposition, i.e., that the choice is between reading the entire paper or reading nothing of it. An understanding of the basic ideas can be obtained by reading only a few selected pages and using the Index as necessary to look up the definitions of terms. The author recommends the following.

“Abstract” on page2;

“Section 1: Tuple-Sets” on page6, through Lemma 1.1;

The initial tables summarizing properties of tuple-sets and rows at the start of
“Summary of Properties of Tuple-sets” on page13;

Initial paragraphs of “The “PushingAway” Strategy ” on page19;

“Strategy of Proving There Is No Minimum Counterexample” on page58;

“.Section 2. Recursive “Spiral”s” on page62, through
“Distance Functions on “Spiral”s” on page63 and the lemmas defining the functions.

“Strategy of “Filling-in” of Intervals” on page68

“Section 3. A Single Structure Combining Tuple-sets and Recursive “Spirals”” on page83.

Section 1: Tuple-Sets

In the first section of this paper, we describe a structure called "tuple-sets" that underlies iterations of the 3x + 1 function. The "spacial", "geometric" nature of this structure is important for the strategies it suggests. Informally, for a given finite sequence of exponents $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, $a_i \geq 1$, the tuple-set T_A consists of:

all tuples of length 1 containing an odd, positive integer that does not map to another odd, positive integer via the exponent a_2; plus

all tuples of length 2 representing all computations of the 3x + 1 function obtained from the exponent sequence $\{a_2\}$ such that the last element of each tuple does not map to another odd, positive integer via the exponent a_3; plus

all tuples of length 3 representing all computations of the 3x + 1 function obtained from the exponent sequence $\{a_2, a_3\}$ such that the last element of each tuple does not map to another odd, positive integer via the exponent a_4; plus ...

all tuples of length i representing all computations by the 3x + 1 function obtained from the exponent sequence A.

Tuples are oriented vertically on the page in a sequence extending infinitely to the right. There is always a first (leftmost) tuple in every tuple-set.

Definitions

3*x* + 1 function, *C*(*x*)

In the literature, the most common definitions of the 3x + 1 function are f and T:

$f(x) = \{3x + 1 \text{ if } x \text{ is odd; } x/2 \text{ if } x \text{ is even}\}$;

$T(x) = \{(3x + 1)/2 \text{ if } x \text{ is odd; } x/2 \text{ if } x \text{ is even}\}$;

(In [3], n is used instead of x to emphasize that the domain of the function is the natural numbers (p. 11).)

A few authors have used the definition (or an equivalent) given at the start of this paper. We designate this representation of the function as $C(x)$, following Crandall ([3], p. 65):

$$C(x) = \frac{3x + 1}{2^{ord_2(3x+1)}}$$

where $ord_2(3x + 1)$ is the largest exponent of 2 such that the denominator evenly divides the numerator. We will show that this definition of the function, in which successive divisions by 2 are collapsed into a single power of 2, brings out two structures underlying the function that are otherwise are not at all evident. Henceforth in this paper, unless otherwise specified, the term "3x + 1 function" will refer to $C(x)$.

Iteration

An *iteration* takes an odd, positive integer, x, to another odd, positive integer, y, via one application of the 3x + 1 function.

Computation

A *computation* is a sequence of one or more successive iterations of C, i.e.,

$$(C^k(x))_{k \geq 0} = (x, C(x), C^2(x), \ldots)$$

The number of successive iterations in a computation may be finite or infinite. The term *computation* corresponds to the term *C-trajectory*, or simply, *trajectory*, in [3] (p. 10), except that a computation contains no even numbers.

Domain Element, Maps to, Is Mapped to, Range Element

If x yields y in one iteration of the $3x + 1$ function, we say that x *maps to* y in one iteration. We also say that x *maps to* any other odd, positive integer occuring in the computation of x. We will sometimes refer to x as a *domain element*, since it is an element of the domain of the $3x + 1$ function. We will say that y, or any subsequent odd, positive integer in the computation of x, is *mapped to* by x. We will sometimes refer to y, or any subsequent odd, positive integer in the computation of any x, as a *range element*, since it is an element of the range of the $3x + 1$ function. It is trivial to show that there are no multiples of 3 in the range of the $3x + 1$ function (see Lemma 0.2.), so the range is a proper subset of the domain.

Power of 2

A *power of 2* is a term 2^{a_j} .

Exponent

An *exponent* is the exponent a_j of 2 in the power of 2 that, in the $3x +1$ function, yields an odd, positive integer in one iteration. Sometimes, by abuse of language, we shall speak of a_j as "mapping to" a range element y, by which we shall mean that $(3x + 1)/2^{a_j} = y$.

A sequence $A = \{a_2, a_3, ..., a_i\}$ of exponents corresponds to an *admissible vector* in [3] (p. 42). (The reason we begin our subscripts with 2 is trivial and will be explained below.)

The 3*x* + 1 Conjecture (Conjecture 1)

The $3x + 1$ Conjecture (referred to in this paper as Conjecture 1) asserts that, for all domain elements x, repeated iterations of the $3x + 1$ function beginning with x eventually terminate in 1. A domain element that does not meet this condition is called a *counterexample*. That is, a counterexample x has the property that, for all z mapped to by x, $z > 1$. A *minimum* counterexample y, with which we will be concerned in certain parts of this paper, has the following two properties: (1) for all z mapped to by y, $z \geq y$, and (2) for all x mapping to y, $x \geq y$.

Tuple-set

A *tuple-set*, T_A, is defined as follows. (The reader may find it helpful to refer to Fig. 1 while reading the following. The integers between arrows are explained below, immediately following the statement of Lemma 1.0.)

Let $A = \{a_2, a_3, ..., a_i\}$ be a finite sequence of exponents, where $a_i \geq 1$, $i \geq 2$. (The reason for beginning our indexes with 2 is trivial, and will become clear below in the definition of a *level* in a tuple-set.) We associate with each A a *tuple-set* T_A that, informally, represents all possible computations such that the sequence of exponents in each computation is given by A. Formally, T_A is the set of all tuples t_k such that:

(a) the first element $t_{k(1)}$ of t_k is a domain element;

(b) the j'th element $t_{k(j)}$ of t_k, $j \geq 2$, if it exists, is the range element produced, in a single iteration, from the (j-1)'th element of the tuple t_k, via the exponent a_j.

The set of tuples in a tuple-set is thus as described in the first paragraph of Section 1.

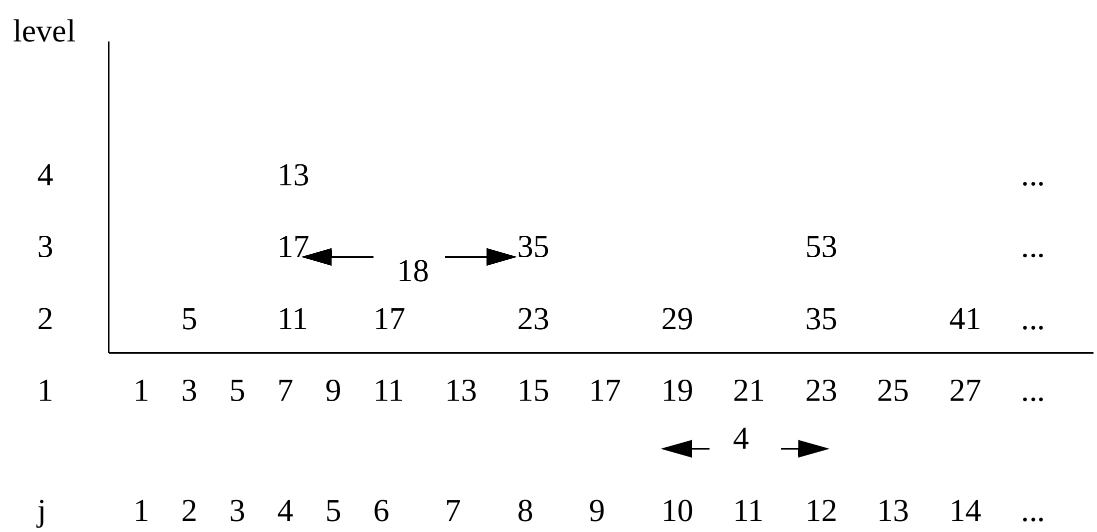

Fig. 1. Part of the tuple-set T_A associated with the sequence A = {1, 1, 2}

Fig. 1 shows part of a tuple-set, namely, the tuple-set T_A associated with the sequence A = {1, 1, 2}.

The 2nd element of the 8th tuple, $t_{8(2)}$, is 23 because 23 is the range element mapped to by the 1st element, 15, in one iteration ($a_2 = 1$).

The 4th element of the 4th tuple, $t_{4(4)}$, is 13 because 13 is the range element mapped to by the 3rd element, 17, in one iteration ($a_4 = 2$).

There is no 2nd element of the 1st tuple because there is no range element mapped to by 1 such that $a_2 = 1$.

There is no 2nd element of the 5th tuple because there is no range element mapped to by 9 such that $a_2 = 1$.

Tuples in a tuple-set are ordered according to their first elements.

Level in a Tuple-set

A *level* j in a tuple-set is defined as follows. If $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, is a finite sequence of exponents, the subscrpt j in a_j, $2 \leq j \leq i$, denotes the *level* j in T_A. As specified under the definition of tuple-set, we begin numbering our levels with 2 so that level 1 is then the level containing the set of all possible tuple first elements {1, 3, 5, 7, ... } in any T_A, that is, the set of odd, positive integers, or, in other words, the set of all domain elements.

If a tuple has an element at level j, but none at level $j + 1$, we will refer to the tuple as a *j-tuple*, or a *j-level tuple*. If the tuple also has an element at level $j + 1$, we will sometimes refer to the tuple as a ($\geq j$)-tuple. The longest tuple in any tuple-set defined by an exponent sequence of length i - 1 is an i-level tuple.

In the case that $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, we will refer to T_A as an *i-level tuple-set*. Clearly, every range element mapped to by a given (i - 1)-long exponent sequence occurs in level i of the corresponding tuple-set.

Tuples Consecutive at Level j

Tuples *consecutive at level j* are defined as follows. Let t_k, t_m be ($\geq j$)-tuples in some T_A. If there is no ($\geq j$)-tuple between t_k and t_m, we say that t_k and t_m are *tuples consecutive at level j*. Here, "between" means relative to the natural linear ordering of tuples based on their first elements.

Thus, for example, in Fig. 1, tuples 4 and 8 are consecutive at level 3.

Row

Let $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, be a sequence of exponents, and let T_A be the corresponding tuple-set. Then a *level-j row*, R_j, $1 \leq j \leq i$, in T_A is the set of all jth tuple-elements in tuples consecutive at level j. We shall see, as a result of the distance functions defined in Lemmas 1.0 and 1.1, that each row is a congruence class.

Extensions of Tuples and of Tuple-sets

Let T_A be a tuple-set defined by the sequence of exponents $A = \{a_2, a_3, ..., a_i\}$. Then any tuple-set $T_{A'}$ defined by a sequence of exponents $A' = \{a_2, a_3, ..., a_i, a_{i+1}\}$ is called an *extension* of T_A. We define extensions of tuples in a similar manner. Thus, a ($\geq i$)-tuple in $T_{A'}$ is an extension of an i-tuple in T_A.

If $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, is a sequence of exponents, then we define an *initial sub-sequence of the exponent sequence A* as the sequence $\{a_2, a_3, ..., a_j\}$, where $2 \leq j \leq i$. Thus, for example, $\{a_2\}$ is an initial sub-sequence of A, and so is $\{a_2, a_3, a_4\}$, but, for example, $\{a_3, a_4\}$ is not. We define an *inititial sub-sequence of a tuple* t_k similarly.

With the concept of extensions of tuples and tuple-sets established, we can see that every j-tuple, $2 \leq j \leq i$, defined by an initial sub-sequence $\{a_2, a_3, ..., a_j\}$ of A is in the tuple-set T_A.

Non-terminating Tuple (n-t-v-1, n-t-v-c)

As stated under "Computation" on page 7, a trajectory (tuple) may be finite or infinite. We will use the term *non-counterexample tuple* to denote a finite tuple whose elements ultimately map to 1, and the term *counterexample tuple* to denote a finite tuple whose elements are counterexamples. We will use the term *n-t-v-1* (non-terminating-tuple-via-1) to denote an infinite tuple whose elements map to 1, and the term *n-t-v-c* (non-terminating-tuple-via-c (*c* for *counterexample*)) to denote an infinite tuple whose elements are counterexamples.

It is possible that a tuple contains an infinite repetition of a sequence of its elements, where the sequence may be of length 1. (The tuple <1, 1, 1, ...> is a trivial example, and the only known example at time of writing.) Clearly, any such tuple is infinite. If the repeated element is not 1, then the tuple contains solely counterexample elements. An infinitely repeated sequence of elements is called a *cycle*.

We remark in passing that, since the statement of the $3x + 1$ problem specifies that iterations are to terminate when and if 1 is reached, counterexamples are sometimes described as giving rise to "infinite" or non-terminating computations. However, by the definition of n-t-v-1 and n-t-v-c, *all* domain elements give rise to "infinite" computations.

Graphical View of a Tuple-set

At this point, it will be helpful if we get an abstract view of the various-length tuples in a tuple-set. Let T_A be any tuple-set, with $A = \{a_2, a_3, ..., a_i\}$. Then, as shown in Fig. 3.05, there is an infinity of tuples consecutive at level i and, indeed, at all levels $1 \leq j \leq i$. Between each pair of i-level tuples there is a finite set of tuples consecutive at level i - 1. Between each pair of these is a finite set of tuples consecutive at level i - 2, etc., down to level 1. The distance (numerical difference) between elements of tuples at each level will be specified in Lemmas 1.0 and 1.1.

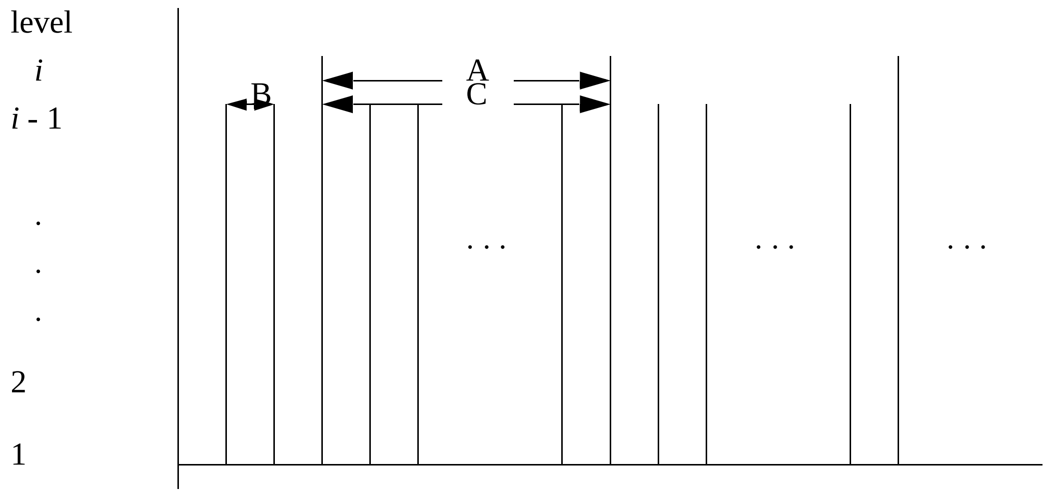

Fig. 3.05. Graphical view of tuples in a tuple-set.

A, the distance (numerical difference) at level i between elements of tuples consecutive at level i, = $2 \cdot 3^{i-1}$ (Lemma 1.0 (a))

B, the distance (numerical difference) at level i - 1 between elements of tuples consecutive at level i - 1, = $2 \cdot 3^{i-2}$ (Lemma 1.0(a))

C, the distance (numerical difference) at level i - 1 between elements of tuples consecutive at level i, =

$$lcm(2 \cdot 2^{a_i}, 2 \cdot 3^{i-2}) = 2 \cdot 2^{a_i} \cdot 3^{i-2}$$

(Lemma 1.1)

Distance Functions on Tuple-sets

Lemma 1.0 (a) Let $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, be a sequence of exponents, and let t_k, t_m be tuples consecutive at level *i*. Let $d(i, i)$, the *distance between* t_k *and* t_m *at level i,* be defined by $|t_{k_i} - t_{m_i}|$, *where* $|x|$ *denotes the absolute value of x. Then d(i, i) is given by:*

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) Let t_k, t_m *be tuples consecutive at level i in* T_A. *Let d(1, i), the distance between* t_k *and* t_m *at level* 1, *be defined by* $|t_{k_1} - t_{m_1}|$. *. Then d*(1, *i) is given by:*

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3})\ldots(2^{a_i})$$

Thus, in Fig. 1, the distance $d(3, 3)$ between $t_{8(3)} = 35$ and $t_{4(3)} = 17$ is $2 \cdot 3^{(3-1)} = 18$. The distance $d(1, 2)$ between $t_{12(1)} = 23$ and $t_{10(1)} = 19$ is $2 \cdot 2^1 = 4$.

Proof:

Since the proof is rather long, but involves only basic algebraic manipulations, it has been placed in Appendix A.

Remarks About the Distance Functions

(1) Strictly speaking, we should include the sequence *A* of exponents as arguments of $d(1, i)$, $d(i, i)$, but this notation would be cumbersome and, since typically this sequence is known, unnecessary.

(2) The distance functions make clear that, for each finite sequence of exponents, there exists an infinity of computations produced by that sequence. (The equivalent of this statement is made in [3] (p. 48).) In particular, there exists an infinity of tuples consecutive at level *i* for all $i \geq 2$.

(3) The formula for $d(1, i)$ implies that it is possible for pairs of tuples consecutive at level *i* in one tuple-set to be the same distance apart, at level 1, as pairs of tuples consecutive at level 1 in another tuple-set. For example, this would occur between tuples consecutive at level 2 in T_A when $A = \{2\}$ ($d(1, 2) = 2 \cdot 2^2 = 8$) and between tuples consecutive at level 3 in $T_{A'}$ when $A' = \{1, 1\}$($d(1, 3) = 2 \cdot 2^1 2^1 = 8$).

(4) The distance between elements of tuples consecutive at level *j*, $2 \leq j < i$, is given in Lemma 1.1.

(5) It is straightforward to prove that the distance functions carry over into the negative integers as well.

Our next task is to prove several lemmas (Lemmas 2.0 through 6.0) required for the proof of Lemma 7.0, which states that for any *y* in the range of the 3*x* + 1 function, and for any sequence of exponents, with the possible addition of a concluding “buffer” exponent, there exists an *x* whose computation yields *y*. But first, we give the distance between tuples consecutive at level *j* and then dispose of the question of the countability of tuple-sets.

Lemma 1.1. *Let T_A be a tuple-set defined by a sequence $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$. Then the distance $d(j, i)$ between elements at level j, $1 \leq j \leq i$, of tuples t_k, t_m consecutive at level i is given by the following table:*

Table 1: Distances between elements of tuples consecutive at level i

Level	Distances between elements of t_k, t_m at level
i	$2 \cdot 3^{i-1}$
i - 1	$2 \cdot 3^{i-2} \cdot 2^{a_i}$
i - 2	$2 \cdot 3^{i-3} \cdot 2^{a_{i-1}} 2^{a_i}$
i - 3	$2 \cdot 3^{i-4} \cdot 2^{a_{i-2}} 2^{a_{i-1}} 2^{a_i}$
...	...
2	$2 \cdot 3 \cdot 2^{a_3} \ldots 2^{a_{i-1}} 2^{a_i}$
1	$2 \cdot 2^{a_2} 2^{a_3} ... 2^{a_{a-1}}\ 2^{a_i}$

Proof:

The distances at levels 1 and i were established by Lemma 1.0. The distance at any other level j follows from the fact that, by Lemma 1.0, the distance must be the least common multiple of

$$2 \cdot 3^{j-1}$$

and

$$2 \cdot 2^{a_{j+1}} \cdot 2^{a_{j+2}} \cdot \ldots \cdot 2^{a_i}$$

□

Lemmas 1.0 and 1.1 are stronger results than Lemma 3.1 in [3] (p. 45), since the latter only deals with congruence mod 3^{i-1} of elements at level i, not at levels 1, 2, 3, ..., i - 1. (Also, congruence mod 3^{i-1} does not imply congruence mod $2 \cdot 3^{i-1}$.)

Summary of Properties of Tuple-sets

For readers with limited time, we now provide a table that summarizes our results — both those above and those to follow — on tuple-sets and rows in a tuple-set. (Recall that a row is simply the set of elements at a given level in a given tuple-set.) We break the properties of rows into three parts: those concerning top rows, those concerning middle rows, and those concerning the bottom (i.e., first) row. The phrase "extension of a top row R_i" means the same thing as "the top row R_{i+1} mapped to by a top row R_i".

The table entry for each property whose value is known includes a reference to definitions or lemma(s) that establish the value.

Following these tables, we present the results themselves. They are organized by the possible strategies they suggest for proving Conjecture 1.

Note: some table-rows may have the same content as other rows, though under different properties. This redundancy is deliberate, the purpose being to aid understanding and to make the looking up of properties easier.

Table 2: Some important properties of tuple-sets

Property	Value of property	Reference
Sequence of exponents, A, that define a tuple-set T_A	$A = \{a_2, a_3, ..., a_{\mathrm{i}}\}$, $a_i \geq 1$.	Definition of tuple-set
Structure of tuple-sets (not of tuples within tuple-sets)	Infinitary tree, equivalent to a $2 \cdot 3^{i-2}$-ary tree. Thus, in the latter, finitary, tree: **level 2** has $2 \cdot 3^{2-2} = 2$ nodes (the 2 top rows of all 2-level tuple-sets), mapped to by 2 equivalence classes of exponents; **level 3** has $2 \cdot 3^{3-2} = 6$ nodes (the 6 top rows of all 3-level tuple-sets), mapped to by 6 equivalence classes of exponents; **level 4** has $2 \cdot 3^{4-2} = 18$ nodes (the 18 top rows of all 4-level tuple-sets), mapped to by 18 equivalence classes of exponents; etc.	Lemma 7.3

Table 2: Some important properties of tuple-sets

Property	Value of property	Reference
$2 \cdot 3^{i-1}$	Distance between elements of tuples successive at level *i* in an *i*-level tuple-set	Lemma 1.0
$2 \cdot 3^{i-2}$	Number of top rows of all *i*-level tuple-sets; also Number of exponent equivalence classes (and the maximum exponent), from which exponents mapping to the top row of any *i*-level tuple-set, from the top rows of all *i*- 1 level tuple-sets, must be selected.	Lemmas 3.055, 3.057 Lemma 7.3

Table 3: Some important properties of the top (i.e., level*i*) row of an *i*-level tuple-set

Property	Value of property	Reference
Distance $d(i, i)$ between successive elements of a top row, i.e., between *i*-level elements of tuples consecutive at level *i*	$d(i, i) = 2 \cdot 3^{i-1}$	Lemma 1.0 (a)
Total number of different top rows over the set of all *i*-level tuple-sets	$\phi(2 \cdot 3^{i-1}) = 2 \cdot 3^{i-2}$ = the number of reduced residue classes mod $2 \cdot 3^{i-1}$	Lemmas 3.055, 3.057
Distance between successive exponents in an exponent equivalence class mapping from an *i*-level top row to an (*i* + 1)-level top row. All members of a class map to the same level-(*i*+1) top row from the same level *i* top row.	$2 \cdot 3^{i-2}$	Lemma 7.3

Table 3: Some important properties of the top (i.e., level*i*) row of an *i*-level tuple-set

Property	Value of property	Reference
Total number of exponent equivalence classes mapping a level-*i* top row to all level-(i + 1) top rows	$\phi(2 \bullet 3^{i-1}) = 2 \bullet 3^{i-2}$ = the number of reduced residue classes mod $2 \bullet 3^{i-1}$	Lemma 7.3
Smallest exponent mapping to any given top row of an (i + 1)-level tuple-set from any top row of an *i*-level tuple-set	≤ 4	Lemma 7.35
Upper bound on exponents mapping from any given top row of an *i*-level tuple-set to the top row of any (i + 1)-level tuple-set	$2 \bullet 3^{i-1}$ (All larger exponents are elements of equivalence classes having smaller minimum elements)	Lemma 7.3
Beginning of sequence of exponents mapping to any given (i + 1)-level top row from *all* *i*-level top rows	*For an (i + 1)-level top row mapped to by odd exponents*: 1,3, *, or 1, *, 5, or *, 3, 5. *For an (i + 1)-level top row mapped to by even exponents*: 2, 4, *, or 2, *, 4, or *, 4, 6, where * denotes a "missing" exponent due to absence of a multiple-of-3 in the *i*-level top row. The * recurs after every two non-* exponents.	Lemma 15.0
Sequence of exponents mapping from any given *i*-level top row to *all* (i + 1)-level top rows	1, 2, 3, ..., $2 \bullet 3^{i-1}$, with each exponent mapping to a unique (i + 1)-level top row. A larger exponent a'_{i+1} then maps to the same row as one of the above exponents a_{i+1} if $a'_{i+1} \equiv a_{i+1} \bmod 2 \bullet 3^{i-1}$.	Lemma 7.3
Minimum element in a top row	Minimum residue in a reduced residue class mod $2 \bullet 3^{i-1}$	Lemmas 3.055, 3.057

Table 3: Some important properties of the top (i.e., level*i*) row of an *i*-level tuple-set

Property	Value of property	Reference
Formula for the minimum element of the top row of an *i*-level tuple-set, given only the sequence of exponents defining the tuple-set	See Lemma 7.38	Lemma 7.38
Formula for the minimum element of the top row of an (*i*+ 1)-level tuple-set mapped to by the top row of an *i*-level tuple-set via an exponent a_{i+1}	See Lemma 7.36	Lemma 7.36
Distance between successive elements of (sub-row of) top row of an *i*-level tuple-set that generates a top row of an (*i* + 1)-level tuple-set via the exponent a_{i+1}	$lcm(2 \cdot 3^{i-1}, 2 \cdot 2^{a_{i+1}})$, where *lcm* denotes least common multiple	Lemma 1.1
Successive elements of (sub-row of) top row of i-level tuple-set map to successive elements elements of top row of (i + 1)-level tuple-set?	Yes.	Lemma 7.40
Set of elements in all top rows of all *i*-level tuple-sets	Set of range elements, i.e., set of odd, positive integers not multiples of 3	Lemma 3.28
Relationship between top rows of all *i*-level tuple-sets and top rows of all (*i* + 1)-level tuple-sets	(1) *Each* top row in an *i*-level tuple-set generates, via all exponents a_{i+1}, the top rows of *all* (*i* + 1)-level tuple-sets. (2) For *each* (*i* + 1)-level top row, if it is desired to generate the row via all possible exponents, then *all* *i*-level top rows are required .	(1) Lemma 7.25 (2) Lemma 7.27.

Table 4: Some important properties of the middle (i.e., levels 1 < *j* < *i*) row of an i-level tuple-set

Distance, $d(j, i)$ between elements at level j of successive tuples consecutive at level i	$d(j, i)$ = $lcm(2 \cdot 3^{i-1}, 2 \cdot 2^{a_{j+1}} \cdot 2^{a_{j+2}} \cdot \ldots \cdot 2^{a_i})$ where *lcm* is the least common multiple.	Lemma 1.1
For each i and each j, minimum elements of level j rows over all i-level tuple-sets	General formula not yet known; must be determined empirically for each given tuple-set	
For each j, set of elements in all j-level rows of all i-level tuple-sets	Set of range elements, i.e., set of odd, positive integers not multiples of 3	Lemma 3.28

Table 5: Some important properties of the bottom (i.e., level 1) row of an*i*-level tuple-set

Property	Value of property	Reference
Distance, $d(1, i)$, between successive tuple elements at level 1 of tuples consecutive at level i	$d(1, i)$ = $2 \cdot 2^{a_2} \cdot 2^{a_3} \cdot \ldots \cdot 2^{a_i}$	Lemma 1.0 (b)
Set of elements in bottom row of all i-level tuple-sets	Set of domain elements, i.e., set of all odd, positive integers	Lemma 3.28

We now proceed to the statements, and proofs, of our results concerning tuple-sets. The results are organized by the possibile strategies they suggest for proving Conjecture 1. But first we prove a few elementary facts concerning multiples of 3.

Lemmas Concerning Multiples of 3

The following lemmas are widely known. They are included because they were not known to several readers of earlier versions of this paper, and because the proof of Lemma 0.4 is related to Lemma 15.0.

Lemma 0.2 *No multiple of 3 is a range element.*

Proof :

If

$$\frac{3x+1}{2^{a_j}} = 3m$$

then $1 \equiv 0 \bmod 3$, which is false. □

Lemma 0.4 *Every odd, positive integer (except a multiple of 3) is generated by a multiple of 3 in one iteration of the 3x + 1 function.*

Proof:
The following is an edited version of a proof by Michael O'Neill.

The only relevant generators are $3(2i + 1)$, for some i. We show that, for each odd, positive integer k not a multiple of 3, there exists an i and a j such that

$$k = \frac{(3(3(2i+1))+1)}{2^j} \qquad (0.4.1)$$

where j is necessarily the largest such j, since k is assumed odd.
Rewriting (0.4.1), we have:

$$k2^{j-1} - 5 = 9i \qquad (0.4.2)$$

Without loss of generality, we can let $k \equiv r \bmod 18$, where r is one of 1, 5, 7, 11, 13, or 17 (k is not a multiple of 3). Or, in other words, for some q, r, $k = 18q + r$. Then, from (0.4.2) we can write:

$$18(2^{j-1})q + (2^{j-1})r - 5 = 9i \qquad (0.4.3)$$

Since the first term is a multiple of 9, $(2^{j-1})r - 5$ must also be, if the equation is to hold. We can thus construct the following table. (Certain larger j also serve equally well, but those given suffice for purposes of this proof.)

Table 6: Values of r, j, for Proof of Lemma

r	j	$(2^{j-1})r - 5$
1	6	27

Table 6: Values of r, j, for Proof of Lemma

r	j	$(2^{j-1})r - 5$
5	1	0
7	2	9
11	5	171
13	4	99
17	3	63

Given q and r, we can use r to look up j in the table, and then solve (3) for integral i. □

Possible Strategies for Proving Conjecture 1 Using Tuple-sets

The "Pushing Away" Strategy

The idea here is to show that every tuple containing an assumed counterexample is "pushed away" from tuples whose elements map to 1, with the result that the counterexample tuples never "find a home". The strategy relies on the distance functions described above, in particular $d(1, i)$, which gives the distance between the first elements of i-level tuples in an i-level tuple-set, and the important Lemma 7.0, below, which asserts that every range element is mapped to by every exponent sequence, with the possible concatenation of one additional "buffer" exponent. There are several versions of this strategy, which we now describe informally.

The "Pushing Away" Strategy: Version 1

We begin by asking, "What is the difference between a tuple containing counterexamples (an n-t-v-c) and one that does not (an n-t-v-1)?" The answer is that extensions of an n-t-v-c never become <..., 1, 1, ..., 1> (defined by an exponent sequence {..., 2, 2, ..., 2}), where the number of 1s (2s) is unbounded).

Now we ask, "What does it mean for a tuple t *not* to have an extension in a given tuple-set? In other words, what does it mean for t to be only a j-level tuple, $j < i$, in an i-level tuple-set?" And we answer that it means simply that j has an extension in *another* tuple-set, i.e., a tuple-set defined by a different exponent sequence. It is important to understand that *every* (finite) sequence of extensions of *every* tuple in *every* tuple-set, defines a tuple-set. And, of course, the first element of each tuple remains the same — remains fixed — throughout all these extensions.

Now, by Lemma 7.0, we know that, for all range elements y (i.e., all odd, positive integers that are not multiples of 3), and for all exponent sequences A, there exists an x that maps to y via A with, possibly, an additional "buffer" exponent following A.

This means that every tuple-set T_A has at least one tuple whose elements ultimately map to 1. In fact, every tuple-set T_A has an *infinity* of tuples whose elements ultimately map to 1. (1 is certainly a range element, and every x mapping to 1 via A must be in the tuple-set T_A. There is an infinity of such x.)

Now, tuples whose elements ultimately map to 1 (n-t-v-1s) look like <*x*, ..., *y*, *b*, 1, 1, 1, ..., 1>, where *b* is the tuple element produced by the optional buffer exponent, and the number of 1's is unbounded. (The corresponding exponent sequences look like (a_2, ..., a_i, B, 2, 2, 2, ..., 2}, where the number of 2s is unbounded.)

Suppose, now, the Devil is trying to find a place for a counterexample tuple (an n-t-v-c). He selects a counterexample tuple that is defined by the exponent sequence $A = \{a_2, a_3, ..., a_i\}$, goes to tuple-set T_A, and says, "I can put my tuple here, as long as I put the first element at a distance $\cdot\ 2^{a_2} \cdot 2^{a_3} \cdot \ldots \cdot 2^{a_i}$ from an *x* that maps to 1 (by Lemma 1.0(b)).

But then he finds that the extensions of the tuples that map to 1 all eventually wind up in tuple-sets defined by $A' = \{a_2,..., a_i, B, 2, 2, 2, ..., 2\}$, where *B* is the optional buffer exponent, so that the counterexample tuple can find no home in any of these tuple-sets. So he says, "OK, I'll select a counterexample tuple defined by a longer *A*, and put my tuple in the tuple-set T_A." But again the same thing happens. Not only can he not keep a tuple in any of these sets, but each time he tries again with a larger *A*, he must put the first element of his counterexample at a greater distance from the *x*'s that map to 1 (by Lemma 1.0 (b)).

In short, it appears that every counterexample tuple is eventually "pushed away" from any fixed starting point in any tuple-set. And if no counterexample tuple has a permanent starting point (a permanent "home"), then there are no counterexample tuples.

We can present this strategy a little less fancifully. By Lemma 7.0 we know that for all exponent sequences *A*, there exists an *x* that maps to 1 with, possibly, an additional buffer exponent *B* following *A*. Then for each such *x*, there exists a tuple-set defined by successive extensions of <*x*> until the first expansion that contains 1, i.e., a tuple-set defined by the tuple <*x*, ..., 1>. There is a countable infinity of such tuple-sets. If $T_{A*\{B\}*\{2\}}$ is any such set, where $A = \{a_2, a_3, ..., a_i\}$ and *B* is the optional buffer exponent, then we know that the first element x of a counterexample

tuple must be at least at distance $2 \cdot 2^{a_2} \cdot 2^{a_3} \cdot \ldots \cdot 2^2$ (*B* is optional) from *x*. Since, by Lemma 10.0, if a counterexample exists, every tuple-set contains an infinity of n-t-v-cs, we know that an infinity of counterexamples x' must be at least at this distance from *x* which map to 1. The question then is, where — in which tuple-set — is the counterexample x' which has unlimited tuple-extensions that never contain 1?

We now present the strategy more precisely.

For each exponent sequence $A = \{a_2, a_3, ..., a_i\}$, there exists an *x* which is the first element of an infinite sequence of tuples:

<*x*>,
<*x*, *C*(*x*)>,
<*x*, *C*(*x*), $C^2(x)$>, ...,
<*x*, *C*(*x*), $C^2(x)$, ..., $C^{i-1}(x)$, $C^i(x)$, 1>,
<*x*, *C*(*x*), $C^2(x)$, ..., $C^{i-1}(x)$, $C^i(x)$, 1, 1>, ...,
<*x*, C(*x*), $C^2(x)$, ..., $C^{i-1}(x)$, $C^i(x)$, 1, 1, ..., 1>
where:

C(*x*) is the result of one iteration of the 3*x* + 1 function applied to *x*, a result which occurs via the exponent a_2;

$C^2(x)$ is the result of the next iteration of the 3*x* + 1 function, a result which occurs via the exponent a_3; ...

$C^i(x)$ is the result of an optional iteration of the function, a result which occurs via an optional "buffer exponent";

the number of 1s is unbounded.

(Lemma 7.0)

Each such tuple defines a tuple-set (definition of *tuple-set*).

In each such tuple-set, the minimum distance to the first element x' of the next tuple after (or before) the tuple having x as first element, is given by

$$2 \cdot 2^{a_2}$$

and

$$2 \cdot 2^{a_2} \cdot 2^{a_3}$$

and

$$\cdot\, 2^{a_2} \cdot 2^{a_3} \cdot \ldots \cdot 2^2$$

respectively (Lemma 1.0 (b)).

Therefore, x' cannot remain the same for the infinite sequence of tuple extensions of x. Therefore there is no counterexample and, in particular, no minimum counterexample.

Remarks:

(1) Any valid implementation of this strategy must deal with the possibility that the first element x of an n-t-v-c can be *smaller* than the first element of any n-t-v-1.

(2) Some people are bothered by this strategy because they can easily conceive of an infinite tuple that does not contain a 1. The author believes, however, that the important question is not whether we can conceive of such a tuple, but whether, instead, we can describe an infinite *sequence* of tuple-sets each of which contains a finite approximation (sub-tuple) of such a tuple, given what we know about tuple-sets and about tuples whose elements ultimately map to 1.

The "Pushing Away" Strategy: Version 2

This version is based on the fact (proved in Lemmas 3.0 and 4.0) that once an extension of a tuple becomes the first *i*-level tuple in an *i*-level tuple-set, all extensions of the tuple remain first tuples in their respective tuple-sets. Assume that an n-t-v-c becomes such a first tuple. This has the effect of "pushing away" n-t-v-1s that are defined by the same sequence of exponents. But this would imply that there exists at least one n-t-v-1 that never becomes the first *i*-level tuple in any *i*-level tuple-set, contradicting Lemma 3.0.

The "Pushing Away" Strategy: Version 3

Another version of the strategy derives from the obvious fact that every tuple that remains "fixed" in a succession of tuple-set extensions has the property that its first element remains the same in the sequence of tuple extensions. Now Lemma 7.38 gives a formula for the last element of the first *i*-level tuple in an *i*-level tuple-set, assuming that only the sequence *A* of exponents defining the tuple-set is known. Lemma 2.13, p. 42 of [3], gives a formula that can easily by converted into a formula for the *first* element of the first *i*-level tuple in an *i*-level tuple-set, assuming that the sequence *A* of exponents defining the tuple-set, *and* the last element of the tuple, are known. Since Lemma 7.38 gives us the last element, we now have a formula for the first element, assuming only the sequence A of exponents is known. Let $f_1(a_2, a_3, ..., a_i)$ be that formula.

Then to prove that the only tuples that remain fixed in a sufficiently long sequence of tuple-set extensions are those that eventually have 1 (hence 1, 1, 1, ..., 1) as their last element, and hence that Conjecture 1 is true, we need only prove that:

For all first elements *x* of tuples, and given that, for each tuple <*x*>, $a_2, a_3, ..., a_j, ...$, is the exponent sequence defined by successive extensions of <*x*>, $f_1(a_2) = f_1(a_2, a_3) = f_1(a_2, a_3, a_4) = ...$ iff there exists a *j* such that $a_j = a_{j+1} = a_{j+2} = ... = 2$, where *j*, of course, depends on the preceding sequence $a_2, a_3, a_4, ...$

We can express this conjecture in more detail.

We begin with the observation that, if the first element of an unlimited series of tuple extensions remains fixed, then so does the second, and the third, and ...

Let *x* be a fixed, odd, positive integer which is the element of a 1-tuple <*x*>. We know that unlimited extensions of <*x*> define an unlimited sequence $\{a_2, a_3, a_4, ...\}$ of exponents.

By a simple inductive argument, we can show that, for arbitrarily large *i*,

$$x = \frac{(2^{a_2}2^{a_3}\ldots 2^{a_i}y_i) - (2^{a_2}2^{a_3}\ldots 2^{a_{i-1}}) - (3 \cdot 2^{a_2}2^{a_3}\ldots 2^{a_{i-2}}) - (3^2 \cdot 2^{a_2}2^{a_3}\ldots 2^{a_{i-3}}) - \ldots - 3^{i-2}}{3^{i-1}}$$

Since the second, third, ... element of each tuple in the sequence of tuple expansions also remains fixed for all expansions, we can write, by a simple inductive argument, that, for all *i*, and for all *k*,

(2)

$$\frac{3^{i-1}x + 3^{i-2} + 3^{i-3} \cdot 2^{a_2} + 3^{i-4} \cdot 2^{a_2}2^{a_3} + \ldots + 2^{a_2}2^{a_3}2^{a_4}\ldots 2^{a_{i-1}}}{2^{a_2}2^{a_3}2^{a_4}\ldots 2^{a_i}} =$$

$$\frac{(2^{a_{i+1}}2^{a_{i+2}}\ldots 2^{a_k}y_k) - (2^{a_{i+1}}2^{a_{i+2}}\ldots 2^{a_{k-1}}) - (3 \cdot 2^{a_2}2^{a_3}\ldots 2^{a_{k-2}}) - (3^2 \cdot 2^{a_2}2^{a_3}\ldots 2^{a_{k-3}}) - \ldots - 3^{k-2}}{3^{k-1}}$$

Conjecture: For all *x*, there exists a *k* such that, for all $n > k$, $y_n = 1$ and $a_n = 2$.

Notes:
(1) The author at present cannot believe that the entire arsenal of modern mathematics does not already contain machinery to prove the Conjecture.

(2) If we multiply the top and bottom of each side of each equation by 2, then all the terms are distances between elements at all levels of tuples consecutive at certain levels (see Lemma 1.1).

It would seem to be of fundamental interest to know *why* these distances appear in a series of expressions for the extensions of a single tuple.

We now prove several lemmas that support the various versions of this strategy.

Lemma 1.2. *The number of tuple-sets is countably infinite*.

Proof:
Simply regard each finite sequence of exponents, with commas, as a base 11 integer. The result follows from the countability of the integers. □

Lemma 1.3. Let x, x' be two odd, positive integers and consider the two infinite "forward" sequences of tuple extensions $X = \{<x>, <x, y>, <x, y, y'>, ... \}$ and $X' = \{<x'>, <x', w>, <x', w, w'>, ... \}$. Each tuple extension defines a finite "forward" exponent sequence. Then the infinite "forward" exponent sequence defined by X cannot equal the infinite "forward" exponent sequence defined by X'.

Proof:
Assume, to the contrary, that x, x' exist such that $X = X'$. But then an i-level extension of $<x>$ is in an i-level tuple-set T_A defined by the extension iff an i-level extension of $<x'>$ is in T_A. But then sooner or later, i.e., for some i, the level 1 distance function between x and x' will be violated. □

Lemma 1.4[1]. Let y, z be two range elements and consider the two infinite sequences of "backward" or inverse or "downward" tuple extensions $Y = \{<y>, <y', y>, <y'', y', y>, ... \}$ and $Z = \{<z>, <z',z>, <z'', z', z>, ... \}$. Each tuple extension defines a finite "backward" or inverse or "downward" exponent sequence. Then the infinite "backward" exponent sequence defined by Y cannot equal the infinite "backward" exponent sequence defined by Z.

Proof:
Assume, to the contrary, that y, z exist such that $Y = Z$. But then an i-level "downward" extension of $<y>$ is in an i-level tuple-set T_A defined by the extension iff an i-level extension of $<z>$ is in T_A. But then sooner or later, i.e., for some i, the level i distance function between y and z will be violated. □

1. This lemma was first stated and proved by a consultant. The given proof is, however, that of the author of this paper.

Lemma 2.0. *Every i-level tuple-set can be extended by any even or odd exponent* a_{i+1}. *In other words, for each even or odd* a_{i+1}, *every i-level row maps to a non-empty tuple-set row .*

The following proof is by Michael O'Neill.

Proof:

The following identities make possible the creation of examples with arbitrary sequences of exponents, which proves the Lemma.

Let

$$\frac{3i_1 + 1}{2^{j_1}} = i_2$$

and

$$\frac{3i_2 + 1}{2^{j_2}} = i_3$$

This can be generalized to longer sequences, but two is enough to show what is going on. We are going to shift i_1 by 2^m . There are three interesting cases:

Case 1:

$$m = j_1 + j_2 - 1$$

Case 2:

$$m = j_1 + j_2$$

Case 3:

$$m > j_1 + j_2$$

In the first case the sequence becomes:

$$\frac{3(i_1 + 2^m) + 1}{2^{j_1}} = i_2 + 3(2^{m - j_1}) = i_2 + 3(2^{j_2 - 1})$$

$$\frac{3(i_2 + 3(2^{j_2 - 1})) + 1}{2^{j_2 - 1}} = 2i_3 + 3^2$$

Since $2i_3 + 3^2$ is odd, no further halving is possible and we have produced a sequence $(j_1, j_2 - 1)$ from (j_1, j_2). Shifting down by 2^n produces the same result.

The second case starts out the same way but the second step is:

$$\frac{3(i_2 \pm 2^{j_2}) + 1}{2^{j_2}} = i_3 \pm 3^2$$

This time the right-hand side is even. The sign of the shift determines how far the exponent increases. The important point is that we can always choose the sign so that the exponent is increased by 1.

If $i_3 = 4n + 1$ then choose an up-shift. This gives $4n + 10 = 2(2n + 5)$. Likewise, if $i_3 = 4n + 3$, a down-shift gives $4n - 6 = 2(2n - 3)$. In both cases, no further factors of 2 can come out.

For longer sequences, 3^2 becomes 3^n, but the same process works.

So this second case allows us to take a sequence (j_1, j_2) and get $(j_1, j_2 + 1)$.

In the third case the general step is:

$$\frac{3(j_2 \pm 2^{m - j_1}) + 1}{2^{j_2}} = i_3 \pm 3^2 2^{m - j_1 - j_2}$$

The right-hand side is now always odd, so the exponent does not change, but we have a way to shift the same exponent sequence to a different position. .

Note that these operations work for negative as well as positive integers.

This third case also provides a proof that the distance functions are valid for negative integers as well. Any finite tuple-set can be moved *as a whole* to another part of the space, positive or negative. A quick glance at the formula for the shift shows that the difference between corresponding elments does not change. □

Lemma 3.0. *For each range element y there exists an i-level tuple-set in which y is an element of the i-level first tuple.*

Proof:

Clearly, y is the first element of a tuple t at level 1. Now consider the sequence of tuple-set extensions defined by tuple-extensions of t. In each extension, the extension of t is present. By Lemma 1.0, the distance between first elements of tuples consecutive at level i increases with i. Therefore, an i must eventually be reached in which the distance at level 1 is greater than y, and thus y will be an element of the first i-level tuple in that i-level tuple-set. □

Lemma 3.055. *The top row of an i-level tuple-set is a residue class modulo* $2 \bullet 3^{i-1}$.

The following proof is by Michael O'Neill.

Proof:

The proof is by induction.

The base case is row 2. Two representatives of row 2 for some exponent j are:

$$\frac{3(2m+1)+1}{2^j} = k$$

and

$$\frac{3(2m'+1)+1}{2^j} = k'$$

since members of row 2 are generated by odd numbers.

Now,

$$k - k' = \frac{6(m - m')}{2^j}$$

and k - k' is thus divisible by 3. It is also divisible by 2 since k and k' are both odd. So, for any single exponent, the members of tuple-row 2 are in a single residue class modulo 6.

The general case is very similar. Assuming that the general member of tuple-row i - 1 is $2 \bullet 3^{i-1}m + r$ for some fixed r, then two members of tuple row i are:

$$\frac{3(2 \cdot 3^{i-1}m + r) + 1}{2^j} = k$$

$$\frac{3(2 \cdot 3^{i-1}m' + r) + 1}{2^j} = k'$$

So,

$$k - k' = \frac{(2 \cdot 3^i)(m - m')}{2^j}$$

So k - k' is divisible by $2 \cdot 3^i$.. □

By Lemma 3.055, there are two level-2 rows (i.e., top rows) in the set of all 2-level tuple-sets. These rows are the residue classes $\{x | x \equiv 5 \bmod 2 \cdot 3^1\}$ for odd a_2 and $\{x | x \equiv 1 \bmod 2 \cdot 3^1\}$for even a_2.

Lemma 3.055 implies that, for any i-level tuple-set, there is only a finite number of i-level-rows, even though there is an infinity of exponent sequences of length i-1. For example, for $i = 2$, there are two 2-level rows, but there is an infinite number of possible exponent sequences of length 1, namely, $\{a_2 | a_2$ is an even or odd positive integer}. For $i = 3$, there are six 3-level rows, but there is an infinite number of possible exponent sequences of length 2, namely, $\{a_2, a_3 | a_2, a_3$ is each an even or odd positive integer}. Elaboration of these facts for all i is given in the Remark immediately following the proof of Lemma 7.0. A closely related fact is given in the next lemma.

Lemma 3.057. *The set of minimum elements of all top rows in all i-level tuple-sets is the set of minimum residues of the reduced residue classes mod* $2 \cdot 3^{i-1}$.

Proof:

By definition, no range element is an even number. By Lemma 0.2, no range element is a multiple of 3. Therefore, by Lemma 3.055, the set of top rows of all i-level tuple-sets is the set of reduced residue classes mod $2 \cdot 3^{i-1}$, and the result follows. □

Any such minimum element is, of course, the last element of the first i-level tuple in some i-level tuple-set. Actually, it is the last element of an infinity of such tuples, as explained in the Remark immediately following the proof of Lemma 7.0.

Lemma 4.0. *If* t_1 *is the first i-level tuple in an i-level tuple-set, then the extension of* t_1 *is the first (i + 1)-level tuple in the tuple-set its extension defines. And so on, recursively.*

Proof:

If t_1 is the first i-level tuple in an i-level tuple-set, then there are no tuple-elements to the left of at least one element of t_1. Therefore there cannot be any tuple-elements to the left of this (these) elements in any extension tuple, and the result follows. □

Lemma 5.0.

If

$$\frac{3x + 1}{2^j} = y$$

Then, for all n ≥ 1,

$$\frac{3(x + (2^{j+2(0)} + 2^{j+2(1)} + \ldots + 2^{j+2(n-1)})y) + 1}{2^{j+2(n)}} = y$$

This Lemma implies that, for each counterexample there exists an infinity of counterexamples, and so on, recursively, for each counterexample in each such infinity. This fact is well-known, but Lemma 5.0, and its proof, are apparently less well-known. In any case, they have important ramifications for our investigation of the structure of the $3x + 1$ function.

Proof:

The proof is a matter of straightforward algebra.

From the antecedent, we have:

$$x = \frac{2^j y - 1}{3}$$

Substituting into the left-hand side of the consequent, multiplying the term in parentheses by 3, cancelling the 1's, and factoring out $(2^j)(y)$ yields:

$$\frac{2^j y(1 + 3(2^0 + 2^2 + 2^4 + \ldots + 2^{2(n-1)}))}{2^{j+2(n)}}$$

The 2^js cancel, the term 1 + 3... is easily shown to equal $2^{2(n)}$, and the result follows. □

Lemma 5.0 immediately allows us to specify the first elements of tuples consecutive at level 2 in all 2-level tuple-sets.

Lemma 3.25. *The first elements of tuples consecutive at level 2 in all 2-level tuple-sets are as described in the following tables.*

Table 7: First elements of tuples consecutive at level 2: odd powers

Exponent a_2	**First elements of tuples consecutive at level 2**
1	$\{x \mid (x \equiv 3 \ \ mod \ \ (2 \cdot 2^1))\}$
3	$\{x \mid (x \equiv 3 + 5(2^1) mod \ \ (2 \cdot 2^3))\}$
5	$\{x \mid (x \equiv 3 + 5(2^1 + 2^3) mod \ \ (2 \cdot 2^5))\}$
7	$\{x \mid (x \equiv 3 + 5(2^1 + 2^3 + 2^5) mod \ \ (2 \cdot 2^7))\}$
...	...
$2k + 1$	$\{x \mid (x \equiv 3 + 5(2^1 + 2^3 + 2^5 + \ldots + 2^{2(k-1)+1}) mod \ \ (2 \cdot 2^{2k+1}))\}$

Table 8: First elements of tuples consecutive at level 2: even powers

Exponent a_2	**First elements of tuples consecutive at level 2**
2	$\{x \mid (x \equiv 1 mod \ \ (2 \cdot 2^2))\}$
4	$\{x \mid (x \equiv 1 + 1(2^2) mod \ \ (2 \cdot 2^4))\}$
6	$\{x \mid (x \equiv 1 + 1(2^2 + 2^4) mod \ \ (2 \cdot 2^6))\}$
8	$\{x \mid (x \equiv 1 + 1(2^2 + 2^4 + 2^6) mod \ \ (2 \cdot 2^8))\}$
...	...
$2k$	$\{x \mid (x \equiv 1 + 1(2^2 + 2^4 + 2^6 + \ldots + 2^{2(k-1)}) mod \ \ (2 \cdot 2^{2k}))\}$

Proof: Follows from Lemma 5.0 and the definition of the function $d(1, i)$. □

We now give two well-known results that relate the "input", x, the exponent, a_j, and the "output", y, of an iteration of the $3x + 1$ function. The author believes that these two lemmas are equivalent to Lemmas 1.0 and 1.1. Because of their generality, they may prove useful in proving the existence of the mapping between tuple-sets and recursive "spiral"s described in Section 3.

The following lemma was brought to my attention by Alan Tyte.

Lemma 5.5. *If, in one iteration of the 3x + 1 function, y is mapped to by an even exponent of 2, then* $y \equiv 1 \ mod \ 3$; *if y is mapped to by an odd exponent of* 2, *then* $y \equiv 2 \ mod \ 3$.

Proof:

Since 2 is a primitive root mod 3, and since $\phi(3) = 2$, we have $2^1, 2^3, 2^5, ..., 2^{2k+1}, ... \equiv 2 \bmod 3$, and $2^2, 2^4, 2^6, ..., 2^{2k} ... \equiv 1 \bmod 3$. If $ord_2(3x + 1)$ is even, then, from the definition of the $3x + 1$ function, we have $1 \equiv 2^{2k}y \bmod 3$, which implies that $y \equiv 1 \bmod 3$; if $ord_2(3x + 1)$ is odd, then, from the definition of the $3x + 1$ function, we have $1 \equiv 2^{2k+1}y \bmod 3$, which implies that $y \equiv 2 \bmod 3$. □

The following lemma was brought to my attention by Xiaoming Huo.

Lemma 5.7. *(a) If, in one iteration of the 3x + 1 function, x maps to y via the exponent 1, then* $x \equiv 3$ *mod* 4; *(b) if, in one iteration, x maps to y via any other exponent, then* $x \equiv 1$ *mod* 4.

Proof:

(a) $(3x + 1)/2 = y$, y odd, implies $3x + 1 = 2(2k + 1)$, or $x = (4k + 1)/3$. Since x must be an integer, we find by trial that the smallest value of k is 2, yielding 3, the second 5, yielding 7, and, in general, if $k = 2 + 3m$, $m \geq 0$, then $(4(2 + 3m) + 1)/3 = (8 + 12m + 1)/3 = 4m + 3$.

(b) Since, by the proof of *(a)*, the x mapping to y via the exponent 1 constitute the entire residue class $\equiv 3 \bmod 4$, and since x must be an odd, positive integer, hence not $\equiv 0$ or $2 \bmod 4$, it follows that all the x mapping to y via any other exponent must be $\equiv 1 \bmod 4$. □

Lemma 6.0. *There exists an explicit construction of the tuple-set produced by a given tuple.*

The following proof is by Michael O'Neill.

Proof:

Let x be the first element of a tuple and let $\{a_2, a_3, ..., a_{n+1}\}$ be the sequence of exponents resulting from the first n extensions of the tuple $\langle x \rangle$. The last element of the tuple will be given by:

$$\frac{3^n x + r}{2^a}$$

where

$$a = \sum_{i=2}^{n} a_i$$

r is most easily calculated by iterating from $x = 0$, then multiplying by the appropriate power of 2, as shown in the table immediately following this construction. We want the integral x that produce odd outputs:

$$\frac{3^n x + r}{2^a} = 2k + 1$$

which gives

$$3^n x - 2^{a+1} k = 2^a - r$$

This is a standard linear Diophantine equation. It has an explicit solution:

$$x_0 = (-(2^a - r))\left(\frac{2^{2 \cdot 3^{n-1} \cdot (a+1)} - 1}{3^n}\right)$$

$$k_0 = (-(2^a - r))(2^{(2 \cdot 3^{n-1} - 1)(a+1)})$$

Note that the ratio in the expression for x_0 is an integer because

$$2^{2 \cdot 3^{n-1}} \equiv 1 \ \ mod \ \ 3^n$$

The general solution is:

$$x = x_0 + t \cdot 2^{a+1}$$

$$k = k_0 + t \cdot 3^n$$

where t ranges over the integers. Thus, the x's are the inputs that iterate with the specified exponents and

$$2k + 1 = 2k_0 + t \cdot 2 \cdot 3^n + 1$$

are the outputs.

Likewise, if we want to extend the set $m + t \bullet 2 \bullet 3^n$ by an exponent j we get:

$$\frac{3(m + t \cdot 2 \cdot 3^n) + 1}{2^j} = 2k + 1$$

or

$$3m + t \cdot 2 \cdot 3^{n+1} + 1 = 2^{j+1}k + 2^j$$

which implies

$$t \cdot 2 \cdot 3^{n+1} - 2^{j+1}k = 2^j - 3m - 1$$

and this equation has the same kind of explicit solution as the above. □

Table 9: Successive values of *n*, the x term, and *r* in proof of Lemma 6.0

n	***x* term**	***r***	**level of tuple element yielded, i.e., *i* in a_i**
1	3^1x	1	2
2	3^2x	$3^1 + 2^{a_2}$	3
3	3^3x	$3^2 + 3^1 2^{a_2} + 2^{a_2}2^{a_3}$	4
4	3^4x	$3^3 + 3^2 2^{a_2} + 3^1 2^{a_2}2^{a_3} + 2^{a_2}2^{a_3}2^{a_4}$	5
. . .	. . .	. . .	. . .

Lemma 7.0. *For each range element y, and for each finite sum a of exponents, a domain element x exists that maps to y via a sum a′ that contains a.*

Proof:

The following proof is by Michael O'Neill.

We are looking for an x such that the sequence of iterations represented by

$$\frac{3^n x + r}{2^a}$$

where n, a, and r are defined as in Lemma 6.0, lead to a computation that ends with y. n, a, and r are determined by the exponent sequence we want. There also has to be a buffer iteration between the above and y, e.g., to allow for parity constraints on the exponent leading to y (see Lemma 5.0). So, we want

$$\frac{3\left(\frac{3^n x + r}{2^a}\right) + 1}{2^j} = y$$

or

$$\frac{3^{n+1} x + 3r + 2^a}{2^{a+j}} = y$$

which gives

$$3^{n+1} x = (2^a y)2^j - 3r - 2^a \quad (7.1)$$

or

$$(2^a y)2^j \equiv 3r + 2^a \mod 3^{n+1}$$

We are looking for x and j. Since y is a range element, it cannot be a multiple of 3 (by Lemma 0.2). Therefore $2^a y$ is relatively prime to 3^{n+1}, as is $3r + 2^a$. Since 2^j for any j is an element of a reduced residue class mod 3^{n+1}, the congruence is solvable. Hence we can find j, and then, from (7.1), x. □

Remark. Since Lemma 7.0 applies to all range elements, it applies to the last elements of all first i-level tuples in all i-level tuple-sets, $i \geq 2$. Since, for each such i, the set S_i of these last elements is the set of minimum residues of the reduced residue classes mod $2 \cdot 3^{i-1}$ (Lemma 3.057), we see that, if y is an element of S_i, then for each exponent sequence A of length i - 1, there must exist a first i-level tuple (hence an i-level tuple-set) defined by A, plus a possible additional buffer exponent. Now if y is an element of S_i, then y is certainly an element of S_{i+1}, and so the same argument applies, only this case for each sequence A of length i. And so on, recursively, for all i. This matter is further discussed under "The "Last/First" Property" on page56.

The Buffer Exponent in the Proof of Lemma 7.0

Let us now consider the buffer exponent in the proof of Lemma 7.0. We have said that it is required to meet parity constraints on the exponent leading to y, since, by Lemma 5.0, y is mapped to only by exponents of one parity. Thus, if the last exponent, a_i, in our sequence A = $\{a_2, a_3, ..., a_i\}$ (whose sum is a), is of the opposite parity from that which maps to y, then the buffer exponent is needed provide the proper parity mapping to y.

But parity is not the only reason that the buffer exponent is required. For it can happen that even if a_i is of the same parity as that which maps to y, a buffer exponent may still be required.

Does this mean that the buffer exponent can be arbitrarily large? If so, this would, among other things, tend to defeat any hope of our proving Conjecture 1 by proving that there is no minimum counterexample. This strategy, using tuple-sets, is discussed in detail under "Strategy of Proving There Is No Minimum Counterexample" on page 58 and, using recursive "spiral"s, under "Strategy of Proving There Is No Minimum Counterexample" on page72. The idea of such a strategy is this: assume a counterexample, hence a minimum counterexample, y exists, and then define an exponent sequence such that the x that maps to y, as promised by Lemma 7.0, is less than y. If we could prove that such an exponent sequence exists for any assumed minimum counterexample y, we would have a proof of Conjecture 1, because it would imply there is no minimum counterexample.

Now, it is easy to define such exponent sequences. For example, we can assign the value 2/3 to each exponent 1, and the value 4/3 to each exponent 2. For, if $(3x + 1)/2^1 = y$, then x is about $(2/3)y$ (the 1 in the numerator becomes negligible for large x), and if $(3x + 1)/2^2 = y$, then x is about $(4/3)y$ (the 1 in the numerator again neglible for large x). Thus, for each sequence of iterations of the $3x + 1$ function involving only the exponents 1 and 2, we can compute the product of the above values. If the result is less than 1, then we know that the x that produced the sequence is less than the final y. However, if the buffer exponent can be arbitrarily large, then all our labors in defining the sequence may be in vain, since the buffer exponent can be so large that x is greater than y.

But the congruence following (7.1) in the proof of Lemma 7.0 can be used to show that the buffer exponent can not be arbitrarily large. The specifics are contained in the following lemma.

Lemma 7.1. *Let A = $\{a_2, a_3, ..., a_i\}$ be a sequence of exponents, and let y be any range element. Then the maximum buffer exponent j (see proof of Lemma 7.0) required to ensure that an x exists that maps to y via A, is* $2 \cdot 3^{i-1}$.

Proof:

If a buffer exponent j in the proof of Lemma 7.0 (usually B elsewhere in this paper) is required, then it can be regarded as a_{i+1}. Since the order of 2 mod $2 \cdot 3^{(i+1)-1} = \phi(2 \cdot 3^{(i+1)-1}) = 2 \cdot 3^{i-1}$, and since the exponent 0 is not allowed in an exponent sequence, the result follows.
□

The following table contains some examples.

Table 10:

$i + 1$	No. of reduced residue classes mod $2 \bullet 3^{(i+1)}$ (top rows of (i + 1)-level tuple-sets)	Possible distinct powers of 2 utilizing buffer exponents $j = B = a_{i+1}$
3	$\phi(2 \bullet 3^{3-1}) = \phi(18) = 6$	$2^1, 2^2, 2^3, 2^4, 2^5, 2^6$
4	$\phi(2 \bullet 3^{4-1}) = \phi(54) = 18$	$2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}, 2^{11}, 2^{12}, 2^{13}, 2^{14}, 2^{15}, 2^{16}, 2^{17}, 2^{18}$
...	...	...
$i + 1$	$\phi(2 \bullet 3^{(i+1)-1}) = 2 \bullet 3^{i-1}$	$2^1, 2^2, 2^3, \ldots, \quad 2^{2 \cdot 3^{i-1}}$

Lemma 7.1 implies that, for each $i + 1$, the buffer exponents are distributed among $2 \bullet 3^{i-1}$ equivalence classes. Thus $2^j \equiv 2^k$ iff $j \equiv k \bmod 2 \bullet 3^{i-1}$, and we need only consider the minimum power of 2 in each equivalence class, as shown in the above table.

Unfortunately, the fact that the largest of these minimum powers of 2 is

$$2^{2 \cdot 3^{i-1}}$$

is not in itself sufficient to give us our hoped-for proof that there is no minimum counterexample. For, assume that this largest minimum buffer exponent is required to map to a given y via the exponent sequence $A = \{1, 1, 1, ..., 1\}$ ((i - 1) 1s). Then any x that is the first element of a tuple defined by
A will be about $(2/3)^{i-1}$ of the last element x' of the tuple, or x' is about $(3/2)^{i-1}x$. However

$$\frac{3((3/2)^{i-1}x) + 1}{2^{2 \bullet 3^{i-1}}} = y$$

implies x is greater than y, thus defeating our purpose.

Next, we observe that, since, by Lemma 5.0, any range element y is mapped to only by exponents of one parity, then for any given y, there are, in fact, only $(2 \bullet 3^{i-1})/2$ possible classes of buffer exponents mapping to y in a single iteration of the $3x + 1$ function..

Next, we observe that $2 \bullet 3^{i-1}$ is an expression we have seen before! It is the distance between successive top-row tuple elements in any i-level tuple-set. $\phi(2 \bullet 3^{i-1}) = 2 \bullet 3^{i-2}$ is the number of elements in the set of minimal elements of all top rows in all i-level tuple-sets, where $\phi(u)$ is Euler's function, which returns the number of numbers relatively prime to u.

Does this fact in some way guarantee that, since, for each $i \geq 2$, 1 is an element of the set of minimal elements of all top rows in all i-level tuple-sets, that all such minimal elements map to 1?

If this were the case, then we would have a proof of Conjecture 1. Consider the following two closely related conjectures, each of which is equivalent to Conjecture 1.

Conjecture 30. Let $\{T_A\}_i$ be the set of all *i*-level tuple-sets, and let $\{T_A\}_{i+1}$ be the set of all *i*+1 level tuple-sets. Then 1 in the set of minimum residues of all top rows in $\{T_A\}_{i+1}$ is mapped to, directly or indirectly, by all the elements of the set of minimum residues of all top rows in $\{T_A\}_i$.

Conjecture 31. Let $\{T_A\}_i$ be the set of all *i*-level tuple-sets. Then 1 in the set of minimum residues of all top rows in $\{T_A\}_i$ is mapped to, directly or indirectly, by all the elements of the set of minimum residues of all top rows in $\{T_A\}_i$.

The author will pay $100 for the first, correct proof or disproof of one of these conjectures.

The reason why each of these conjectures is equivalent to Conjecture 1 is that, if true, each implies that all first *i*-level tuples in all *i*-level tuple-sets are n-t-v-1s.

A proof of Conjecture 31 by induction suggests itself.

Basis Step: the Conjecture is certainly true for $i = 2$, since the set of all minimum residues of all top rows in $\{T_A\}_2$ is {1, 5}, and, by actual trial, we know that each of these maps directly or indirectly to 1.

Induction Step: Assume the Conjecture is true for all levels $2 \leq i \leq k$, but that it is false for level $k + 1$. We ask if this is possible, given what we have just established regarding buffer exponents. By Lemma 7.0, we know that, for 1, and for each sequence *A* of length $k - 1$ (these establish *k*-level tuples), there exists an *x* that maps to 1 via *A*, with the possible addition of a buffer exponent (thus, possibly, making a $(k + 1)$-level tuple. Is it possible that these tuples are not the tuples giving rise to the minimum residues of all top rows in $\{T_A\}_{k+1}$? If so, then what tuples give rise to these minimum residues?

Lemma 3.28. *For any $i \geq 2$, let $\{T_A\}_i$ denote the set of all i-level tuple-sets, i.e., the set of all tuple-sets defined by exponent sequences $A = \{a_2, a_3, ..., a_i\}$ where a_j is a positive integer. Then (a) for level 1, the set of all elements in all 1-level rows is the set of domain elements; (b) for level $1 < j \leq i$, the set of all elements in all j-level rows in $\{T_A\}_i$ is the set of all range elements.*

Proof:

(a) follows from the definition of the $3x + 1$ function.

For (b) we use an inductive proof.

Basis Step: (b) certainly holds for level 2, since, for $\{T_A\}_2$, where $A = \{a_2\}$, if a_2 is even, the set of 2-level elements is {1, 7, 13, 19, 25, ...} and if a_2 is odd, then the set of 2-level elements is {5, 11, 17, 23, ...}. The union of the two sets is the set of range elements.

Induction Step: Now assume that (b) is true for all *i* from $i = 2$ through *k*, but that it is false for $i = k + 1$. But then at least one range element *y* must be missing from the set of all elements in all $(i + 1)$-level rows. But by Lemma 5.0, we know that each range element is mapped to by an infinity of exponents, either all even or all odd, and by Lemma 2.0 we know that each tuple-set has extensions via each positive integer. Therefore *y* cannot be missing. □)

Now we proceed with a lemma that answers question 5, above. It shows that, for each $i \geq 2$, the infinity of domain elements mapping to a given range element (in accordance with Lemma 5.0), fall into $2 \bullet 3^{i-1}$ classes.

Lemma 3.24. *Let x be a range element that is a minimum residue mod* $2 \cdot 3^{i-1}$*, and let*

$$\frac{3x+1}{2^j} = h$$

Then if

$$j \equiv k \ \ mod \ \ (2 \cdot 3^{i-1})$$

there exists an x′ such that

$$\frac{3x'+1}{2^k} = h$$

and furthermore

$$x \equiv x' \ \ mod \ \ (2 \cdot 3^{i-1})$$

Proof:
The following proof is by Michael O'Neill.

The required x' is:

$$x' = x + h2^j\left(\frac{2^{m2(3^{j-1})}-1}{3}\right)$$

where m comes from

$$k = j + m2(3^{i-1})$$

This gives:

$$\frac{3x+1}{2^k} = \frac{3\left(x + h2^j\left(\frac{2^{m2(3^{i-1})}-1}{3}\right)\right)+1}{2^k} = \frac{3x+1+h2^j(2^{m2(3^{i-1})}-1)}{2^k}$$

and

$$h2^j\left(\frac{2^{m2(3^{i-1})}}{2^k}\right) = \frac{h2^{j+m2(3^{i-1})}}{2^k} = h$$

□

Thus, for example, $x = 3$ maps to $h = 5$ via the exponent 1, 5 being a minimal top row element for 2-level tuple-sets whose defining sequence is $A = \{a_2\}$, a_2 odd. Since $1 \equiv 7 \bmod 2 \cdot 3^{2-1}$, we should find an x' such that $x \equiv x' \bmod 2 \cdot 3^{2-1}$ that maps to 5 via the exponent 7, and, indeed, this is the case for $x' = 213$, since $3 \equiv 213 \bmod 2 \cdot 3^{2-1}$.

Again we see, as in our discussion of buffer exponents, the presence of the term $2 \cdot 3^{i-1}$ in connection with exponents. The close relationship between Lemma 3.24 and Lemma 7.1 is clear. In fact, we can say that **the two lemmas show that, for each range element y in the set of minimum top row elements for all (*i* + 1)-level tuple-sets, all *x* mapping to *y* via exponents in a given equivalence class, are congruent mod $2 \cdot 3^{i-1}$**. The two lemmas suggest a strategy for a proof of Conjecture 31, namely: assume that for some first *i*, an element *y* of the set of minimal elements of all top rows in all *i*-level tuple-sets does *not* map, directly or indirectly, to 1. Then this means that such an element is not an x' (in Lemma 3.24) for any *x* in the set that does map, directly or indirectly, to 1, and furthermore that this is true for all *i* starting with 2. Is such a thing possible? In other words, speaking informally, is it possible for an element of the set of minimal elements to have no connection with the set that maps, directly or indirectly, to 1?

We now investigate in more detail the process by which *i*-level tuple-sets become (*i* + 1)-level tuple-sets.

Generating Level-(*i* + 1) Top Rows from Level-*i* Top Rows

To understand the generating of top rows of tuple-sets, we can begin with the domain elements themselves, i.e., the odd, positive integers. Let each domain element be the element of a 1-tuple. We imagine these level-1 tuples oriented vertically (so that extensions go in the vertical direction), and situated on a horizontal line infinite to the right, as in Fig. 1.

We now consider all possible extensions, under one iteration of the 3*x* + 1 function, of this initial, i.e., level-1, row of tuples. That is, we consider all tuple-sets T_A, $A = \{a_2\}$, where a_2 is an even or odd positive integer. By Lemms 3.055 and 3.057, we know that the top row, i.e., the level-2 row, of each of the resulting tuple-sets, is either the set {1, 7, 13, 19, ... } or the set {5, 11, 17, 23, ... }. These sets are the two reduced residue classes mod $2 \cdot 3^{(2-1)} = 6$, as required by Lemma 3.055.

We now consider all possible extensions, under one iteration of the 3*x* + 1 function, of each of these level-2 rows, and then all possible extensions, under one iteration of the 3x + 1 function, of the level-3 rows. The next two tables gives a summary of the results. Following these two tables are tables giving the details. A discussion of these results follows.

Table 11: Summary of extensions, under one iteration of the 3x + 1 function, of the top rows of all 2-level tuple-sets (see explanatory notes below)

level-2 top row	**1**	**2**	**3**	**4**	**5**	**6**
1	11	1	5	7	17	13
5	17	13	11	1	5	7

Notes: Numbers 1, 2, ..., 6 running horizontally are generating exponents. All exponents that are congruent mod 6 generate the same level-3 top row. Numbers 1, 5 in the left-hand column are the minimum elements of the two top rows of all 2-level tuple-sets. Remaining numbers are the minimum elements of the six top rows of all 3-level tuple-sets. Thus, e.g., the table shows that the level-2 top row {1, 7, 13, ... } generates the level-3 top row {7, 25, 43, ... } via the exponent 4.

The sequence of minimum elements of level-3 top rows generated by any level-2 top row is the sequence of minimum reduced residues mod $2 \bullet 3^{3-1} = 18$ generated by the primitive root 5. Thus, e.g., for the level-3 top rows generated by the level-2 top row whose minimum element is 1, we observe that $5^1 \equiv 5 \bmod 18$; $5^2 \equiv 7 \bmod 18$; $5^3 \equiv 17 \bmod 18$; etc. Does this phenomenon generalize to all *i*? We conjecture that it does. See discussion of Conjecture 6 below.

Table 12: Another version of the previous table. Here, level-3 top rows run along the top. Exponents are in boldface. Thus, e.g., the level-2 top row {1, 7, 13, ...} generates the level-3 top row {17, 35, 53, ...} via the exponent 5.

level-2 top row	1	5	7	11	13	17
1	**2**	**3**	**4**	**1**	**6**	**5**
5	**4**	**5**	**6**	**3**	**2**	**1**

Table 13: Summary of extensions, under one iteration of the 3x + 1 function, of the top rows of all 3-level tuple-sets (see explanatory notes below)

level 3 top row	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**
1	29	1	41	7	17	49	11	19	23	25	53	13	47	37	5	43	35	31
5	35	31	29	1	41	7	17	49	11	19	23	25	53	13	47	37	5	43

Table 13: Summary of extensions, under one iteration of the 3x + 1 function, of the top rows of all 3-level tuple-sets (see explanatory notes below)

level 3 top row	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
7	11	19	23	25	53	13	47	37	5	43	35	31	29	1	41	7	17	49
11	17	49	11	19	23	25	53	13	47	37	5	43	35	31	29	1	41	7
13	47	37	5	43	35	31	29	1	41	7	17	49	11	19	23	25	53	13
17	53	13	47	37	5	43	35	31	29	1	41	7	17	49	11	19	23	25

Notes: Numbers 1, 2, ..., 18 running horizontally are generating exponents. All exponents that are congruent mod 18 generate the same level-4 top row. Numbers 1, 5, ..., 17 in the left-hand column are the minimum elements of the six top rows of all 3-level tuple-sets. Remaining numbers are the minimum elements of the 18 top rows of all 4-level tuple-sets. Thus, e.g., the table shows that the level-3 top row {7, 25, 43, ...} generates the level-4 top row {35, 89, 143, ... } via the exponent 11.

The sequence of minimum elements of level-4 top rows generated by any level-3 top row is the sequence of minimum reduced residues mod $2 \cdot 3^{4-1} = 54$ generated by the primitive root 41. Thus, e.g., for the level-4 top rows generated by the level-3 top row whose minimum element is 13, we observe that $41^1 \equiv 41$ mod 54; $41^2 \equiv 7$ mod 54; $41^3 \equiv 17$ mod 54; etc. Does this phenomenon generalize to all i? We conjecture that it does. See discussion of Conjecture 6 below.

Conjecture 6. *The sequence of top rows R_{i+1} generated by top rows R_i under successive exponents, is the sequence of top rows R_{i+1} generated by successive powers of (-13), a primitive root mod $2 \cdot 3^{i-1}$ for all $i \geq 3$.*

The author will pay $50 for the first proof or disproof of this conjecture.

Experiment suggests that, for all $i \geq 2$, -13 is a primitive root mod $2 \cdot 3^{i-1}$. (I am indebted to Alan Tyte for bringing this to my attention.) This is in fact true, since, $(-13)^2$ is not $\equiv 1$ mod 3^2, and therefore, by a well-known result in elementary congruence theory, -13 is a primitive root mod 3^k for all $k \geq 1$, hence, by another well-known result, a primitive root mod $2 \cdot 3^{i-1}$ for all $i \geq$ 2.

Then to prove the Conjecture, we need only show that:

$$\frac{2^k \cdot (-13)^n - 1}{3} \equiv r_0 \ \ mod \ \ (2 \cdot 3^{i-1})$$

implies

$$\frac{2^{k+1} \cdot (-13)^{n+1} - 1}{3} \equiv r_0 \ \ mod \ \ (2 \cdot 3^{i-1})$$

where r_0 is a minimum residue mod $2 \bullet 3^{i-1}$, i.e., the minimum residue of a row R_i.

Table 14: Another version of the previous table. Here, the level-4 top row runs across the top, and exponents are in boldface. Thus, e.g., the level-3 top row {5, 11, 17, ...} generates the level-4 top row {13, 31, 49, ...} via the exponent 14.

level 3 top row	1	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47	49	53
1	**2**	**15**	**4**	**7**	**12**	**5**	**8**	**9**	**10**	**1**	**18**	**17**	**14**	**3**	**16**	**13**	**6**	**11**
5	**4**	**17**	**6**	**9**	**14**	**7**	**10**	**11**	**12**	**3**	**2**	**1**	**16**	**5**	**18**	**15**	**8**	**13**
7	**14**	**9**	**16**	**1**	**6**	**17**	**2**	**3**	**4**	**13**	**12**	**11**	**8**	**15**	**10**	**7**	**18**	**5**
11	**16**	**11**	**18**	**3**	**8**	**1**	**4**	**5**	**6**	**15**	**14**	**13**	**10**	**17**	**12**	**9**	**2**	**7**
13	**8**	**3**	**10**	**13**	**18**	**11**	**14**	**15**	**16**	**7**	**6**	**5**	**2**	**9**	**4**	**1**	**12**	**17**
17	**10**	**5**	**12**	**15**	**2**	**13**	**16**	**17**	**18**	**9**	**8**	**7**	**4**	**11**	**6**	**3**	**14**	**1**

Table 15: Extensions, under one iteration of the 3*x* + 1 function, of the top rows of all 2-level tuple-sets

Top row of 2-level tuple-set	**Exponent**	**Generating elements in top row of 2-level tuple-set**	**Top row of 3-level tuple-set generated via exponent**
{1, 7, 13, 19, ... }	**1**	{7, 19, 31, 43, ...}	{11, 29, 47, 65, ...}
{1, 7, 13, 19, ... }	**2**	{1, 25, 49, 73, ...}	{1, 19, 37, 55, ...}
{1, 7, 13, 19, ... }	**3**	{13, 61, 109, 157, ...}	{5, 23, 51, 69, ...}
{1, 7, 13, 19, ... }	**4**	{37, 133, 229, 325, ...}	{7, 25, 43, 61, ...}
{1, 7, 13, 19, ... }	**5**	{181, 373, 565, 757, ...}	{17, 35, 53, 71, ...}
{1, 7, 13, 19, ... }	**6**	{277, 661, 1045, 1429,...}	{13, 31, 49, 67, ...}
{1, 7, 13, 19, ... }	**7**	{469, 1237, 2005, 2773, ...}	{11, 29, 47, 65, ...}

Top row of 2-level tuple-set	**Exponent**	**Generating elements in top row of 2-level tuple-set**	**Top row of 3-level tuple-set generated via exponent**
{5, 11, 17, 23, ... }	**1**	{11, 23, 35, 47, ...}	{17, 35, 53, 71, ...}
{5, 11, 17, 23, ... }	**2**	{17, 41, 65, 89, ...}	{13, 31, 49, 67, ...}
{5, 11, 17, 23, ... }	**3**	{29, 77, 125, 173, ...}	{11, 29, 47, 65, ...}
{5, 11, 17, 23, ... }	**4**	{5, 101, 197, 293, ...}	{1, 19, 37, 55, ...}
{5, 11, 17, 23, ... }	**5**	{53, 245, 437, 629,...}	{5, 23, 51, 69, ...}
{5, 11, 17, 23, ... }	**6**	{149, 533, 917, 1301, ...}	{7, 25, 43, 61, ...}
{5, 11, 17, 23, ... }	**7**	{725, 1493, 2261, 3029, ...}	{17, 35, 53, 71, ...}

Table 16: Extensions, under one iteration of the 3x + 1 function, of some top rows of 3-level tuple-sets

Top row of 3-level tuple-set	**Exponent**	**Generating elements in top row of 3-level tuple-set**	**Top row of 4-level tuple-set generated via exponent**
{13, 31, 49, 67, ...}	**1**	{31, 67, 103, 139, ...}	{47, 101, 155, 209,...}
{13, 31, 49, 67, ...}	**2**	{49, 121, 193, 265, ...}	{37, 91, 145, 199, ...}
{13, 31, 49, 67, ...}	**3**	{13, 157, 301, 445, ...}	{5, 59, 113, 167, ...}
{13, 31, 49, 67, ...}	**4**	{229, 517, 805, 1093, ...}	{43, 97, 151, 205, ...}
{13, 31, 49, 67, ...}	**5**	{373, 949, 1525, 2101, ...}	{35, 89, 143, 197, ...}
{13, 31, 49, 67, ...}	**6**	{661, 1813, 2965, 4117, ...}	{31, 85, 139, 193, ...}
{13, 31, 49, 67, ...}	**7**	{1237, 3541, 5845, 8149, ...}	{29, 83, 137, 191,...}

Table 16: Extensions, under one iteration of the 3x + 1 function, of some top rows of 3-level tuple-sets

Top row of 3-level tuple-set	Exponent	Generating elements in top row of 3-level tuple-set	Top row of 4-level tuple-set generated via exponent
{13, 31, 49, 67, ...}	**8**	{85, 4693, 9301, 13909, ...}	{1, 55, 109, 163, ...}
{13, 31, 49, 67, ...}	**9**	{6997, 16213, 25429, 34645, ...}	{41, 95, 149, 203, ...}
{13, 31, 49, 67, ...}	**10**	{2389, 20821, 39253, 57685, ...}	{7, 61, 115, 169, ...}
{13, 31, 49, 67, ...}	**11**	{11605, 48469, 85333, 122197, ...}	{17, 71, 125, 179, ...}
{13, 31, 49, 67, ...}	**12**	{66901, 140629, 214357, 288085,...}	{49, 103, 157, 211, ...}
{13, 31, 49, 67, ...}	**13**	{30037, 177493, 324949, 472405,... }	{11, 65, 119, 173, ...}
{13, 31, 49, 67, ...}	**14**	{103765, 398677, 103765, 988501,...}	{19, 73, 127, 181, ...}
{13, 31, 49, 67, ...}	**15**	{251221, 841045, 1430869, 2020693, ...}	{23, 77, 131, 185, ...}
{13, 31, 49, 67, ...}	**16**	{546133, 1725781, 2905429, 4085077, ...}	{25, 79, 133, 187, ...}
{13, 31, 49, 67, ...}	**17**	{2315605, 4674901, 7034197, 9393493, ...}	{53, 107, 161, 215, ...}
{13, 31, 49, 67, ...}	**18**	{1135957, 5854549, 10573141, 15291733, ...}	{13, 67, 121, 175, ,...}
{13, 31, 49, 67, ...}	**19**	{8213845, 17651029, 27088213, 36525397, ...}	{47, 101, 155, 209,...}

Observations Concerning the Generation of Level-3 Top Rows by Level-2 Top Rows

We observe, first of all, that, in conformity with Lemma 1.0, the distance between successive level-3 row elements is $2 \cdot 3^{3 - 1} = 18$. The distance between successive *generating* elements of each level-2 row is given by Lemma 1.1. Thus, for example, the distance for exponent 4 (with {5, 11, 17, 23, ...} as generating level-2 row) is the least common multiple of 6 and $2 \cdot 2^4 = [6, 32] = 96$.

Next, we observe that, with exponent 7, the same level-3 row is generated as was generated for exponent 1. This is in conformity with Lemma 7.3, since $1 \equiv 7 \bmod (2 \cdot 3^{2 - 1} = 6)$.

Next, we observe that the top rows of *all* 3-level tuple-sets are generated, via the exponents 1 through 6, by *each* of the top rows in 2-level tuple-sets. This phenomenon, in fact, applies for all levels $i \geq 2$, as we shall now prove.

Lemma 7.25. *Let R_i be the top-level row of an i-level tuple-set, $i \geq 2$. Let $f(R_i, a_{i+1})$ denote the row produced by applying the 3x + 1 function to all elements x of R_i, and then selecting only those y yielded by $ord_2(3x + 1) = a_{i+1}$. Then the set $\{ f(R_i, a_{i+1}) | a_{i+1} \geq 1\}$ is the set of top rows of all (i + 1)-level tuple-sets, i.e., the set of reduced residue classes mod $2 \cdot 3^{(i + 1) - 1}$.*

Proof:

The following is an edited version of a proof by Michael O'Neill.

Given y, an element of a level $(i + 1)$ top row (y is thus an odd, positive integer not divisible by 3), and R_i, a top row of an i-level tuple-set, we wish to find an element of the latter that maps to the former.

By definition, $R_i = \{x \mid x = r_0 + k \cdot 2 \cdot 3^{i - 1},\ k \geq 0\}$, where r_0 is the least element of the row R_i. We want to find k and j such that

$$\frac{3(r_0 + k \cdot 2 \cdot 3^{i-1}) + 1}{2^j} = y$$

This equation gives

$$2^j y = (3r_0 + 1) + k \cdot 2 \cdot 3^i$$

or

$$2^j y - (3r_0 + 1) = k \cdot 2 \cdot 3^i$$

$(3r_0 + 1)$ is part of an iteration, so it is equal to $2^h m$, where m is an odd, positive integer not divisible by 3, and h is established by r_0. Let $k = 2^{h - 1}k'$. Then

$$2^j y - 2^h m = 2 \cdot 3^i \cdot 2^{h-1} \cdot k'$$

and

$$2^{j-h} y - m = 3^i k'$$

or

$$2^{j-h} y \equiv m \mod 3^i$$

which can be solved for 2^{j-h}. Since h is known, we can find j. k can be obtained from k'. □

Remark. It is important to keep in mind what it means to say that a top row R_i *generates* a top row R_{i+1} via an exponent a_{i+1}. It means that a *subset* of R_i yields *all* the elements of R_{i+1} via the exponent a_{i+1}. The first element of this subset — examples are given in table Table 15, "Extensions, under one iteration of the 3x + 1 function, of the top rows of all 2-level tuple-sets," on page41 and in Tabl e16, " Extensions, under one iteration of the 3x + 1 function, of some top rows of 3-level tuple-sets," on page42 — can be determined by taking the inverse, under a_{i+1}, of the smallest element of R_{i+1}, and then applying Lemma 1.1 to obtain succeeding elements. Thus each succeeding element will be some multiple k of the distance

$lcm(2 \bullet 3^{i-1}, \ 2 \cdot 2^{a_{i+1}})$, where *lcm* denotes the least common multiple.

Next, we observe that it seems to require *all* i-level top rows to generate a *given* (i + 1)-level row via *all* exponents that *can* generate that (i + 1)-level top row. For example, as we can see in the table, in order to generate the level-3 row {11, 29, 47, 65, ...} via the exponent 1, the level-2 top row {1, 7, 13, 19, ...} is required. But in order to generate the same level-3 row via the exponent 3, the level-2 row {5, 11, 17, 23, ...} is required. We can generalize this observation.

Lemma 7.27. *For all $i \geq 2$, and for all (i + 1)-level top rows R_{i+1}, the minimum set of i-level top rows required to generate R_{i+1} via all possible exponents that can generate R_{i+1}, is $\{R_i\}$, the set of all i-level top rows. In other words, for all $i \geq 2$, and for all (i + 1)-level top rows R_{i+1}, if we generate R_{i+1} by any proper subset of $\{R_i\}$, then some elements of R_{i+1} will be generated by a proper subset of the set of exponents that can generate these elements*.

Proof:

The following is an edited version of a proof by Michael O'Neill.

Since each (i + 1)-level top row R_{i+1} is generated either by even or by odd exponents only (by Lemma 5.0), and since, by Lemma 15.0, "a third" of the exponents are excluded because they would imply mapping from a multiple of 3, there are $2 \bullet 3^{i-2}$ possible exponents mapping to R_{i+1}. Since this is the same as the number of *i*-level top rows, to prove the lemma we have to show that no two elements from different *i*-level top rows can map to the same element of R_{i+1} via the same exponent.

Let the two *i*-level rows be:

$r_0 + k_0 \bullet 2 \bullet 3^{i-1}$

and

$r_1 + k_1 \bullet 2 \bullet 3^{i-1}$

Mapping these to the $(i + 1)$ level and setting the results equal gives

$$\frac{3r_0 + 3k_0 \cdot 2 \cdot 3^{i-1} + 1}{2^j} = \frac{3r_1 + 3k_1 \cdot 2 \cdot 3^{i-1} + 1}{2^j}$$

or

$$r_0 + k_0 \cdot 2 \cdot 3^{i-1} = r_1 + k_1 \cdot 2 \cdot 3^{i-1}$$

and

$$r_0 - r_1 = (k_1 - k_0) \cdot 2 \cdot 3^{i-1}$$

So

$$r_0 \equiv r_1 \mod (2 \cdot 3^{i-1})$$

which imples that the two *i*-level rows are the same row. □

The Relationship Between Top Rows of All *i*-level and (*i* + 1)-level Tuple-sets

Let us pause at this point, and, first, get a clearer idea of the relationship between the top rows in all *i*-level and of all (*i* + 1)-level tuple-sets, and then, second, see if we can explain why this relationship seems to work out so nicely.

We begin by showing graphically, in Fig. 7, the generating relationship between the top rows of all 2-level tuple-sets, and the top rows of all 3-level tuple-sets. Each arrow represents the generating function via all exponents. The arrow points to the row generated. Note that, even though each row is identified by its first element, the contents of rows with the same first element at different levels are not identical, because of the distance function $d(i, i)$ (Lemma 1.0). By Lemma 7.25, the same generating relationship between successive top levels holds for all higher levels.

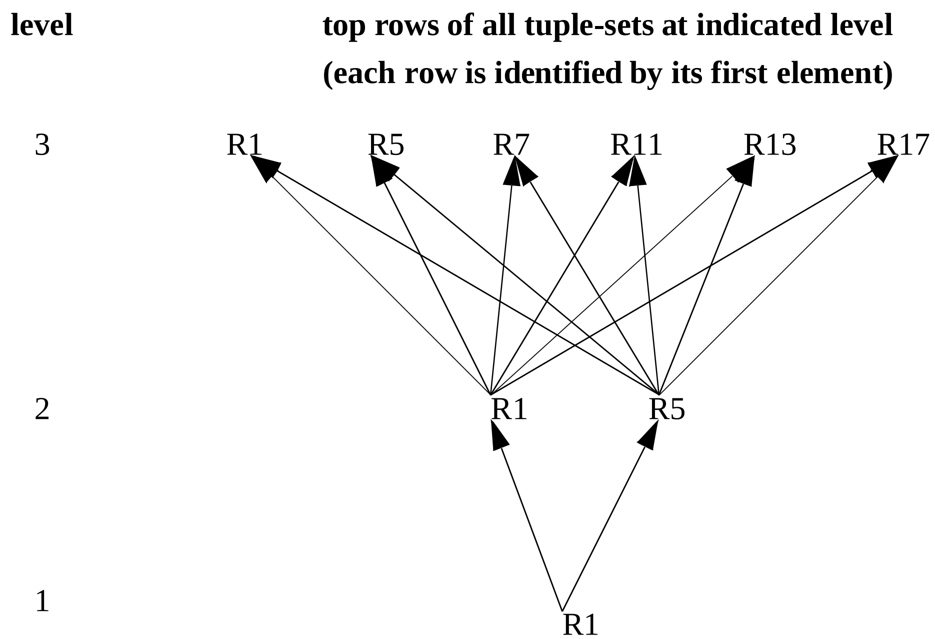

Fig. 7. Generating relationship between top levels of all 2-level tuple-sets and top levels of all 3-level tuple-sets

The Reduction of the Infinitary Tuple-sets Tree to an Equivalent Finitary Tree

Next, we observe that Lemma 3.24 and Lemma 7.25 imply that the infinitary tree of all tuple-sets can, without loss of generality, be reduced to a finitary tree, namely, a $(2 \cdot 3^{i-2})$ -ary tree, $i \geq 2$. ($(2 \cdot 3^{i-2})$ is the number of reduced residue classes mod $(2 \cdot 3^{i-1})$.) We state and prove this as Lemma 7.3.

Lemma 7.3. *Let R_i be a top-level row of an i-level tuple-set, $i \geq 2$. Then all exponents $a_{i+1} \geq 1$ can be partitioned into $(2 \cdot 3^{i-2})$ equivalence classes such that all a_{i+1} which are in a given class, generate the same $(i + 1)$-level row $R_{i+1} = f(R_i, a_{i+1})$, where f is as defined in Lemma 7.25.*

Proof:

The following is an edited version of a proof by Michael O'Neill.

We want to show that any $(i + 1)$-level top row R_{i+1} generated by an exponent sequence A with final exponent j also has a generating sequence with final exponent $j + 2 \cdot 3^{i-2}$.

Let y be a member of such a row and x be its generator. Then

$$\frac{3x+1}{2^j} = y$$

So we want another generator x' such that

$$\frac{3x'+1}{2^{j+2\cdot 3^{i-1}}} = y$$

This gives

$$\frac{3x+1}{2^j} = \frac{3x'+1}{2^{j+2\cdot 3^{i-1}}}$$

Then:

$$(3\cdot 2^{(2\cdot 3^{i-1})})x + 2^{2\cdot 3^{i-1}} - 1 = 3x'$$

Since, by Fermat's Little Theorem, $2^{2m} - 1 \equiv 0 \bmod 3$, we can divide this equation through by 3. Call the resulting second term on the left-hand side, k. So we have:

$$2^{2\cdot 3^{i-1}}x + k = x'$$

So x' always exists and the elements of the row R_{i+1} are generated by exponents separated by $2 \cdot 3^{i-2}$. □

Lemma 7.3 essentially proves the existence of the similarity classes of feasible vectors defined in Definition 3.6 of [3] (p. 48).

Remark 1 The finitary tree of tuple-sets has the property that, by Lemma 7.3, the number of exponent equivalence classes at each level i increases with i. Or, in other words, we may say, informally, that as i increases, the "branches" (exponent equivalence classes) grow at the same time *"thinner"* (because there is a greater distance between successive elements) *and more plentiful*, so that the larger that i is, the more closely do the uppermost branches "approximate" individual exponents.

Lemma 7.31. *Let a_{i+1}, $i \geq 2$, be an exponent that is "missing" from the set of exponents that generate a top row R_{i+1} because a_{i+1} is the exponent for a multiple-of-3. Then all exponents congruent to a_{i+1} mod $2 \cdot 3^{i-2}$ are likewise missing from the set of exponents that generate R_{i+1}.*

Proof:

We will have our result if we can prove:

if

$$\frac{3 \cdot 3m + 1}{2^{a_{i+1}}} = y$$

then there exists a k such that

$$\frac{3 \cdot 3k + 1}{2^{a_{i+1} + 2 \cdot 3^{i-2}}} = y$$

Setting the two left-hand sides equal and multiplying through by the larger denominator, we get:

$$2^{2 \cdot 3^{i-2}}(3 \cdot 3m) + 2^{2 \cdot 3^{i-2}} = 3 \cdot 3k + 1$$

which implies

$$2^{2 \cdot 3^{i-2}} \equiv 1 mod \ \ 3$$

Since $2^2 \equiv 1$ *mod* 3, this congruence holds, hence k exists. □

An illustration of the truth of Lemma 7.31 is given by the following table, which is derived from exponents mapping to the level-3 row {11, 29, 47, 65, ... } in Tabl e15, “Extensions, under one iteration of the 3x + 1 function, of the top rows of all 2-level tuple-sets,” on page 41. Here *m*3 denotes an exponent resulting from a multiple-of-3.

Table 17: Multiples-of-3 in a set of odd-exponent equivalence classes

Minimum residue of exponent equivalence class	Exponent congruent to minimum residue	Exponent congruent to minimum residue	Exponent congruent to minimum residue	...
1	7	13	19	...
3	9	15	21	...
*m*3	*m*3	*m*3	*m*3	...

Lemma 7.32. *Let R_{i+1} be the top row of an (i + 1)-level tuple-set. Then R_{i+1} is generated by exponents of one parity only.*

Proof:

Let y be the minimum element of R_{i+1}. Let x be an element of a top row R_i which maps to y via an exponent of, say, even parity $2k$. Then we have:

$$\frac{3x+1}{2^{2k}} = y$$

The next element in R_{i+1} is $y + 2 \cdot 3^{(i+1)-1}$ (Lemma 1.0). So adding $2 \cdot 3^{(i+1)-1}$ to both sides, we get:

$$\frac{3x+1}{2^{2k}} + 2 \cdot 3^{(i+1)-1} = y + 2 \cdot 3^{(i+1)-1}$$

The left-hand side is equal to:

$$\frac{3x + 1 + 2^{2k} \cdot 2 \cdot 3^{(i+1)-1}}{2^{2k}}$$

or

$$\frac{3(x + 2^{2k} \cdot 2 \cdot 3^{i-1}) + 1}{2^{2k}}$$

thus showing that $y + 2 \cdot 3^{(i+1)-1}$ is mapped to from an element of row R_i and via an exponent of even parity. A similar argument applies to odd exponents. The result follows by repetition of the argument for successive elements of row R_{i+1}. □

Is There a Fixed Upper Bound on the Smallest Generating Exponent for Any Top Row?

We will now consider a phenomenon exhibited by the first part of Table 15, “Extensions, under one iteration of the 3x + 1 function, of the top rows of all 2-level tuple-sets,” on page41. The phenomenon is that each top row in a 3-level tuple-set is generated by (among other exponents) an exponent which is ≤ 4. The details are summarized in the following table.

Table 18: Minimum exponents generating level-3 top rows

Level-3 top row generated	**Minimum exponent in equivalence class of exponents generating level-3 top row by, respectively, the level-2 top rows {1, 7, 13, ...}, {5, 11, 17, ...}**
{1, 19, 37, ...}	2, 4
{5, 23, 41, ...}	3, 5
{7, 25, 43, ...}	4, 6
{11, 29, 47, ...}	1, 3
{13, 31, 49, ...}	6, 2
{17, 35, 53, ...}	5, 1

The generalization of this phenomenon turns out to be true.

Lemma 7.35. *Let R_{i+1} be the top row of any (i + 1)-level tuple-set. Let* $\{min\{a_{i+1}\}_{R_{i+1}}\}$ *be the set of minimum residues of all exponent congruence classes (i.e., equivalence classes) whose exponents map to R_{i+1}. (By Lemmas 7.1 and 7.3 we know that each such class is a residue class mod $2 \cdot 3^{(i+1)-1}$.) Then at least one element of* $\{min\{a_{i+1}\}_{R_{i+1}}\}$ *is ≤ 4.*

Proof:
The following is an edited version of a proof by Michael O’Neill.

By Lemma 5.0 we know that the exponents mapping to a given tuple-set are either all even or all odd. In the following we deal with the even case; the odd case is similar.

The main thing to prove is that every third exponent is left out of the sequence of exponents mapping to a given top row. Let x, x' be two numbers that map, by successive even exponents, to the top row R_{i+1} whose minimum element is r_0. Then:

$$\frac{3x+1}{2^{2m}} = k2 \cdot 3^i + r_0$$

and

$$\frac{3x' + 1}{2^{2m+2}} = k'2 \cdot 3^i + r_0$$

Subtracting, we get:

$$\frac{3x + 1}{2^{2m}} - \frac{3x' + 1}{2^{2m+2}} = (k2 \cdot 3^i + r_0) - (k'2 \cdot 3^i + r_0)$$

which implies

$$4 \cdot 3x + 4 - 3x' - 1 = 2^{2m+3}3(k - k')$$

Dividing by 3, we get:

$$4x + 1 - x' = 2^{2m+3}3^{i-1}(k - k')$$

which gives

$$4x + 1 \equiv x' \quad mod \quad 3$$

or

$$x + 1 \equiv x' \quad mod \quad 3$$

This implies that successive even powers of two map numbers that have successive residues mod 3, so every third exponent is forbidden.

To see what happens when every third exponent is removed, consider the three possibilities for the case where there are 18 possible exponents (which is the number of equivalence classes of exponents mapping from a 3-level top row to a 4-level top row):

Table 19: All possiblites of even exponents through 18 mapping from a level-3 top row to a level-4 top row

2	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**
2	4		8	10		14	16	
2		6	8		12	14		18
	4	6		10	12		16	18

Obviously, in each case all of the available exponents must be used to provide 6 different exponents, so the least element in the set of exponents must be ≤ 4. It can easily be seen that this applies to odd exponents and levels other than 3. □

Instances of this lemma can be seen in Table 15, "Extensions, under one iteration of the 3x + 1 function, of the top rows of all 2-level tuple-sets," on page41 and Table 16, "Extensions, under one iteration of the 3x + 1 function, of some top rows of 3-level tuple-sets," on page42.

Lemma 7.36. *Let R_i be the top row of an i-level tuple-set and let r_{0i} be its first element. Then the the first element, $r_{0(i+1)}$, of the (i + 1)-level top row $R_{(i+1)}$ mapped to by R_i via the exponent a_{i+1} is given by*:

$$r_{0(i+1)} \equiv (2^{(j(2 \cdot 3^{i-1}) - a_{i+1})}(3r_{0i} + 1) - 3^i) mod (2 \cdot 3^{i-1})$$

where *j* is chosen to make the exponent positive.

Proof:

The following is an edited version of a proof by Michael O'Neill.

The mapping of R_i to $R_{(i+1)}$ is expressed by the following equation:

$$\frac{3(k_{0i}(2 \cdot 3^{i-1}) + r_{0i}) + 1}{2^{a_{i+1}}} = k_{i+1}(2 \cdot 3^i) + r_{0(i+1)}$$

This gives:

$$3r_{0i} + 1 - 2^{a_{i+1}}r_{0(i+1)} = k_{i+1}(2^{a_{i+1}+1} \cdot 3^i) - k_{0i}(2 \cdot 3^i) = g(2 \cdot 3^{i-1})$$

where *g* is an integer whose actual structure is irrelevant.

Now we multiply through by

$$2^{(j(2 \cdot 3^{i-1}) - a_{i+1})}$$

where *j* is chosen to make the exponent positive. This gives:

$$2^{j(2 \cdot 3^{i-1}) - a_{i+1}}(3r_{0i} + 1) - 2^{j(2 \cdot 3^{i-1})}r_{0(i+1)} = g(2 \cdot 3^i)2^{j(2 \cdot 3^{i-1}) - a_{i+1}}$$

And since

$$2^{j(2\cdot 3^{i-1})} \equiv 1 mod \ \ 3^i$$

this gives:

$$2^{j(2\cdot 3^{i-1})-a_{i+1}}(3r_{0i}+1)-(k_3 3^i+1)r_{0(i+1)} \ = \ g(2\cdot 3^i)2^{j(2\cdot 3^{i-1})-a_{i+1}}$$

Note that k_3 must be odd, since $k_3 3^i + 1$ is even (a power of 2). We now have:

$$2^{j(2\cdot 3^{i-1})-a_{i+1}}(3r_{0i}+1) \equiv ((k_3 3^i+1)r_{0(i+1)}) \ \ mod(2\cdot 3^i)$$

which, since k_3 and $r_{0(i+1)}$ are odd, and since, if x is odd,

$$x3^i \equiv 3^i \ \ mod(2\cdot 3^i)$$

gives us our result, namely:

$$r_{0(i+1)} \equiv (2^{(j(2\cdot 3^{i-1})-a_{i+1})}(3r_{0i}+1)-3^i)mod \ \ (2\cdot 3^{i-1})$$

□

Lemma 7.38. *Let $A = \{a_2, a_3, ..., a_i\}$ be an exponent sequence, and let $a = a_2 + a_3 + ... + a_i$. Let r be as defined in the proof of Lemma 6.0. Then the smallest element r_{0i} of the top row of the tuple-set T_A is given by*

$$r_{0i} \equiv (2^{j(2\cdot 3^{i-1})-a}r - 3^i) \ \ mod(2\cdot 3^i)$$

Proof:

The following is an edited version of a proof by Michael O'Neill.

We begin with an equation whose right-hand side represents the top row (see proof of Lemma 6.0):

$$\frac{3^i x + r}{2^a} \ = \ k(2\cdot 3^i) + r_{0i}$$

It will turn out that x is irrelevant.

We perform various manipulations and multiply through by

$2^{j(2 \cdot 3^{i-1}) - a}$

and get:

$$2^{j(2 \cdot 3^{i-1}) - a} 3^i x + 2^{j(2 \cdot 3^{i-1}) - a} r - 2^{j(2 \cdot 3^{i-1}) - a} r_{0i} = 2^{j(2 \cdot 3^{i-1}) - a} k(2 \cdot 3^i)$$

The first term is a multiple of $2 \cdot 3^i$ and so is congruent to 0 mod $2 \cdot 3^i$. Using an argument similar to that used in the proof of Lemma 7.36, we arrive at our result:

$$r_{0i} \equiv (2^{j(2 \cdot 3^{i-1}) - a} r - 3^i) \ \ mod(2 \cdot 3^i)$$

□

Next, we ask (looking at Tabl e15, "Extensions, under one iteration of the 3x + 1 function, of the top rows of all 2-level tuple-sets," on page41) if it is the case that successive elements of the generating sub-row, generate successive elements of the generated row. The answer is yes, as this next lemma establishes.

Lemma 7.4. Let x, $x + lcm(2 \cdot 3^{i-1}, 2 \cdot 2a_i)$, where *lcm* denotes the least common multiple, $i \geq 2$, be successive elements of the sub-row R'_i of the top row R_i that maps to the top row R_{i+1} via the exponent a_i. Then these successive elements map to successive elements of R_{i+1}. In other words, any sub-row of a top row R_i maps to an entire top row R_{i+1}.

Proof:
The result follows because

$$\frac{3(x + lcm(2 \cdot 3^{i-1}, 2 \cdot 2^{a_i})) + 1}{2^{a_i}} = \frac{3x + 1}{2^{a_i}} + 2 \cdot 3^i$$

which we obtain from the equality

$$\frac{3x + (2 \cdot 3^i)(2^{a_i}) + 1}{2^{a_i}} = \frac{3x + (2 \cdot 3^i)(2^{a_i}) + 1}{2^{a_i}}$$

. □

Lemma 8.0. *Let T be a 2-level tuple-set. Then the first 2-level tuple of T is an n-t-v-1.*

Proof:

Consider all tuple-sets defined by an even exponent. The tuple <1, 1> is in the tuple-set defined by the sequence {2 }, the tuple <5, 1> is in the tuple-set defined by the sequence { 4 }, the tuple <21, 1> is in the tuple-set defined by the sequence { 6 }, etc., for each even exponent, in accordance with Lemma 5.0. Clearly, each of these tuples is an n-t-v-1. But each of these tuples must be the first 2-level tuple in its tuple-set, because there does not exist an odd, positive integer 1 – 2 • 3, and therefore by Lemma 1.0 there does not exist a previous tuple in the standard ordering of tuples.

Now consider all tuple-sets defined by an odd exponent. The tuple <3, 5> is in the tuple-set defined by the sequence { 1 }, the tuple <13, 5> is in the tuple-set defined by the sequence { 3 }, the tuple <53, 5> is in the tuple-set defined by the sequence { 5 }, etc., for each odd exponent, in accordance with Lemma 5.0. Clearly, each of these tuples is an n-t-v-1, since 5 maps to 1. But each of these tuples must be the first 2-level tuple of its tuple-set, because there does not exist an odd, positive integer 5 – 2 • 3, and therefore by Lemma 1.0 there does not exist a previous tuple in the standard ordering of tuples. □

Lemma 10.0. *Assume a counterexample exists. Then every tuple-set contains an infinity of n-t-v-cs and an infinity of n-t-v-1s*.

Proof:

If a counterexample exists, then by Lemma 5.0 there exists an infinity of counterexamples. By Lemma 0.4, we can eliminate all multiples of 3 from the infinity of counterexamples and still have an infinity of counterexamples *y*, each of which is now a range element. Choose any finite sequence *A* of exponents. For each counterexample range element *y*, there exists an *x* which maps to *y* via *A* with a possible buffer exponent following *A*, by Lemma 7.0. Since, by definition, a tuple-set contains all tuples defined by a given sequence of exponents, it follows that for each *y*, there exists an n-t-v-c in the tuple-set T_A. A similar argument applies for n-t-v-1s. □

The “Last/First” Property

We now define a property of range elements that will be of importance in attempts to prove Conjecture 2 (equivalent to Conjecture 1), below. The term “last/first” is an abbreviation for “last element of the first *i*-level tuple in an *i*-level tuple-set”.

The “last/first” property of a range element collects the following properties:

(*a*) For each range element *y* (counterexample or not) there is a least *i* such that *y* is the last element of the first *i*-level tuple of an infinity of *i*-level tuple-sets. This follows from Lemma 3.0, i.e., from the fact that for some least *i*, *y* is a minimum residue of the reduced residue set mod 2 • 3^{i-1}, hence the least element in the top row of an *i*-level tuple-set. The fact that this applies to an infinity of tuple-sets follows from Lemma 5.0.

(b) Furthermore *y* is the last element of the first (*i* + 1)-, (*i* + 2)-, ...tuple in an infinity of (*i* + 1)-, (*i* + 2), ... -level tuple-sets. This follows from the fact that *y* is a minimum residue of the

reduced residue sets mod $2 \cdot 3^{(i + 1) - 1}$, mod $2 \cdot 3^{(i + 2) - 1}$... hence the least element of the corresponding top rows of the corresponding tuple-sets.

(c) Furthermore, every extension of every tuple containing y beginning at level i also defines first (i + 1)-, (i + 2)-, ... -level tuples in an infinity of (i + 1)-, (i + 2)-, ... -level tuple-sets. This follows from Lemmas 4.0 and 5.0.

We now state and discuss Conjecture 2.

Conjecture 2. *Let T be an i-level tuple-set. Then the first i-level tuple of T is an n-t-v-1.*

Discussion of Conjecture 2

Conjecture 2 is equivalent to Conjecture 1 because if Conjecture 2 is true, then the assumption of a counterexample implies, by Lemmas 3.0 and 4.0, that the first i-level tuple in some i-level tuple-set is both an n-t-v-1 and an n-t-v-c, contradicting the definition of these tuples.

Are there grounds for optimism that Conjecture 2 can be proved? The author believes there are, and that the following are some of them:

1. Lemma 8.0 proves that Conjecture 2 holds for all 2-level tuple-sets.

2. Lemmas 5.0 and 10.0 show that, if a counterexample exists, then not only is there an infinity of counterexamples, but there also is, in *each tuple-set*, an infinity of tuples which contain counterexamples.

We remark in passing that if we could prove that at least one tuple-set contained nothing but n-t-v-1s, or that the assumption of a counterexample implied that at least one tuple-set contained nothing but n-t-v-cs, then Conjecture 2 would be proved, because this would mean that (respectively) no n-t-v-c could be a first i-level tuple in an i-level tuple-set, or that no n-t-v-1 could be a first i-level tuple in an i-level tuple-set, contradicting Lemma 3.0.

3. A criticism of tuple-sets has been that the lemmas do not discriminate between n-t-v-1s and n-t-v-cs. In others words, that tuple-sets are “too coarse a net” to catch a proof of Conjecture 1. However, countering this criticism are two facts: (1) that each i-level tuple-set has one and only one first i-level tuple, and (2) that this tuple must be either an n-t-v-1 or an n-t-v-c, but not both.

Lemma 10.5. *Let T_A be an i-level tuple-set defined by an exponent sequence $A*a_i$, where $i \geq 4$, “*” denotes concatenation of exponents, A is any exponent sequence of length i - 2, and a_i is even if the last exponent of A is odd, and a_i is odd if the last exponent of A is even. Then the first i-level tuple of T_A is an n-t-v-1.*

Proof:

We know that 1 and 5 map to 1. Since, for all $i \geq 2$, 1 and 5 are minimal elements of the set of i-level tuple-set top rows (by Lemma 3.055), and hence are elements of first i-level tuples, and since 1 is mapped to by even exponents, and 5 is mapped to by odd exponents, we can apply Lemma 7.0 at each level i, being assured that a_i is the buffer exponent that is “forced” by the constraints on the last exponent in A. □

We will now show how the lemmas in this sub-section might be used to prove that there is no minimum counterexample and, hence, that Conjecture 1 is true.

Strategy of Proving There Is No Minimum Counterexample

Assume a counterexample exists. Without loss of generality (by Lemma 0.4), let it be the minimum counterexample that is a range element, i.e., not a multiple-of-3.

A minimum counterexample y that is a range element has the following properties: (1) for all z resulting from computations of y, $z \geq y$; and (2) for all x mapping directly or indirectly to y, $x \geq y$. Our possible strategy will be to show that no range element has property (2), hence that there is no minimum counterexample.

Now, for each $i \geq 2$, there exists a finite number of exponent sequences A of length i - 1 having the property that the first element x of any tuple defined by such a sequence is less than the last element y. Call this the "less-to-greater" property of a sequence. For example, all sequences {1, 1, 1, ..., 1} have this property.

(See also the discussion of "Conjecture 12. On the Existence of Paths With the "Less-to-greater" Property" on page76, as well as " Question 2. On the Number of Sequences, For Each i, Having the "Less-to-greater Property" on pa ge77, and the sub-section, "A Way to Reduce Computation Time in Computer Testing of Conjecture 1" on page77.)

We let $P(i - 1)$ denote the (finite) set of all sequences of length i - 1 having the less-to-greater property.

By Lemma 3.0 there exists a level i — call it $\underline{i}$ — at which our assumed minimum counterexample y first becomes the last element of the first i-level tuple of an i-level tuple-set (in fact, by Lemma 5.0, it becomes such an element in an infinity of tuple-sets). y is thus a minimum element of a reduced residue class mod $2 \cdot 3^{i - 1}$, and, by Lemma 3.055, y remains such a minimum element for all larger i, that is, y is a minimum element of a reduced residue class mod $2 \cdot 3^{(i + 1) - 1}$, $2 \cdot 3^{(i + 2) - 1}$, $2 \cdot 3^{(i + 3) - 1}$, ... (By known results — see discussion of "Conjecture 4. On the Filling-in of Intervals in the Base Sequence" on pa ge70 — we know that $\underline{i}$ must be > 21.)

By Lemma 5.0, we know that y is mapped to either by all even or by all odd exponents. Now even though no multiples of 3 are present in any i-level rows if $i > 1$, hence no exponent mapping from a multiple of 3 can map to y, we know, by Lemma 7.35 (and Lemma 15.0), that the smallest exponent mapping to the row containing y must be ≤ 4.

By Lemma 7.25, for all $i \geq 2$, *each* i-level top row generates *all* $(i + 1)$ - level top rows. Therefore for each $(i + 1)$ there exists a table in which the top of each column represents a reduced residue class mod $2 \cdot 3^{(i + 1) - 1}$, and the left-hand end of each row represents a reduced residue class mod $2 \cdot 3^{i - 1}$. (Our convention has been to represent each such class by its minimum element.) The intersection of row R_i and column $R_{i + 1}$ contains the exponent by which the reduced residue class R_i generates the residue class $R_{i + 1}$. Examples of such tables are given in "Generating Level-(i + 1) Top Rows from Level-i Top Rows" on page38.

Therefore we can proceed as follows:

1. Choose a $k \geq 1$ such that (a) there exists an exponent sequence A of length $\underline{i} + k - 1$ having the less-to-greater property; (b) the last exponent $a_{\underline{i} + k}$ of A is of the same parity as that which maps to y; and (c) $a_{\underline{i} + k} \leq 4$.

2. Now proceed down the y column in the table for $\underline{i} + k$ to a row $R_{\underline{i} + k - 1}$ containing the exponent $a_{\underline{i} + k}$.

3. In the table for $\underline{i} + k - 1$, proceed down the column for $R_{\underline{i} + k - 1}$ to a row containing the second exponent from the last in A, i.e., the exponent $a_{\underline{i} + k - 1}$.

4. Continue in this manner down to R_1.

We would like to argue that this process proves the existance of an x that is less than y, and hence that there is no minimum counterexample.

Of course, if this strategy is to work at all, we must show that "*each i*-level top row generates *all* (i + 1) - level top rows" has the implications we want it to have (see Remark following proof of Lemma 7.25).

The Buffer Exponent in the Proof of Lemma 7.1

Next, we investigate the behavior of the buffer exponent (defined in the proof of Lemma 7.1). We will begin with the shortest exponent sequence, namely, the exponent sequence $A = \{a_2\}$. It is clear that, if $y = 11$, then for any sequence A = $\{a_2\}$, where a_2 is odd, we can find an x that maps to y via a_2. (This we knew as soon as we knew the two top rows of all 2-level tuple-sets, namely the row {1, 7, 13, 19, ...], which is defined by even a_2, and the row {5, 11, 17, 23, .. }, which is defined by odd a_2.) For $y = 11$, and $A = \{a_2\}$, where a_2 is even, we need buffer exponents. The following table sets forth a few examples.

Table 21: Examples of buffer exponents

***y*, to be produced by exponent sequence A**	**Exponent sequence, *A*, desired to map to *y***	**x that maps to y via sequence *A***	**Buffer exponent following *A* required to produce *y***
11	$\{a_2\}$, a_2 odd	7, 29, 117, ...	None
11	{2}	9	1
11	{4}	37	1
11	{6}	149	1

Question 3. For each range element y and each exponent sequence A of length 1, 2, 3, ..., what is the buffer exponent required to have A produce y? In other words, what is the function $g(y, A) = a_i$, where y is a range element, A is an exponent sequence of length i - 2, and a_i is the buffer exponent required to guarantee the existence of an x such that x maps to y via A followed by a_i (which, of course, may be the null exponent).

The author will pay $75 for the first correct answer to this question.

It should be pointed out that the observations, conjectures, and question in this sub-section constitute steps toward "converting" recursive "spiral"s, described in the next section of this paper, into tuple-sets. This statement should become clearer when the reader has read the section on recursive "spiral"s.

Cycles

We conclude the tuple-sets section of this paper with a few results on cycles, even though these do not so far contribute to a possible strategy for proving Conjecture 1.

In the literature on the $3x + 1$ Problem, the term "cycle", or "loop", denotes the equivalent of a tuple in which a domain element repeats. If we allow x to be an odd, *negative* integer, then known cycles include:

Table 24: Known Cycles

Sequence of odd integers	Corresponding sequence of exponents of 2
1, 1, 1, 1, ...	2, 2, 2, 2, ...
-1, -1, -1, -1, ...	1, 1, 1, 1, ...
-5, -7, -5, ...	1, 2, 1, ...
-17, -25, -37, -55, -41, -61, -91, -17, ...	1, 1, 1, 2, 1, 1, 4, 1, ...

We will refer to the cycles containing the domain elements 1 and -1 as the "trivial cycles". A simple algebraic argument shows that the only possible cycles of length 1, i.e., the only cycles containing only one domain element, are the trivial cycles.

Lemma 3.06. *If a cycle exists, it must be of length at least 17,087,915.*

The proof is in (Eliahou 1993). Note: the author of the present paper does not know if Eliahou's result is based on a definition of the $3x + 1$ function in which, if $3x + 1$ is even, then $3x + 1/2$ is considered an iteration.

Clearly, a cycle of domain elements implies that the sequence of exponents, A, defining the cycle must also be cyclical. However, the existence of a cycle of exponents in A does not necessarily imply a cycle of domain elements, as we shall see in Lemma 3.1.

Lemma 3.07. *At most one cycle exists having a given sequence of exponents.*

Proof :

Assume that, in a tuple, the first element u occurs again at level i, $i > 2$. But there is no h such that (as required by the distance functions defined in Lemma 1.0)

$$u + h \cdot 2 \cdot 3^{i-1} = u + h \cdot 2 \cdot 2^{a_2} 2^{a_3} \dots 2^{a_i}$$

Therefore there can be no other tuple whose first element and ith element are the same. □

Thus we know that there cannot exist another cycle having the sequence of exponents defined by the cycle beginning with -17 (see Table 5).

The fact that there are only a finite number of integers in a cycle in no way implies that the existence of a cycle implies only a finite number of counterexamples to Conjecture 1. See Lemma 5.0.

Clearly, no cycle can exist in the sequence of tuple-sets defined by unlimited, successive concatenations of any fixed exponent a_i, $a_i > 1$, because each iteration under a_i produces a result y that is less than the argument x. The following lemma shows that no cycle can exist in the sequence of tuple-sets defined by unlimited, successive concatenations of the exponent 1 either.

Lemma 3.08. *No cycle exists in the sequence of tuple-sets defined by unlimited, successive concatenations of the exponent 1.*

Proof:

Assume such a cycle exists. By a simple algebraic argument, it must contain at least two elements. These elements, a and b, must be the first elements of different tuples in every tuple-set in the sequence of tuple-sets established by the successive repetitions of the exponent 1. But then, at some level i, the distance function $d(1, i)$ defined in Lemma 1 must be violated. □

Lemma 3.1. *Let A consist of an infinitely repeating cycle of exponents, i.e., let* $A = \{a_2, a_3, ..., a_m = a_2, a_{m+1} = a_3, ... \}$, $m \geq 3$. *Then, informally, no tuple such that elements don't repeat when elements of A repeat, can be an infinite-tuple. Formally,*

If $a_{i+1} = a_{m+i+1}$ *for all* $i \geq 1$ *and* $t_{j_i} \neq t_{j_{m+i}}$, *then* t_j *is not an infinite-tuple.*

Proof:

We use an indirect proof.

Assume that t_j is an infinite-tuple in T_A. Since, by hypothesis, elements don't repeat when elements of A repeat, this means there exists in T_A at least two infinite-tuples. But this implies that at some level i, the distance functions defined in Lemma 1.0 would fail to hold. Hence the Lemma is proved. □

It is obvious that no A containing an infinite repetition of any exponent other than 1 can define a tuple-set containing a counterexample, for, in an iteration under such an exponent,x always produces a $y < x$.

From Lemma 3.1 we see that, although in an iteration under the exponent 1, x always produces a $y > x$, no tuple-set T_A contains a counterexample if $A = \{a_2, a_3, ..., a_k, 1, 1, 1, ...\}$, k ≥ 2, where $a_2, a_3, ..., a_k$ may, of course, each = 1 also. A second proof of this statement follows from fact that -1 results in an infinite computation all of whose exponents are 1, hence by Lemma 3.07, no other infinite loop can exist in that infinite tuple-set.

If x is replaced by y in Equation (7.1) in the proof of Lemma 7.0, we obtain an equation that expresses the existence of a cycle. If present-day Diophantine equation theory is able to determine the number of solutions to this modified equation, then the open question of the existence of non-trivial cycles is answered. However, as Michael O'Neill has pointed out to the author, it must be remembered that r and a in the modified equation are not independent variables but short-hand for the result of several iterations of the $3x + 1$ function. Hence, it does not seem likely that existing Diophantine theory will provide a ready-made answer.