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CHAPTER 8

For Future Mathematicians Only

(from William Curtis’s *How to Improve Your Math Grades,* on occampress.com)[1]

1. This line has been added to the first page of the chapter because Google does not reveal this information when it makes the chapter accessible following a user’s search.

Myths Sustaining the Current Mathematics Culture

This book was written out of frustration with the current methods of presenting mathematical knowledge, whether in the classroom, in textbooks, or in mathematical journals. During the course of doing background reading for the book, especially in the history of mathematics, and discussing some of the book's ideas with graduate students and professional mathematicians, and then during the actual writing, it began to dawn on me how rigid — and I will even say how out-of-step with reality — the current mathematics culture is.

I began to accumulate a list of some of the myths that sustain this culture — myths that young mathematicians-to-be learn in graduate school, where they also learn that the myths are not to be questioned.

Following the list is evidence that I believe shows that the myths are false. Fortunately, I can be very brief in this chapter, because the case against these myths was made clearly and forcefully and in detail in a book by Morris Kline called *Why the Professor Can't Teach* (St. Martin's Press, N.Y., 1977). The book is unfortunately out of print, but copies are readily available on the Internet. (See, e.g., the web sites www.mxbf.com, www.booksold.com, www.abe.com, and www.abebooks.com.) If you plan to become a professional mathematician, you need to read this book. You may disagree with some or all of it, but you need to read it.

Here are the myths. Evidence that the myths are not true follows.

- All top-rank mathematicians were child prodigies.
- No top-rank mathematician is interested in anything but pure mathematics — not applied mathematics, and certainly not literature or art (with the possible exception of music) or any of the humanities.
- No one can hope to be a really good mathematician unless he or she goes to one of the best schools.
- Mathematical talent can only be discovered through difficult, highly competitive exams
- Every top-rank mathematician has held a tenured position on the faculty of a major university.
- Whatever is published in refereed journals is probably important.
- Every competent mathematician reads the literature on a problem he or she is trying to solve.
- No top-rank mathematician publishes his own work.
- No top-rank mathematician ever publishes incomplete or faulty proofs, or publishes conjectures that turn out to be false.
- No top-rank mathematician is interested in teaching or in writing textbooks.

• No top-rank mathematician is interested in writing popularizations of his, or others', work.

• The way that mathematical knowledge is currently presented — in classrooms, textbooks, and journals — is the best way it can be presented.

• In general, the length of time that is required to earn a PhD in mathematics is the proper length of time.

• No top-rank mathematician is interested in the history of his subject.

• Mathematicians lose their talent/ability with age.

Evidence that the Myths are not True

Myth: All top-rank mathematicians were child prodigies.

• "... precocity in a mathematician has no particular significance one way or the other, and there are examples both ways."— Littlewood, J. E. [one of the great mathematicians of the 20th century, who happened to be precocious], in *Littlewood's Miscellany*, Cambridge University Press, N.Y., p. 80.

• "[Isaac] Newton (1642-1727) ... was educated at local schools of low educational standards and as a youth showed no special flair, except for an interest in mechanical devices." — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p. 357

"It appears that Newton's interest in mathematics began in 1664 [when he was 22]. In that year he embarked on a course of self-instruction..." — *The Princeton Companion to Mathematics*, ed. Gowers, Timothy, Princeton University Press, Princeton, N.J., 2008, p. 742.

• René Descartes (1596 - 1650), discoverer of analytical geometry, "was graduated from the University of Poitiers as a lawyer and went to Paris. There he met Mydorge and Father Marin Mersenne and spent a year with them in the study of mathematics." — ibid., p. 304

• "Before Newton and Leibniz, the man who did most to introduce analytical methods in the calculus was John Wallis (1616-1703). ... he did not begin to learn mathematics until he was about twenty — his university education at Cambridge was devoted to theology..." — ibid., p. 353

• Gottfried Wilhelm Leibniz (1646-1716), co-discoverer of the calculus and a pioneer logician, “knew almost no mathematics up to 1672 [when he was 26].” — ibid., p. 370.

• “We have no data to tell us when Fermat[1] became interested in mathematics, but this must have occurred at the latest during his stay in Bordeaux in the late twenties.” — Weil, Andre, *Number Theory: An approach through history*, Birkhäuser, Boston, 1984, p. 39. In other words, Fermat, who was born in 1601, may have become interested in the subject as late as his late twenties.

• “Hermann Günther Grassmann (1809-77) [discoverer of vector algebra], ... showed no talent for mathematics as a youth and ... had no university education in mathematics, but later became a teacher of mathematics in the gymnasium (high school) at Stettin, Germany...” — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p. 782

• George Boole (1815-1864), discoverer of boolean algebra, which governs the logical operations of modern computers. His “extraordinary mathematical talents ... did not manifest themselves in early life. At first, his favorite subject was classics. By his teens, he had learned Latin, Greek, German, Italian, and French.” — “George Boole”, Wikipedia, May 21, 2011.

• “Karl Weierstrass (1815-1897) [who was instrumental in establishing a rigorous logical basis for analysis] entered the University of Bonn to study law. After four years in this effort he turned to mathematics in 1838 ...”— ibid., p. 643

• Sophus Lie (1842-1899) [discoverer of Lie groups and Lie algebras], “was twenty-six when he discovered that, in his own words, he ‘harbored a mathematician’...

“Lie graduated in general science from the university in Oslo in 1865 [age 23] but without showing any special aptitude for mathematics. It was not until 1868, when he attended a lecture by the Danish geometer Hieronymus Zeuthen on the work of Chasles, Möbius and Plücker, that he becamse inspired by modern geometry.” — *The*

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1. Pierre de Fermat (1601 - 1665), generally considered the founder of modern number theory and, with Pascal, of probability theory, also made fundamental discoveries in what was to become the integral and differential calculus.

Princeton Companion to Mathematics, ed. Gowers, Timothy, Princeton University Press, Princeton, N.J., 2008, p. 737.

• I cannot neglect to mention Edward Witten, who, though a physicist (he is one of the major forces behind the development of string theory), won the 1990 Fields Metal, the most prestigious prize in *mathematics*. He "graduated from Brandeis College in 1971 with a degree in history and plans to become a political journalist. He succeeded in publishing articles in the *New Republic* and the *Nation*. He nonetheless soon decided that he lacked the 'common sense' for journalism (or so he told one reporter); he entered Princeton to study physics and obtained his doctorate in 1976." — Horgan, John, *The End of Science*, Broadway Books, N.Y., 1996, p. 67.

Myth: No top-rank mathematician is interested in anything outside his or her specialty — not in applied mathematics, and certainly not literature or art (with the possible exception of music) or any of the humanities.

I once knew a mathematician whose contempt for all subjects outside of mathematics went to the point that he was *proud* of his inability to play Scrabble well. (That inability was only further proof of his mathematical genius.) Yet it has not always been this way. Some of the very greatest mathematicians of the past exhibited outstanding ability in subjects having nothing to do with numbers.

Consider:

• Francois Viète (1540-1603), who laid the foundations for the algebraic approach to geometry, spent most of his life in high public office; "it was only during the time he had free from official duties that he was able to devote himself to mathematics." — *The Princeton Companion to Mathematics*, ed. Gowers, Timothy, Princeton University Press, Princeton, N.J., 2008, p. 737.

• Girolamo Cardano (1501 - 1576), a leading mathematician of the Renaissance, who was "famous throughout Europe as a physician, mathematician, humanist, and astrologer...The volume and range of his writing was immense: although much is probably lost, the extant material has been estimated to fill about 7000 pages." — Hollingdale, Stuart, *Makers of Mathematics*, Penguin Books, N.Y., 1989, p. pp. 111-112.

• "...Fermat had enjoyed an excellent classical education; he was well versed in Latin, Greek, Italian and Spanish, and generally praised for his skill in writing verse in several languages, a skill which he transmitted to his son Samuel. He collected manuscripts; his advice was eagerly sought on the emendation of Greek texts." — Weil, Andre, *Number Theory: An approach through history*, Birkhäuser, Boston, 1984, p. 38.

• René Descartes (1596 - 1650), who discovered analytic geometry, was equally interested in philosophy, and is known primarily to most educated laymen as a philosopher, via his famous statement, *cogito ergo sum* (I think, therefore I am).

• Blaise Pascal (1623 - 1662) , geometer, inventor of a mechanical calculating machine, co-discoverer, with Fermat, of the theory of probability, was as interested in religion, theology, and philosophy, as he was in mathematics. His *Pensées* is one of the classics of philosophy.

• Isaac Newton (1642 - 1727), co-discoverer of the differential and integral calculus, and of the law of gravity: "...if you want a picture of the complete man, you must remember that theological questions occupied much of his attention. Archbishop Tenison said to him, 'You know more divinity than all of us put together', and Locke wrote of [his knowledge of] 'divinity too, and his great knowledge in the Scriptures, wherein I know few his equals'...The works of the Church Fathers were prominent in his library. His two books the *Chronology of Ancient Kingdoms Amended* and *Observations upon the Prophecies of Daniel and the Apocalypse of St John* probably cost him as much effort as the Principia. There were over 1,300,000 words in manuscript on theology in [his] Portsmouth papers..." — Newman, James R., *The World of Mathematics*, Vol. 1, Simon and Schuster, N.Y., 1956, p. 274.

• Gottfried Wilhelm Leibniz (1646-1716), known to mathematicians as one of the co-discovers, along with Newton, of the calculus, and to philosophers as one of the greatest of all philosophers, was in addition a diplomat, lawyer, historian, philologist, and pioneer geologist. — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p. 370.

• Leonhard Euler (1707 - 1783), ranked among the top four or five mathematicians of all time, knew by heart "innumerable poems and the entire *Aneid*." — ibid., p. 401.

• Paulo Ruffini (17650 - 1822), was not only a mathematician but also a philosopher and a medical doctor (he conducted scientific research on typhus). He is best known for his work on the Abel-Ruffini theorem, which proved that polynomial equations of degree 5 and higher cannot be solved by radicals, and for his pioneering work on group theory.

• Karl Friedrich Gauss (1777 - 1855), considered to be the greatest mathematician who ever lived, possessed "abilities in classical languages [that] matched those in mathematics. Indeed, when he left [the Caroline College] in 1795 to enter the University of Göttingen, he was undecided whether to devote himself to mathematics or philology... the study of languages was to remain a lifelong hobby..." — Hollingdale, Stuart, *Makers of Mathematics*, Penguin Books, N.Y., 1989, p. 315.

•Augustin-Louis Cauchy (1789-1857), one of the greatest 19th-century mathematicians, was at first a student of engineering. Even after turning to mathematics, he had universal interests. While in political exile in Italy, he "taught Latin and Italian for several years... He knew the poetry of his time and was the author of a work on Hebrew prosody." — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p. 633.

• William Rowan Hamilton (1805 - 1865), considered the greatest English mathematician after Newton "... at the age of five ... could read Latin, Greek, and Hebrew. At eight, he added Italian and French; at ten he could read Arabic and Sanskrit and at fourteen, Persian. A contact with a lightning calculator inspired him to study mathematics...

"He was deeply religious and this interest was most important to him. Next in order of importance were metaphysics, mathematics, poetry, physics, and general literature. He also wrote poetry." — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, pp. 777-778.

• Hermann Grassmann (1809 - 1877), discoverer of vector algebra, "wrote books on German grammar, collected folk songs, and learned Sanskrit. His dictionary and his translation of the *Rigveda* (still in print) were recognized among philologists. He devised a sound law of Indo-European languages, named "Grassmann's Law" in his honor. [Unlike his mathematical accomplishments] these philological accomplishments were honored during his lifetime; he was elected to the American Oriental Society and in 1876 he received a honorary doctorate from the University of Tübingen." — Wikipedia, June, 2009.

• Arthur Cayley (1821-1895), "a prolific writer and creator in various subjects, notably the analytic geometry of *n* dimensions, determinant theory, linear transformations...and matrix theory" in addition to "his fine work in law and [his] prodigious accomplishments in mathematics, ... found time to develop interests in literature, travel, painting, and architecture." — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p. 805.

• Bernhard Riemann (1826-1866), one of the greatest mathematicians of the 19th century, "though he made contributions to mathematics proper, he was deeply concerned with physics and the relationship of mathematics to the physical world. He wrote papers on heat, light, the theory of gases, magnetism, fluid dynamics, and acoustics. He attempted to unify gravitation and light and investigated the mechanism of the human ear. His work on the foundations of geometry sought to ascertain what is absolutely reliable about our knowledge of physical space... He himself states that his work on physical laws was his chief interest. ... It seems very likely, on the basis of evidence given by Felix Klein, that Riemann's ideas on complex functions were suggested to him by his studies on the flow of electrical currents along a plane." Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, pp. 655-656.

• Henri Poincaré (1854-1912), "the extraordinary French mathematician, philosopher, and physicist ... A cultured intellectual, Poincaré was widely acclaimed as one of the greatest of nineteenth-century mathematicians for his invention of a great part of topology." He was also acclaimed for "his celestial mechanics, his enormous contributions to the electrodynamics of moving bodies. Engineers lauded his writings on wireless telegraphy. The wider public devoured his best-selling books on the philosophy of conventionalism, science and values, and his defense of 'science for science'." — Galison, Peter, *Einstein's Clocks, Poincaré's Maps*, W. W. Norton & Co., N.Y., 2003, p. 32.

He also "produced, quite independently of Einstein, a detailed mathematical physics incorporating the relativity principle." — ibid., p. 28.

"Who would guess from *Science and Hypothesis*, his best-selling book of 1902, that he had trained as a mining engineer and served as an inspector in the dangerous, hard-pressed coal mines of eastern France?" — ibid., p. 41.

He was also a "scientific rapporteur of a major scientific expedition and had served as president of the Bureau of Longitude." — ibid., p. 48.

• Émile Borel (1871-1956), the French mathematician who created the first effective theory of the measure of sets of points and who shares credit with Baire and Lebesgue for launching the modern theory of functions of a real variable[1], "at different times...was a brilliant young mathematician; professor; socialite in Paris; director of the École Normale Supérieure; journalist; publisher of the *Revue du Mois* (which played a critical role in the formation of the Radical Left); mayor of his small home town; chief of scientific and technical services for the Ministry of War during the First World War; Minister of the Navy for six months; activist in the French Resistance [during the Second World War]..." — Graham, Loren, and Kantor, Jean-Michel, *Naming Infinity*, The Belknap Press of Harvard University Press, Cambridge, Mass., 2009, p. 37.

• Felix Hausdorff (1868-1942), who made major contributions to set theory and general topology, "had broad intellectual interests and moved in Nietzschean circles of artists and poets at Leipzig. Under the pseudonym Paul Mongré he wrote two long philosophical essays of which the more prominent was "Das Chaos in kosmischer Auslese" ("The chaos in cosmic selection"). Until 1904 he regularly contributed cultural critical essays to a renowned German intellectual review of the time, continuing to contribute, although less frequently, until 1912. He also published poems and a satirical play." — *The Princeton Companion to Mathematics*, ed. Gowers, Timothy, Princeton University Press, Princeton, N.J., 2008, p. 793.

• J. E. Littlewood (1885-1977), one of the great mathematicians of the 20th century, "was as unlike as possible to what used to be and still is regarded as a typical mathematician... He was one of the best gymnasts at school, loved to swim, stroked a College eight at Cambridge, excelled at rock climbing and skiing, was a hard hitting cricketer (though he did not play regularly). He was an elegant dancer and passionaately keen on music [especially Bach, Beethoven, and Mozart — "He considered life too short to waste on other composers."].... He was well read and could talk on almost everything in an engaging fashion. He was very witty, had a lovely sense of humour and was full of fascinating stories." — Bollobás, Béla, Foreword to *Littlewood's Miscellany*, Cambridge University Press, N.Y., 1990, pp. 17, 20, 21.

In physics, I will also mention in passing:

• Albert Einstein (1879 - 1955), who loved classical music, particularly Mozart, and was an amateur violinist. "With...his devotion to literature and music and philosophy, Einstein in those days [just prior to the publication of his first four, great

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1. *Britannica Online Encyclopedia*, April, 2009

papers in 1905] resembled a poet as much as a scientist." — Ferris, Timothy, *Coming of Age in the Milky Way*, Anchor Books, N.Y., 1988, p. 183.

Einstein wrote to the philosopher Moritz Schlick in 1917:

"Your representations that the theory of rel[ativity] suggests itself in positivism, yet without requiring it, are ... very right. In this also you saw correctly that this line of thought had a great influence on my efforts, and more especially, E. Mach, and even more so Hume, whose *Treatist of Human Nature* I had studied avidly and with admiration shortly before discovering the theory of relativity. It is very possible that without these philosophical studies I would not have arrived at the solution."[1]

• Richard Feynman(1918 - 1988), noted for his contempt for all things cultural, nevertheless thought it worth the trouble to learn how to draw portraits.

• Paul Dirac (1902 - 1984), about whom his wife, Margit, wrote: "Paul Dirac adored music. Even my knitting had to stop for complete silence when he was listening. He was also a great admirer of art. Not only did he like beautiful things in our home, he was also a tireless museum fan. He made me read *War and Peace*, and he read a great many books that I suggested to him. Theater, movies, ballet: we never missed a good performance, even if we had to go to London or from Princeton to New York." — Dirac, Margit W., letter to *Scientific American*, Aug., 1993, p. 11.

In spite of this testimony, the authors of the article to which Dirac's wife was responding, for some reason held to their belief that Dirac had only a "nodding acquaintance" with the arts and humanities, but they did not deny the depth of interest in these subjects displayed by such physicists as Bohr, Heisenberg, Oppenheimer and Schrödinger.

• Steven Weinberg, Nobel-Prize-winning physicist: "His spacious office [at the University of Texas at Austin] was cluttered with periodicals that testified to the breadth of his interests, including *Foreign Affairs*, *Isis*, *Skeptical Inquirer*, and *American Historical Review*, as well as physics journals." — Horgan, John, *The End of Science*, Broadway Books, N.Y., 1996, p. 73.

• Murray Gell-Mann, Nobel-Prize-winning physicist: "According to a 'personal statement' that he distributes to reporters, his interests include not only particle physics and modern literature, but also cosmology, nuclear arms-control policy,

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1. Quoted in Galison, Peter, *Einstein's Clocks, Poincaré's Maps*, W. W. Norton & Company, N.Y., 2003, pp. 239-240.

natural history, human history, population growth, sustainable human development, archaelogy, and the evolution of language. Gell-Mann seems to have at least some familiarity with most of the major languages of the world and with many dialects." — Horgan, John, *The End of Science*, Broadway Books, N.Y., 1996, p. 211.

Carl Sagan's remarks are appropriate here. "At the University of Chicago I also was lucky enough to go through a general education program devised by Robert M. Hutchins, where science was presented as an integral part of the gorgeous tapestry of human knowledge. *It was considered unthinkable for an aspiring physicist not to know Plato, Aristotle, Bach, Shakespeare, Gibbon, Malinowski, and Freud — among many others.* In an introductory science class, Ptolemy's view that the Sun revolved around the Earth was presented so compellingly that some students found themselves re-evaluating their commitment to Copernicus. The status of the teachers in the Hutchins curriculum had almost nothing to do with their research; perversely — unlike the American university standard of today — teachers were valued for their teaching, their ability to inform and inspire the next generation." — Sagan, Carl, *The Demon-Haunted World*, Random House, N.Y., 1995, pp. *xiv-xv*. (Italics mine)

As far as applied mathematics is concerned, the modern mathematician's contempt for practical applications of his subject would have appalled most mathematicians of the past. Archimedes, Newton, Euler, and Gauss, considered to be the greatest mathematicians who ever lived, each worked extensively in applied mathematics (see, e.g., Morris Kline's *Mathematical Thought from Ancient to Modern Times*).

All four of these mathematicians would have been in complete agreement with Gauss's words, "Thou, nature, art my goddess; to thy laws my services are bound...", as would virtually every mathematician until the end of the 19th century. Joseph Fourier said, "The profound study of nature is the most fertile source of mathematical discoveries."

The vast majority of good and great mathematicians would have regarded with incredulity remarks like Paul Erdös' to a colleague who had published an "applied" paper, "I am praying for your soul" (Alexander, James, review of Hoffman, Paul, *The Man Who Lived Only for Numbers*, in *The New York Times Book Review,* p. 33, Sept. 27, 1998). And they would have been utterly appalled at the remark that one famous 20th century mathematician is said to have made, namely, that it is not a question of whether applied mathematics is good mathematics or bad mathematics, because applied mathematics isn't mathematics at all.

A similar elitism is present in parts of the scientic community. But not all scientists agree. "I detest and abhor the academic snobbery which places pure scientists on a higher cultural level than inventors." — Dyson, Freeman, *Infinite in All Directions*, Harper & Row, Publishers, N.Y., 1988, p. 138.

It is perfectly legitimate for students to ask of a mathematical subject they are studying, or of any part of that subject, "*What good is this?*", "*Does this have any practical applications?*" The scorn with which such questions are greeted by many professional mathematicians only comes back at them in the scorn that students soon develop for the subject, and indeed for all of mathematics. Knowing that a subject has practical applications helps many students to feel that they are not being asked to waste their time on a mere collection of difficult abstractions. A really *good* math teacher (whether in high school or college) makes part of his own ongoing attempt to improve his teaching, the collecting of descriptions of practical applications of his subject, no matter how few and far between these may be. "This idea is used in modern cryptography in the following way..."; "This idea is used to speed up certain types of computations, in the following way..."; "This idea is useful in some of the most advanced theoretical physics, in the following way..."... Of course, "practical applications" can also refer to other subjects in pure mathematics in which the subject at hand has been found to be useful.

Myth: No one can hope to be a really good mathematician unless he or she goes to one of the best schools.

• Gerard Desargues (1591-1661), one of the pioneers of projective geometry, was self-educated. Yet "Desargues was one of the most original mathematicians in a century rich in genius." — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, pp. 288, 295.

• "In June 1662 [when he was 20] [Isaac] Newton [1642-1727] set out for Cambridge...The teaching given to undergraduates at Cambridge in the 1660s had remained virtually unchanged since the Middle Ages; there was nothing to stimulate a budding mathematician. During his first term Newton picked up a book on astrology, but was unable to understand some of the mathematics it contained. So he bought a copy of Euclid's *Elements*, but was disappointed to find that many of the propositions were 'obvious'. He went on to teach himself mathematics by reading the works of Kepler, Viète and Wallis, among others, but above all a Latin translation of *La*

Géométrie of Descartes, which he found heavy going." — Hollingdale, Stuart, *Makers of Mathematics*, Penguin Books, N.Y., 1989, p. 170.

Gottfried Wilhelm Leibniz (1646-1716), co-discoverer of the calculus and a pioneer logician, was self-taught in mathematics. — *The Princeton Companion to Mathematics*, ed. Gowers, Timothy, Princeton University Press, Princeton, N.J., 2008, p. 743.

• James Bernoulli (1655-1705), a member of the remarkable Bernoulli family that made so many contributions to the early development of the calculus, was self-taught. — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p. 381.

• The two men who first showed how complex numbers, and their arithmetical operations, could be represented graphically (not at all obvious at the time!), were both self-taught. They were the Norwegian-born surveyor Caspar Wessel (1745-1818), and the Swiss bookkeeper Jean-Robert Argand (1768-1822), after whom the graphical representations are named (Argand diagrams). — ibid., p. 630.

• Sophie Germain (1776-1831), probably the greatest female mathematician prior to the 20th century, taught herself mathematics from books in her father's library. Later, she communicated by letter with famous mathematicians, among them Legendre and Gauss. She did pioneering work in elasticity theory and Fermat's Last Theorem.

• Perhaps the most extreme example is that of Galois, unquestionably one of the founding fathers (though he died at 21) of modern mathematics, specifically, of modern algebra.

"[Evariste Galois (1811-32)] sought to enter the Ecole Polytechnique but possibly because he failed to explain in sufficient detail the questions he had to answer orally at the entrance examination or because the examining professors did not understand him he was rejected in two tries. He therefore entered the Ecole Préparatoire (the name then for the Ecole Normale and a much inferior school at the time)."— ibid., p. 756

• "Hermann Günther Grassmann (1809-77) [discoverer of vector algebra], ... showed no talent for mathematics as a youth and ... had no university education in mathematics..." — ibid., p. 782

• George Green (1793-1841), English mathematician who is known to calculus students for the theorem that carries his name, was self-taught. — ibid., p. 683.

"He worked full-time in his father's bakery from the age of nine and taught himself mathematics from library books. In 1828 he published privately *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, but only 100 copies were printed and most of those went to his friends. This pamphlet contained a theorem that is equivalent to what we know as Green's Theorem, but it didn't become widely known at that time. Finally, at age 40, Green entered Cambridge University as an undergraduate but died four years after graduation. In 1846 William Thompson (Lord Kelvin) located a copy of the essay, realized its significance, and had it reprinted. Green was the first person to try to formulate a mathematical theory of electricity and magnetism. His work was the basis for the subsequent electromagnetic theories of Thomson, Stokes, Rayleigh, and Maxwell." — Stewart, James, *Calculus: Fourth Edition*, Brooks/Cole Publishing Company, N.Y., 1999, p. 1103.

• George Boole (1815-1864), discoverer of boolean algebra, which governs the logical operations of modern computers, was largely self-taught. — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p. 1189. In fact he "never attended secondary school, college, or university." — *The Princeton Companion to Mathematics*, ed. Gowers, Timothy, Princeton University Press, Princeton, N.J., 2008, p. 769.

• Ramanujan (1887 - 1920), the extraordinary Indian mathematical genius, was self-taught.

• Greg Chaitin, discoverer of algorithmic information theory, is almost entirely self-taught as far as I know. In any case, he did not even have a bachelor's degree — in any subject — at the time he began publishing his first papers.

In passing, I will mention that Ernest Rutherford (1871-1937), one of the great physicists of the early 20th century, was largely self-taught. — Dyson, Freeman, *Infinite in All Directions*, Harper & Row, Publishers, N.Y., 1988, p. 41.

Myth: Mathematical talent can only be discovered through difficult, highly competitive exams

There are two counterarguments to this myth. The first is that some of the best mathematicians in the past were entirely, or largely, self-taught, and thus became mathematicians without having first excelled in a series of difficult tests. (See the previous myth.) The second is that some of the best of the best have openly criticized systems that attempt to select mathematical talent solely on the basis of competitive exams. For example, one of the great mathematical logicians of the 20th century wrote:

"I was encouraged in my transition to philosophy by a certain disgust with mathematics, resulting from too much concentration and too much absorption in the sort of skill that is needed in examinations. The attempt to acquire examination technique had led me to think of mathematics as consisting of artful dodges and ingenious devices and as altogether too much like a cross-word puzzle." — Russell, Bertrand, *The Basic Writings of Bertrand Russell*, Simon and Schuster, N.Y., 1961, pp. 57-58.

These words may remind us of Einstein's well-known remark:

"One had to cram all this stuff into one's mind for the examinations, whether one liked it or not. This coercion had such a deterring effect on me that, after I had passed the final examination, I found the consideration of any scientific problems distasteful to me for an entire year."

G. H. Hardy, one of the great mathematicians of the 20th century, sought to change the Tripos system of difficult mathematics exams at Cambridge University.

"He was a secretary of the committee [at Cambridge University] which forced the abolition of the order of merit in the Mathematical Tripos through a reluctant Senate in 1910, and many years later he fought hard for the abolition (not reform!) of the Mathematical Tripos itself, which he considered to be harmful to mathematics in the United Kingdom." — *The Princeton Companion to Mathematics*, ed. Gowers, Timothy, Princeton University Press, Princeton, N.J., 2008, p. 798.

There have been other top-rank mathematicians (e.g., J. E. Littlewood, who ranks with Hardy) who did not think competitive exams should have the prestige that is usually accorded them — or, at least, who did not think that the huge effort required to perform well on these exams was justified.

The words of David Brooks are appropriate here:

"Think about the traits that creative people possess. Creative people don't follow the crowds; they seek out the blank spots on the map... Instead of being fastest around the tracks everybody knows, creative people move adaptively through wildernesses nobody knows. [One thinks of Galois here.]

"Now think about the competitive environment that confronts the most fortunate people today and how it undermines those mind-sets.

"First, students have to jump through ever-more demanding, pre-assigned academic hoops. Instead of developing a passion for one subject, they're rewarded for becoming professional students, getting great grades across all subjects, regardless of their intrinsic interests...

"Then they move into a ranking system in which the most competitive college, program, and employment opportunity is deemed to be the best..." — Brooks, David, "The Creative Monolpoly", in *The New York Times*, Apr. 24, 2012, p. A21.

Further observations can be found in "Against Competitive Mathematics Exams" in chapter 2, "Mathematics in the University".

Myth: Every top-rank mathematician has held a tenured position on the faculty of a major university.

There was a time when most mathematicians were what is unthinkable today, namely, *amateurs*, pursuing their interests in the subject *without being college professors*."It was not until the nineteenth century that the universities became the major centers of scientific thought, and many contributors to the scientific revolution (such as Kepler and Descartes) had no university affiliations." — *The Columbia History of the World*, ed. Garraty, John A., and Gay, Peter, Harper & Row, Publishers, N.Y., 1972, p. 691.

• "Until the latter part of the seventeenth century, mathematics had sometimes bestowed high reputation upon its adepts but had seldom provided them with the means to social advancement and honorable employment. VIÈTE had made his living as a lawyer, FERMAT as a magistrate; even in Fermat's days, endowed chairs for mathematics were few and far between...CARDANO had been active as a physician; BOMBELLI was an engineer, and so was Simon STEVIN in the Netherlands. NAPIER, the inventor of logarithms, was a Scottish laird, living in his castle of Merchiston after coming back from the travels of his early youth. Neighboring disciplines did not fare better. COPERNICUS was an ecclesiastical dignitary...KEPLER plied his trade as an astrologer and maker of horoscopes...Among Fermat's scientific correspondents, few held professorial rank...his talented younger friend and collaborator William BROUNCKER, ..., was a nobleman whose career as commissioner of the Navy...[is] abundantly documented in Pepys's diary...René de SLUSE, a mathematician in high esteem among his contemporaries...was a canon at Liège. DESCARTES, as he tells us, felt himself, by the grace of God, above the need of gainful employment; so were his friends Constantin HUYGENS and Constantin's son, the great Christian HUYGENS. LEIBNIZ was in the employ of the Hanoverian court; all his life he preserved his love for mathematics, but his friends marveled sometimes that his occupations left him enough leisure to cultivate them." — Weil, Andre, *Number Theory: An approach through history*, Birkhäuser, Boston, 1984, pp. 159-160. (Names capitalized as in text)

• Blaise Pascal (1623 - 1662) , geometer, inventor of a mechanical calculating machine, co-discoverer, with Fermat, of the theory of probability, never taught at a university. — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p. 398.

• Sophie Germain (1776-1831), probably the greatest female mathematician prior to the 20th century, taught herself mathematics from books in her father's library. Later, she communicated by letter with famous mathematicians, among them Legendre and Gauss. She did pioneering work in elasticity theory and Fermat's Last Theorem. Yet she never held a university position.

• Joseph Fourier (1768-1830), best known to mathematics students as the discoverer of Fourier series, in his late thirties did groundbreaking work on heat diffusion in his spare time from employment as Prefect of the Department of Isère, based at Grenoble, where he oversaw road construction and other projects, and from his hobby of Egyptology. — Grattan-Guinness, Ivor, *The Rainbow of Mathematics*, W. W. Norton & Company, N.Y., 2000, p. 454, and Wikipedia, "Joseph Fourier", 5/20/13

• Evariste Galois (1811-32), founder of what is now known as Galois theory, one of the most important branches of modern algebra, died at the age of 20. He never held a university position.

• "Oliver Heaviside (1850-1925) [one of the pioneers of vector algebra] was in the early part of his scientific career a telegraph and telephone engineer. He retired to country life in 1874 and devoted himself to writing, principally on the subject of electricity and magnetism." — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p. 787.

• Karl Weierstrass (1815-1897) [who was instrumental in establishing a rigorous logical basis for analysis] "entered the University of Bonn to study law. After four years in this effort he turned to mathematics in 1838 but did not complete the doctoral work. Instead he secured a state license to become a gymnasium (high school) teacher and from 1841 to 1854 he taught youngsters such subjects as writing and gymnastics. During these years he had no contact with the mathematical world, though he worked hard in mathematical research." — ibid., p. 643

• “Richard Dedekind (1831-1916), [another mathematician who was instrumental in establishing a rigorous logical basis for analysis] [was] a pupil of Gauss, [but] spent fifty years of his life as a teacher at a technical high school in Germany...” — ibid., p. 820.

• Arthur Cayley (1821-1895), “a prolific writer and creator in various subjects, notably the analytic geometer of n dimensions, determinant theory, linear transformations...and matrix theory” spent fifteen years in the legal profession. “During this period he managed to devote considerable time to mathematics and published close to 200 papers.” — ibid., p. 805.

• Greg Chaitin, discoverer of algorithmic information theory, which (among other things) gives us the most profound theory to date as to the nature of random numbers, was a teen-ager working for IBM when he published his first paper on the subject. He did not, as far as I know, even have an undergraduate degree at the time.

• Benoit Mandelbrot, discoverer of fractal geometry, was an IBM researcher when he published his first papers on the subject — a subject that many academic mathematicians seem to disdain, despite (or perhaps because of!) its wealth of scientific applications. Among the reasons I have heard and read mathematicians give are: that the subject is “merely graphical” and that the proofs are “too easy” (!)

• In passing, I should mention that none of the founders of thermodynamics (19th century) held academic positions at the time of their discoveries. Sadi Carnot was an engineer. Neither Mayer nor Joule ever held an official position but were private experimenters all their lives. Helmholtz did not obtain a chair in physics until twenty-four years after his first discoveries in thermodynamics.

• I should also mention Charles Darwin, discoverer of the theory of evolution, who never held an academic position but, being financially independent, worked at home after returning from the voyage of the *Beagle*.

And again, I must mention a famous physicist:

• “Einstein...started work at the [Swiss] Patent Office on 23 June 1902 [at the age of 23] as a probationary technical expert, third class. He now had a steady job and quickly mastered the undemanding work; he had time to spare for his own researches. And so, in the unlikely surroundings of a minor government office, his genius was able to mature. His seven years at the Patent Office were among the most creative of his life." — Hollingdale, Stuart, *Makers of Mathematics*, Penguin Books, N.Y., 1989, p. 373.

It was in 1905, while at the Patent Office, that Einstein published the three revolutionary papers that established his reputation: "The first paper...deals with the energy of light and explains the photoelectric effect, while the second gives a molecular explanation of Brownian motion...the third paper, 'On the electrodynamics of moving bodies', gives the first presentation of what we now know as the Special Theory of Relativity." — ibid., p. 374.

• "Pierre Ramond was denied tenure at Yale in 1976, a few years after having solved several of string theory's central problems. It turns out that inventing a way to put fermions into string theory, discovering supersymmetry, and removing the tachyon — all in one blow — was not enough of an achievement to convince his colleagues that he deserved a professorship at an Ivy League insttituion.

"John Schwartz, meanwhile, had been denied tenure at Princeton, 1972, in spite of *his* fundamental contributions to string theory. He then moved to Caltech, where he was a research associate for the next twelve years, supported by temporary funds that had to be renewed periodically. He didn't have to teach if he didn't want to — but neither did he have tenure. He discovered the first good idea about how gravity and the other forces could be unified, but apparently Caltech remained unconvinced that he belonged on their permanent faculty." — Smolin, Lee, *The Trouble With Physics*, Houghton Mifflin Company, N.Y., N.Y., 2006, p. 111.

And yet, I know of recent PhDs who, after years of hard work, upon being turned down for a tenure track at Princeton or Harvard, have taken this as a sign from on high that they were not fit to spend their lives in mathematics, or at least not in doing mathematical research. An appalling, a dreadful example of the hold that the myths we are discussing have on the minds of young mathematicians.

Myth: Whatever is published in refereed journals is probably important.

A detailed refutation of this myth by a long-time inside observer is given in Morris Kline's *Why the Professor Can't Teach*, along with a history of how we got to the present state of affairs, in which, to put it bluntly, mediocre (or worse) research is used to justify incompetent teaching.

If we read the history of mathematics, we find that there was a time when publish-or-perish did not rule the lives of mathematicians. For example, Newton only *reluctantly* agreed to the publication of his masterpiece, *Philosophiae Naturalis Principia Mathematica*, considered one of the most important scientific works ever written. The almost exclusive emphasis on research and on publication is filling the archives with mediocre if not worthless papers and giving students, who, after all, are paying to be taught, a very poor return on their money.

I once heard a brilliant young computer science graduate student who was soon to have his pick of tenure tracks in some of the country's best computer science departments, remark, "Ninety-nine percent of computer science papers are shit!" I may disagree a little with the precise percentage, but I think there is no doubt that an extraordinary number of mediocre and trivial papers are published in the computer science journals. A similar trend seems to be occuring in mathematics. The reason in both cases is well-known. Publications — the more the better — are the only means of advancement. Publications are the coin of the realm. No one can blame academic mathematicians for playing the game, since their future depends on it. But one can try to make sure that young mathematicians don't take their place in this culture without knowing the truth.

Myth: Every competent mathematican reads the literature on a problem he or she is trying to solve.

The notion that you will always want to read as much as possible on a problem before or while working on a solution has the odor of the plodding academic grinding out of papers that will in turn be read by other plodders — other obedient members of the Club. Many journals all but force an author to supply a bibliography, preferably a long one, to prove that he or she has not dared to think of something that no one else has thought of — to prove that he or she is a Team Player. The rule should be, If you don't have any good ideas at the start, then read the literature until you do get a good idea. Then don't waste time reading the literature for any purpose except to find out things you haven't time to figure out for yourself.

But by no means is it the case that competent mathematicians, much less the greats, always read the literature.

"It is one of the tasks of the historian of mathematics to discover the persons and the ideas which influenced a great mathematician and prepared the way for his own researchers. This task is peculiarly difficult in the case of Poincaré [(1854-1912)], especially in relation to his vast contributions to topology. The fact is that although Poincaré does mention Riemann and Betti by name, it is almost certain, according to Freudenthal ... speaking at the centennial celebrations of the birth of Poincaré, that the celebrated mathematician had never read Riemann and the authors who continued his theory." — Temple, George, *100 Years of Mathematics*, Springer-Verlag, N.Y., 1981, pp. 125-126.

"Poincaré spent little, if any, time in searching the literature of mathematics." — ibid., p. 120.

"[Einstein's] efforts to keep abreast with the scientific literature were impaired by the fact that the technical library [in Berne, Switzerland] generally was closed when he got off work." — Ferris, Timothy, *Coming of Age in the Milky Way*, Anchor Books, N.Y., 1988, p. 183.

"The 1905 paper [of Einstein] that first enunciated the theory [of special relativity] resembles the work of a crank; it contains no citations whatever from the scientific literature, and mentions the aid of but one individual, [Michele Angelo] Besso, who was not a scientist. (At the time, Einstein knew no scientists.)" — Ferris, Timothy, *Coming of Age in the Milky Way*, Anchor Books, N.Y., 1988, p. 191.

Myth: No top-rank mathematician publishes his own work.

• Carl Friedrich Gauss (1777-1855), generally considered the greatest of all mathematicians, self-published the masterpiece of his youth, *Disquisitiones Arithmeticae*.

"Up to the nineteenth century the theory of numbers was a series of isolated though often brilliant results. A new era began with Gauss's *Disquisitiones Arithmeticae* which he composed at the age of twenty. This great work had been sent to the French Academy in 1800 and was rejected but Gauss published it on his own." — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p. 813.

• George Green (1793-1841), English mathematician who is known to calculus students for the theorem that carries his name, "worked full-time in his father's bakery from the age of nine and taught himself mathematics from library books. In 1828 he published privately *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, but only 100 copies were printed and most of those went to his friends. This pamphlet contained a theorem that is equivalent to what we know as Green's Theorem, but it didn't become widely known at that time. Finally, at age 40, Green entered Cambridge University as an undergraduate but died four years after graduation. In 1846 William Thompson (Lord Kelvin) located a copy of the essay, realized its significance, and had it reprinted. Green was the first person to try to formulate a mathematical theory of electricity and magnetism. His work was the basis for the subsequent electromagnetic theories of Thomson, Stokes, Rayleigh, and Maxwell." — Stewart, James, *Calculus: Fourth Edition*, Brooks/Cole Publishing Company, N.Y., 1999, p. 1103.

• In passing, I should point out that there have been top-rank *scientists* who published their own work. For example, Peter Mitchell (1920-1992), who "received the Nobel Prize in Chemistry in 1978 for his development of the chemiosmotic theory of cellular energy transfer" and "who conducted his work at the Glynn Research Institute, a private research laboratory financed by a family inheritance, and typically disseminated his results in self-published white papers (known as the Grey Books) rather than in peer-reviewed scientific journals." — Wolfe, Audra J., "Innovators and Iconoclasts", review of Harman, Oren and Dietrich, Michael R., *Rebels, Mavericks, and Heretics in Biology*, Yale University Press, 2008.

Myth: No top-rank mathematician ever publishes incomplete or faulty proofs, or publishes conjectures that turn out to be false.

• "Up to about 1650 no one believed that the length of a curve could equal exactly the length of a line. In fact, in the second book of *La Géométrie*, Descartes says the relation between curved lines and straight lines is not nor ever can be known." — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p. 354.

• "In his *Algebra* [1685], John Wallis [1616-1703] regarded a higher-dimensional space as a 'monster in nature, less possible than a chimera or a centaure.' He says 'Length, Breadth, and Thickness take up the whole of space. Nor can Fansie imagine how there should be a Fourth Local Dimension beyond these Three.' Cardan, Descartes, Pascal, and Leibniz had also considered the possibility of a fourth dimension and rejected it as absurd." — ibid., p. 1028

• "Fermat did make some mistakes. He believed that he had found a solution to the long-standing problem of producing a formula that would yield primes for values of the variable n. Now it is not hard to show that $2^m + 1$ cannot be a prime unless m is a power of 2. In many letters dating from 1640 on Fermat asserted the converse — namely, that

$$2^{2^n} + 1$$

represents a series of primes — though he admitted that he could not prove this assertion. Later he doubted its correctness. Thus far only the five primes 3, 5, 17, 257, 65,537 yielded by the formula are known. — ibid., p. 277.

• "To answer [Berkeley's criticisms of the lack of logical rigor in the calculus], Colin Maclaurin (1698-1746), in his *Treatise of Fluxions* (1742), attempted to establish [this rigor]. It was a commendable effort but not correct." — ibid., p. 428.

• In 1759 Lagrange published a paper on the solution to the wave equation in one spatial dimension which, he claimed, put Euler's theory on the problem

> "beyond all doubt and established on direct and clear principles which rest in no way on the law of continuity [analyticity] which Mr. d'Alembert requires; this, moreover, is how it can happen that the same formula that has served to support and proved the theory of Mr. Bernoulli on the mixture of isochronous vibrations when the number of bodies is ... finite shows us its insufficiency ... when the number of these bodies becomes infinite. In fact, the change that this formula undergoes in passing from one case to the other is such that the simple motions which made up the absolute motions of the whole system annul each other for the most part, and those which remain are so disfigured and altered as to become absolutely unrecognizable. It is truly annoying that so ingenious a theory ... is shown false in the principal case, to which all the small reciprocal motions occurring in nature may be related."

About this passage, Kline says, "All of this is almost total nonsense." — ibid., pp. 511-512.

• "Papers on the vibrating string and the hanging chain...were published by many other men [besides d'Alembert and Euler] up to the end of the [eighteenth] century. The authors continued to disagree, correct each other, and make all sorts of errors in doing so, including contradicting what they themselves had previously said and even proven. They made assertions, contentions, and rebuttals on the basis of loose arguments and often just personal predilections and convictions. Their references to papers to prove their contentions did not prove what they claimed." — ibid., p. 518.

• Euler was wrong about several fundamental concepts in the calculus. "In his *Institutiones* of 1755 he argued, 'There is no doubt that every quantity can be diminshed to such an extent that it vanishes completely and disappears. But an infinitely small quantity is nothing other than a vanishing quantity and therefore the thing itself equals 0. It is in harmony also with that definition of infinitely small things, by which the things are said to be less than any assignable quantity; it certainly would have to be nothing; for unless it is equal to 0, an equal quantity can be assigned to it, which is contrary to the hypothesis.'

"Since Euler banished differentials he had to explain how dy/dx, which was 0/0 for him, could equal a definite number. He does this as follows: Since for any number n, $n \bullet 0 = 0$, then $n = 0/0$. The derivative is just a convenient way of determining 0/0." — ibid., p. 429.

• "Euler made mistakes with complex numbers. In [his] *Algebra* he writes

$$\sqrt{-1} \bullet \sqrt{-4} = \sqrt{4} = 2$$

because

$$\sqrt{a} \bullet \sqrt{b} = \sqrt{ab}$$

[instead of the correct

$$\sqrt{-1} \bullet \sqrt{-4} = i \bullet i\sqrt{4} = -1 \bullet 2 = -2\,]$$

]"

— ibid., p. 594.

• Euler's proof of the case $n = 3$ of Fermat's Last Theorem, published in 1770, contained a large gap. (However, since it could be filled by one of Euler's earlier lemmas, he is generally given credit for the first proof of this case.) — Wikipedia, "Fermat's Last Theorem", Aug. 19, 2010.

• "The kernel of the problem of factoring a real polynomial into linear and quadratic factors with real coefficients was to prove that every such polynomial had at least one real or complex root. The proof of this fact, called the fundamental theorem of algebra, became a major goal.

"Proofs offered by d'Alembert and Euler were incomplete." — ibid., p. 598.

• "Euler once conjectured that an *n*th power cannot be expressed as the sum of fewer than *n* smaller *n*th powers. Today...the simplest known counterexample to Euler's conjecture is ... $422{,}481^4 = 95{,}800^4 + 217{,}519^4 + 414{,}560^4$. — Pickover, Clifford A., *A Passion for Mathematics*, John Wiley & Sons, Inc., Hoboken, N.J., 2005, p. 105.

• "In 1785 Legendre [1752-1833] announced the law [of quadratic reciprocity] independently... His proof was incomplete. In his *Théorie de nombres* he again stated the law and gave another proof. However, this one, too, was incomplete, because he assumed that there are an infinite number of primes in certain arithmetic expressions." — ibid., p. 612.

"...in article 151 of his *Disquisitiones* Gauss ... refers to ... work of Euler including [a] paper in the *Opuscula* and to Legendre's work of 1785. Of these papers Gauss says rightly that the proofs were incomplete." — ibid., p. 815.

• Both Euler and Legendre conjectured that the set $\{a + kb \mid k \geq 0$, a and b relatively prime$\}$ contains an infinite number of primes. "In 1808 Legendre gave a proof that was faulty."[1]

• In most of [Gauss's] work he called a series convergent if the terms from a certain one on decrease to zero. But in his 1812 paper ["General Investigations of Infinite Series"] he noted that this is not the correct concept." — ibid., p. 962.

• "[The Jordan Curve Theorem] was first stated by Camille Jordan (1838-1922) in his famous *Cours d'Analyse*, from which a whole generation of mathematicians learned the modern concept of rigor in mathematics. Strangely enough, the proof given by Jordan was neither short nor simple, and the surprise was even greater when it turned out that Jordan's proof was invalid, and that considerable effort was necessary to fill the gaps in his reasoning." — Courant, Richard, and Robbins, Herbert, "Topology", in *The World of Mathematics*, ed. Newman, James R., Simon and Schuster, N.Y., Vol. 1, p. 589.

• E. E. Kummer, A. L. Cauchy and G. Lamé (leading mathematicians of the early 19th century) all wrongly assumed that the integers of the algebraic number field $k(\rho)$, where $\rho = e^{2\pi i/p}$, p an odd prime, had unique factorization. (If the integers did possess

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1. Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p 830.

this property, then a straightforward proof of Fermat's Last Theorem would have been possible.) — Shanks, Daniel, *Solved and Unsolved Problems in Number Theory*, Chelsea Publishing Company, N.Y., 1985, p. 152.

• Both Gauss and Riemann are said to have conjectured that

$$G(N) > 0$$

where:

$$G(N) = Li(N) - \pi(N)$$

$$Li(N) = \int_2^x dx / \log x + 1.045$$

and $\pi(N)$ is the number of positive primes $\leq N$.

But Littlewood disproved the conjecture. — Shanks, Daniel, *Solved and Unsolved Problems in Number Theory*, Chelsea Publishing Company, N.Y., 1985, pp. 242-243.

• "For thirty-five years [after the 1830s] both of Jacobi's conclusions [about important points in the calculus of variations] were accepted as correct. During this period, the papers on the subject were imprecise in statement and dubious in proof; problems were not sharply formulated and all sorts of errors were made." — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p. 747.

• In 1878, Cantor published "Ein Beitrag zur Mannigfaltigkeitslelhre ("A contribution to the theory of aggregates"), in which he proved the invariance of dimension. But the argument was partly faulty. The theorem was first correctly proved by Brouwer in 1911. — *The Princeton Companion to Mathematics*, ed. Gowers, Timothy, Princeton University Press, Princeton, N.J., 2008, p. 779.

• In 1887 a prize was announced for a solution to the *n*-body problem, that is, the problem of mathematically describing the orbits of three or more bodies in the solar system. Poincaré submitted a paper containing a revolutionary new geometric approach to the problem, later called "Poincaré maps", and was judged the winner.

But after a few preliminary copies of his paper had been published and distributed, a gap in his solution was discovered. He was unable to repair it (although he was still awarded the prize)[1].

• "In [an] 1895 paper, Poincaré introduced a basic theorem, known as the duality theorem. It concerns the Betti numbers of a closed manifold... The theorem states that in a closed orientable n-dimensional manifold the Betti number of dimension p equals the Betti number of dimension $n - p$. His proof, however, was not complete. — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, p. 1174.

Kurt Hensel (1861-1941) "gave an easy and elegant p-adic proof that e is transcendental...Unfortunately, it contained a subtle error." — Gouvêa, Fernando Q., "Local and Global in Number Theory" in *The Princeton Companion to Mathematics*, ed. Gowers, Timothy, Princeton University Press, Princeton, N.J., 2008, p. 243.

• David Hilbert (1862-1943), certainly one of the greatest of 20th century mathematicians, often stated his belief that every mathematical truth could be proved. But this belief was proved wrong by the famous incompleteness theorem of Kurt Gõdel (1906-1978) in the early 1930s.

Myth: No top-rank mathematician is interested in teaching or in writing textbooks.

This one is really amazing, considering that the most famous, most long-lived textbook in the history of mathematics, namely, Euclid's *Elements*, was written by one of the leading mathematicians of ancient Greece. Other counterexamples:

• The man who is ranked among the top four or five mathematicians who ever lived, namely, Leonhard Euler (1707 - 1783), "wrote texts on mechanics, algebra, mathematical analysis, analytic and differential geometry, and the calculus of variations that were standard works for a hundred years and more afterward."

"His *Vollständige Anleitung zur Algebra* (Complete Introduction to Algebra)...is the best algebra text of the eighteenth century." — Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, N.Y., 1972, pp. 402, 594.

1. Only in 1912 was a complete solution (for the case $n = 3$) published (by another mathematician). A complete solution for all n greater than 3 was not discovered until the 1990s.

• Gaspard Monge (1746-1818), analyst, "helped to organize the Ecole Polytechnique and as a professor there founded a school of geometers. He was a great teacher..." — ibid., p. 565.

• William R. Hamilton (1805-65), considered the greatest English mathematician after Newton, was noted as a fine teacher. — ibid., p. 778.

And in physics:

• Nobel Prize winner Richard Feynman's *Lectures on Physics*, published in the sixties, is now considered by many to be the best presentation of undergraduate physics ever written.

• One of Feynman's teachers, John A. Wheeler (1911 -), made no bones about the importance of teaching: " [a] favorite Wheelerism is 'one can only learn by teaching.'...Technically [because of age], Wheeler can teach no longer. 'If you know of a school that lets its professors teach after they reach 70,' he says, 'let me know.' " — Horgan, John, *Scientific American*, June 1991, p. 38.

I will repeat in part the words of Carl Sagan quoted above: "The status of the teachers in the Hutchins curriculum [at the University of Chicago] had almost nothing to do with their research; perversely — unlike the American university standard of today — teachers were valued for their teaching, their ability to inform and inspire the next generation." — Sagan, Carl, *The Demon-Haunted World*, Random House, N.Y., 1995, pp. *xiv-xv*. (Italics mine)

For a detailed history of how the current contempt for teaching evolved, see Morris Kline's *Why the Professor Can't Teach*.

Myth: The way that mathematical knowledge is currently presented — in classrooms, textbooks, and journals — is the best way it can be presented.

This myth is, I believe, dispelled in the chapter, "Proofs", in the section, "How to Write Up Proofs in an Environment".

Myth: In general, the length of time that is required to earn a PhD in mathematics is the proper length of time.

Ask yourself: Suppose there were no minimum time and no minimum set of courses that were required for a PhD in mathematics at a certain university. A person with the equivalent of a bachelor's degree in mathematics would ask to work on a research problem he or she was interested in. He or she would discuss with the appropriate department authority what would have to be accomplished in order for the person to be awarded a PhD. If an agreement were reached, then *the rest would be up to the PhD candidate*: what courses (if any) to take, what papers to read. He or she would be allowed a specified number of minutes each week to ask questions of knowledgeable faculty members. If the candidate completed the required work in a year, that would be fine: he or she would be awarded the degree. If the candidate needed three years, or five years, or ... then that too would be fine.

Of course, such a degree policy would almost certainly lower the prestige of the mathematics department. Just as students and their parents believe that the higher the tuition at a university, the better the education the students will receive, so do students and parents and, more important, the heads of mathematics departments who will be the future potential employers of the candidate, believe that the longer and more difficult the PhD process, the greater the value of the PhD. Efficiency is the last thing anyone wants to think about! And yet...

Myth: No top-rank mathematician is interested in writing popularizations of his, or others', work.

Henri Poincaré (1854-1912), certainly one of the greatest mathematicians of modern times, was famous for his books for the educated layman. "The wider public devoured his best-selling books on the philosophy of conventionalism, science and values, and his defense of 'science for science'." — Galison, Peter, *Einstein's Clocks, Poincaré's Maps*, W. W. Norton & Co., N.Y., 2003, p. 32.

Bertrand Russell, one of the leading mathematical logicians of the 20th century, wrote many books for the educated layman, including his celebrated *The ABC of Relativity.*

And, in passing, I must mention Einstein, who wrote one of the best popularizations of his theories of relativity, namely, his *Relativity*[1], first published in 1920.

Myth: No top-rank mathematician is interested in the history of his subject.

• "Andre Weil, who is considered one of the leading mathematicians of [the 20th] century... is known for his work in number theory and algebraic geometry. A lifelong interest in the history of mathematics finds its culmination in this publication..." — cover of Weil, Andre, *Number Theory: An approach through history*, Birkhäuser, Boston, 1984.

• Einstein and Leopold Infeld wrote *The Evolution of Physics*, certainly a historical work, and even though it is generally believed that Infeld did most or all of the writing, it is hard to believe that Einstein would have allowed his name to be listed as co-author of a book whose subject he considered unimportant.

"Researchers who place high value on their work should be obliged to read a somewhat detailed history of mathematics. They would be amazed to find out how much that was regarded as vital and central in the past has been dropped so completely that even the names of those activities or branches are no longer known." — Kline, Morris, *Why the Professor Can't Teach*, St. Martin's Press, N.Y., 1977, p. 57.

Perhaps one reason that the history of mathematics is so widely regarded by the mathematics Establishment as being of no real interest to a promising mathematician, is that the Establishment fears it as a threat to the current culture.

But the truth is, if you don't know the history of your subject, then you are in a real sense uneducated in your subject — you are a mere technician, and you will be easily impressed by the latest fads, by arcane virtuoso performances, by the prejudices of the Establishment in your field.

Having said that, I must also say that the vast majority of histories of mathematics are little more than dull recitations of who proved what when. In fact, the only really good history of mathematics I have ever come across is the one that I have quoted from so often in this chapter, namely, Morris Kline's *Mathematical Thought from Ancient to Modern Times*[2]. If you plan to make a career in mathematics, you need to

1. Einstein, Albert, *Relativity*, tr. Robert W. Lawson, Prometheus Books, Amherst, N.Y., 1995.
2. Oxford University Press, N.Y., 1972.

read this book. Not only will it dispel some of the mythology that the current mathematics culture wants you to believe, but it will also advance your understanding of mathematics as few books will.

Myth: Mathematicians lose their talent/ability with age.

The power of this myth — which seems to be an outgrowth of modern academic culture — was amply portrayed in David Auburn's play, *Proof* (first produced in 2000), in which a significant portion of the dialogue is concerned with whether one of the characters has lost his mathematical powers as a result of having passed age 25. The paralyzing grip of this nonsense on ambitious young mathematicians is hard to overestimate. Let us look at the historical record.

• By all accounts, Archimedes was at the peak of his powers when he died at age 75.

• "...neither old age nor blindness could induce [Euler] to take a well-deserved rest. He had assistants, one of whom was his own son; others were sent to him from Basel...Hundreds of memoirs were written during [the last decade of his life]; enough, as Euler had predicted, to fill up the academy publications for many years to come. He died suddenly on 18 September 1783 [at the age of 76] having preserved excellent general health and his full mental powers until that very day." — Weil, Andre, *Number Theory: An approach through history*, Birkhäuser, Boston, 1984, p. 169.

• "Gauss enjoyed good health throughout his life and preserved his mental vigour and originality to the end...he died peacefully...in his seventy-eighth year." — Hollingdale, Stuart, *Makers of Mathematics*, Penguin Books, N.Y., 1989, p. 318.

• Paul Erdös (1913 - 1996), one of the great mathematicians of the 20th century, retained his powers until his death at 83.

• We might also mention Andrew Wiles, who was in his forties when he proved the part of a conjecture that implied the truth of Fermat's Last Theorem.

What Keeps the Myths Alive?

The reason that the myths persist is understandable. Mathematics is not only the language of science, it is also the one subject that can claim to be in possession of eternal beauty and eternal truth — the dream of every priesthood that has ever existed! Therefore it is understandable that those in this particular priesthood should assume that anything that they agree is right among themselves, must be beyond criticism.

But there is a further reason for the persistence of the myths, and that is that nowadays, the only way you can be a mathematician — the only way you can have any reasonable hope that your discoveries will be published in professional journals — is by having a tenured position in an academic institution. I am speaking here of so-called "pure" mathematicians. It is true that there are journals in which mathematicians employed in industry can reasonably hope to have their results in applied mathematics published. But in pure mathematics, if you are not a tenured professor — if you are not a member of the Club — you are automatically assumed to be either incompetent or a crackpot. You do not get published without superhuman efforts stretching over years.

The more exclusive the Club, the better for its members. I have known mathematicians who clearly loved the subject because it was incomprehensible to the vast majority of mankind, and who were proud of the fact that their speciality was incomprehensible even to other mathematicians. Mathematical knowledge is power, is a kind of magic, in the sense that, if you know the trick, the method, you can produce results that those who do not know it, cannot.

Another reason that the myths persist is that the current mathematics culture is largely intolerant of criticism.

"The pace of innovation in curriculum methods and teaching methods is positively medieval. Any proposal for change has to be approved by the faculty, and in general most professors see nothing wrong with how they have been teaching for decades." — Smolin, Lee, *The Trouble With Physics*, Houghton Mifflin Company, N.Y., 2006, p. 264.

I have heard graduate students speak resignedly of the "Mathematical Mafia". A computer scientist who had published several papers on mathematical subjects told me that any book that was critical of the current mathematics culture would be ignored. The only permissible criticism was to demonstrate an alternative that worked. Thus, e.g., the only way to criticize, say, current mathematics teaching methods was by getting a PhD, then a tenured position, then becoming dean of a math dept., then instituting the new teaching methods, and developing sufficient data to prove that the methods were better than the old. An astounding example of the degree to which the current culture has stifled all challenges to the status quo.

On the other hand, no mathematical specialty would tolerate such conservatism within the specialty itself. It is hard to imagine a specialty that would say, in effect, "No significant advances — certainly no revolutionary advances — are possible or necessary in this specialty. Things are as they should be."

When I consider the stifling atmosphere of the present mathematical culture, and its truly shameful attitude toward applied mathematics — a branch of the subject that captured the interest of the greatest mathematicians up to the start of the 20th century — I sometimes wonder if mathematics will be able to survive the prestige-obsessed bureaucrats who constitute the vast majority of the current generation of mathematicians.

A Hard Look at the Mathematics Culture in the University

Even though I strongly recommend *against* your becoming a reformer if you manage to get a position in a mathematics department (see final section in this chapter) I think it is important for your own well-being and creativity that you have a clear idea of the Culture you will be getting into. The myths listed above are certainly a good start. But let me list some other characteristics of this Culture. Details on several are given in chapter 2, "Mathematics in the University".

• Emphasis on competitive exams as a good means of separating out future Winners from future Losers; (See what two of the greatest 20th-century English mathematicians thought of these exams, and what they did to cut back on them, in "Against Competitive Mathematics Exams" in chapter 2, "Mathematics in the University".)[1]

• Obsession with credentials (no one without a PhD in mathematics can possibly be a mathematician[2]);

1. At present, I do not know of a good substitute for these exams. The chances of a very bright high school student coming up with an original scientific idea for presentation in one of the annual Science Fair competitions, are much greater than the chances of a very bright high school student coming up with a solution to an unsolved math problem. Competitions involving the writing of computer programs to solve *classes* of problems, is one possibility.for math students.

2. A belief that reveals an embarrassing ignorance of mathematics history, since some of the best of the best did not have a PhD in mathematics and did not work in the university: Descartes, Pascal, Fermat, Leibniz, and Galois, to name only the best known.

• Unquestioning faith in publish-or-perish, and in Citation Indexes that record the number of times a given paper is cited by later papers.

"...this measure is almost childish. Very good papers are often soon superseded by ones that advance the subject still further. Even when the advances are minor, the later papers will surely be the ones cited. A fad will be cited many times over a period of years. Many young researchers cite their professors, even at the expense of the true creator, in order to curry favor...." — Kline, Morris, *Why the Professor Can't Teach*, St. Martin's Press, N.Y., 1977, p. 65.

• Indifference to new approaches to the presentation of mathematics, like the one set forth in this book, and instead the endless fussing over trivialities (see almost any issue of AMS[1] *Notices*) as ways of improving mathematics education, all of them based on the assumption that education requires above all a teacher standing in front of a classroom, teaching each course in strict logical order (if the student is to understand the content of page 2 in the textook, the teacher must first have taught him the content of page 1, if the student is to understand the content of page 3 in the textook, the teacher must first have taught him the content of page 2, etc.).

• Indifference to a much clearer, more efficient format for proofs, namely, structured proof, as described in chapter 5, "Proofs", and instead the stubborn adherence to a format that is some 2,300 years old, and consists of a succession of paragraphs that requires the reader to hold everything in his mind as he proceeds.

• Shameful failure to make the writing-down of *procedures* for solving *classes* of problems the primary emphasis in undergraduate education, and instead encouraging students to believe that only the endless grinding through homework problems will enable them to master the material.

• Shameful failure on the part of textbook authors to provide *complete* indexes, including complete indexes of symbols, in each textbook. This failure is part of the naive, schoolmasterly belief that to learn is to memorize. If the student can't remember a definition he or she learned in class, well, then perhaps the student is not mathematical material.

1. American Mathematical Society

• Shameful failure on the part of textbook authors to provide the justification for *each* statement when that justification is part of the textbook subject. The reason for this failure is all too clear, namely, the assumption that not always providing these justifications is a way to exercise the minds of the students, and, in the process, to separate the Winners from the Losers — the Winners will be able to figure out the justifications, or quickly find them in the book, the Losers won't. But the place to exercise the minds of students is in the homework problems and, if offered, semester projects.

• Shameful failure to teach students how to manage their time, especially when they are under pressure, and have more to do in the way of studying, homework, and taking exams, than they have time for. Instead, teaching them that suffering (staying up half the night, working in constant state of anxiety and near-exhaustion) is the only way to excellence;

- Blind faith that what is difficult, in fact, virtually incomprehensible, is good;

• Refusal to face the fact that the sheer quantity of mathematical knowledge, including tens of thousands of new theorems and lemmas being published each year, demands a way for mathematicians to obtain a hierarchical Big Picture of the whole of mathematics; The quantity of mathematical knowledge also renders how much detail a mathematician *knows* — that old reliable source of the power to intimidate students (and even some members of the faculty!) — less and less relevant. What matters more and more is "knowing where things are" — where to *look for* the details.

- Shameful disdain for the history of mathematics.

Don't Become a Reformer!

And yet, the facts and opinions expressed in this chapter notwithstanding, I am *not* suggesting that you become a reformer, because, for one thing, it would almost certainly be professional suicide if you try to do so in the early years of your career, and, second, because it is highly doubtful whether you will accomplish anything for all your pains anyway. My hope is that, after reading this chapter, and reading the two books of Morris Kline that I have frequently mentioned, you at least (1) will not allow yourself to believe you are somehow not "meant" to do mathematical research if you are turned down for a tenure track (or tenure!) at one of the "right" universities; and

(2) will spend some of your time — even if only a small proportion — making a genuine effort to give your students what they are paying for with their money and labor, namely, the best teaching job you can deliver, including, possibly, an Environment in the subject, or the beginnings of an Environment, with appropriate instruction as to how they can add to it on their own!