## Not Problems But *Classes* of Problems

I have emphasized in this book the importance of thinking in terms of *classes* of problems, not just individual problems, and of learning to put trick questions in their place.

### The Jumble, Lack Of Hierarchy, In Most Students' Minds Regarding Problems

I sometimes get the impression in talking to students that, for them, mathematics is a wilderness no different from, say, the tax laws, except there are more numbers and abstract terms. "You guys just keep making this stuff up as you go along!" an exasperated jr. high school student I was tutoring told me. I think that, for many students, precisely what is missing is any sense of hierarchy, of how each subject is *shaped*, of what each subject "looks like" from afar, of what is subordinate to what, of what is more important than what, of how this problem relates to that one. Everything seems to be equally important. In short, what is missing is any sense of the lay of the land.

### Importance Of Thinking In Terms Of Classes Of Problems

Personally, I no longer like to spend time trying to solve a problem unless I have reason to believe that, in the process, I will learn how to solve a *class* of problems. I think students should adopt the same attitude: the goal is not to somehow learn how to solve problems with sufficient struggle, thrashing, agony. The goal is to be able to explain to yourself and others how to solve any problem in that *class* of problem. In the section, "The Principal Classes of Homework and Exam Problems" in chapter 1, we set forth the major classes of problems. But it is also a good practice to underline, in your textbook, the words in each exercise that describe the *type* of problem it is, and this means not only, "Prove that..." but also "Find the derivative of...", "Find the integral" (or "Integrate") ...", "Determine if *x* is a *y* or not", and, of course, in physics courses, things like, "Find the work done in the following process...", "Compute the change in entropy of...", "Find the rotational inertia of...", etc. These underlinings will help you to decide what goes where in your Environment.

A *really* worthwhile calculus class — a one-in-a-million, worth-killing-to-get-into calculus class — would begin by going through the textbook and *classifying* (not solving, initially, just classifying) all the different types of problems. That same class would also, at some time, tackle the very interesting problem of devising a heuristic to solve at least the vast majority of integration problems *without* knowledge of the chapter that the problem came from. A few additional remarks on this subject are given below under "Speculations" on page 165.

(A really *good* professor will always say, when introducing a subject, “The classes of problems we will be dealing with are the following...”)

**“What Problems Does This Problem Lie ‘Near To’?”**

I am against “difficult” problems at the wrong time, i.e., at a time when the student doesn’t know where the problem “fits” in the overall hierarchy, or, still worse, when the student doesn’t understand what is difficult about the problem. Is it difficult because of the size of the numbers involved? Is it difficult because the finding of a solution in anything like a reasonable amount of time requires knowledge of theorems or formulas from another subject? I think that a professor hasn’t earned the right to assign such problems until it is clear where these problems fit relative to other problems — in other words, until it is clear what all the classes of problems in the course are.

To me, possibly because I spent several years as a programmer, trick problems — certain kinds of difficult problems — are like goto’s in a program. (A goto is a command that specifies the next step to be executed, the step having been previously labelled. The step may be anywhere at all in the program.) The trouble with the unconstrained use of goto’s is that they make very difficult the task of proving that the program is correct, or even of arriving at a reasonable assurance that the program is correct. (See Dijkstra, Edsger W., “Go To Statement Considered Harmful, letter, *Communications of the ACM*, March, 1968.) Similarly, trick problems and certain other kinds of difficult problems make it difficult for you to develop your Environment because they are so remote from the classes of problems your Environment is designed to solve. It is not that difficult problems are bad. It is that difficult problems presented too soon are bad. (Here is an interesting question to ask your professor if he or she is the author of a textbook: How did you go about devising the problems at the end of each chapter?)

Your most important question is always, “*Where* does this problem fit relative to all the others in its class?” (What = Where.) Personally, I am always more interested in where a problem *fits* than I am in the solution, because I know that if I know where a problem fits, I know something about its solution even before I begin.

**Use Of Spread Sheets To Represent Classes Of Physics Problems**

In my opinion, most of the problems in elementary physics can and should be presented in spread sheet format, by which I mean, as a collection of interrelated formulas into which you substitute values. *In fact, why should you, the student, have to do more than correctly label a drawing representing the problem?* Why should you have to prove, over and over again, that you can do algebra? Once you have correctly labeled a drawing — e.g., written down expressions for relevant vectors — the rest is mechanical. It should be handed over to a computer. If you don't supply enough variables, then you will get an infinity of solutions to the problem (and the computer can explicitly tell you that when it happens). If you have a sufficient number of variables, then you get one solution. If you have too many variables (e.g., contradictory values for a given variable) then, until you resolve the contradiction, you don't get any solutions (and the computer can explicitly tell you that when it happens). I don't see why this is not a sensible, rational, way to teach physics. I am certainly not suggesting that the student just be handed the spread sheet without any understanding of what the formulas represent. I am suggesting that in the vast majority of physics courses, a clear distinction is never made between, on the one hand, determining which values have to be substituted for which variables, and on the other, the algebra needed to arrive at a solution. (Some students *never* arrive at an understanding of the distinction. It's all just one big wilderness of words and numbers and symbols, like the tax laws.)

The author of a physics text has a section, "Notes on problem solving", in which he says:

"Solving problems, except in the simplest cases, can not always be done by a set procedure. Instead, it often requires creativity, *for each problem is different*." — Giancoli, Douglas C., Physics, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1980, p. 55. (My emphasis.)

I disagree that each problem is different. Or, rather, I think his statement is too imprecise.

He then gives a perfectly good set of rules for solving problems.

"1. Read written problems carefully!...

"2. Draw a picture or diagram of the situation...

"3. Determine what the unknowns are...

"4. Decide what you need in order to find the unknowns...

"5. Solve the problem...

"6. Ask yourself: Is the answer reasonable?..." — ibid., p. 55.

It is a major argument of the book you are now reading that this set of rules should come first in the book! In an Environment, these rules are folded into the Environment itself.

For most jr. high and high school students, word problems are difficult. Why shouldn't teachers begin by placing number one priority on developing a procedure for solving *all* word problems, explaining that (a) the procedure will attempt to put the problems into various categories, and then (b) within each category, attempt to put the problems into sub-categories, etc. In short, to reduce the amount of intellectual labor each student needs to go through. The procedure comes first! The procedure — the classification of word problems — is what it's all about!

Similarly, in elementary physics, I say that all projectile problems are, in a sense, the same! In fact, for a large class of these problems, we can draw the picture once and for all, fill in the formulas and thereafter let the spread sheet do the rest.

## "To Think Is To Classify"

I don't know what most mathematicians would say to the assertion, "To think is to classify" — maybe they would argue that the assertion should be, "To classify is to think" — but I think you will find that working with Environments makes you see thinking more and more as a process of successive classification. You have this problem before you; you don't know how to solve it. Well, what does the problem deal with? (Possibly several things, sets of things.) For each thing, set of things, what are some theorems that might apply, what are some concepts that might apply? You have begun narrowing your igorance to smaller and smaller sets of possibilities; you have begun classifying possible solutions, possible approaches into those that are likely to lead to a solution, and those that are not.

To say that problem-solving is searching (classifying) seems to me a far more accurate and useful statement than that problem-solving is thinking.

## Is Mathematics Easy for Mathematicians?

In their worst moments, students sometimes imagine that mathematics is easy for mathematicians, if for no one else. This is not true. David Hilbert, one of the twentieth century's great mathematicians once remarked,

"That I have been able to accomplish anything in mathematics is really due to the fact that I have always found it to be so difficult. When I read, or when I am told about something, it nearly always seems so difficult, and practically impossible to understand, and then I cannot help wondering if it might not be made simpler. And on several occasions it has turned out that it really was more simple." — David Hilbert to Harald Bohr in Reid, Constance, *Hilbert*,

Carl Friedrich Gauss, the nineteenth-century mathematician who is generally considered to have been the greatest mathematician who ever lived, referred to mathematics as "this thorny subject".

On the other hand, we must keep in mind that the problems these men were wrestling with were not the textbook and classroom problems of their day.

In this book, I am not claiming that mathematics is “really” easy. What I am claiming is that solving mathematical problems can be easier than it normally is for most students.

## Why Are Other Subjects Easier than Math?

Most people consider math to be the most difficult of all subjects. This is true even of people with advanced degrees in the hard, much less the soft, sciences. True, medicine is also considered to be very difficult, but there the reason is the amount that the student has to memorize and the stresses of internship, with its long hours without adequate sleep and the life-and-death nature of the work.

What is it about math that puts it in a class by itself? My answer is that in all the other subjects we know *the space* in advance. Specifically, we know that, no matter what the parts are labelled (learning these labels is most of the work of learning the subject) the space will be “things interacting with other things” and furthermore that the things have insides and outsides — probably most of the things will have other things inside them. This applies to subjects from anatomy to physiology to geography to phyics (at least above quantum mechanics) to astronomy, chemistry, biology, computer science, all branches of engineering, architecture, sociology, political science. All the subjects are *geometrically similar* in the sense that everything in the subject could be presented in terms of enclosures (drawn with closed curves of one sort or another, or with boxes) and arrows connecting them. The enclosures represent things, the arrows represent causes, flows, movements.

But, at least until this book was published, mathematics was different. Here we have all kinds of strange entities not only in name but in their lack of a geometrical space that they fit into. I am speaking here of non-geometric mathematical subjects, e.g., of group theory, number theory.

## On Mathematicians’ Low Opinion of the Computer

Many, perhaps most, pure mathematicians have a low opinion of the computer, and hence of anything, e.g., the Environment idea, that makes use of computer-based ideas. More than one mathematician has told me, “The computer is a big calculator, no more, no less.”

This prejudice arises from two sources: first, the fact that, for a mathematician, once the solution of a class of problems has been reduced to an algorithm, it loses its mathematical interest, apart from questions about the computational complexity of the class of problems, i.e., questions about the theoretical limits to the speed with which a problem in the class can be solved. The second source of the prejudice is the belief that none of the truly creative work in mathematics is in any way “mechanizable”, i.e., capable of being taken over by a machine. Human ingenuity is necessary.

In reply to this latter belief, I will merely point out that many mathematicians use calculation as an exploratory tool.

“[Ramanujan] worked, far more than the majority of modern mathematicians, by induction from numerical examples.” — Hardy, T. H., quoted in Newman, James R., “Srinivasa Ramanujan”, in *The World of Mathematics*, Vol. 1, Simon and Schuster, N.Y., 1956, p. 375.

Computers permit mathematicians (and students) to generate numerical examples with a speed never dreamed of in the past. In fact, much of the research on fractals and chaos theory would be impossible without the computer.

I have a standing bet with a mathematician that by the year 2010, a world-famous mathematician will say that an important result that he achieved in pure mathematics could not have been achieved without the aid of the computer. (The proof in the seventies of the Four-Color Theorem — that four colors suffice to color any map such that no two adjacent regions have the same color — is often cited as an example of a very difficult proof that could not have been accomplished without the computer, but some mathematicians are still skeptical about its validity, arguing that the correctness of the programs themselves has never adequately been proved, and that, in any case, neither the theorem nor the proof seem to have far-reaching implications in any branch of mathematics.)

If I were going to invest money in making the computer more useful for mathematicians, I would not invest it in artificial intelligence (e.g., in trying to develop systems that could prove real-life theorems), but instead I would invest it in the development of *interactive* systems, i.e., systems that made it very easy for mathematicians to perform such tasks as:

- display graphical representations of large portions of complicated functions;
- generate numerical (and symbolic) examples and do searches for simple patterns in these examples;

• do exploratory proofs of low-level theorems, e.g., the mathematician might tell the computer, "See if you can prove the following theorem using the following existing theorems, but don't spend more than an hour on it." Or even, "See if you can find, anywhere in your database" — we assume here that the entire literature of the particular branch of mathematics is in the program's database — "a theorem which looks like the following", where "looks like" might mean, "having the same terms in the antecedent and the consequent as the theorem in question".

• help generate Environments for mathematical subjects, the program at the least doing all the formatting and keeping track of all the references and cross-references (this is possible now, of course), sprucing up the graphics the mathematician enters, notifying him of undefined terms which the mathematician has not assumed the student already knows the meaning of, etc.

## A Word to Overachievers

If you come from a certain kind of achievement-obsessed family, you often think, "If I were *really* good, I wouldn't need to read books on improving my grades."

My reply is that the most successful people I know are students of just such techniques. They are constantly trying to become smarter than they already are, and they have no compunctions about where they pick up new techniques. They are always trying to improve their ability at what they do, they are continually monitoring their, and others', problem-solving performance. Their attitude toward those who believe that, if you're any good, you already know such things or else can develop them on your own, is "Fine. Let them spend their time re-inventing the wheel. I'll be miles ahead of them by the time they are where I am now."

One of the most important ideas you will ever come across is so obvious that almost everyone ignores it. I mean, the idea of learning from experience (from "feedback"). Particularly if you are an overachiever, you will dismiss it because it is well-known, it is lying around for everyone to see; it isn't new. But one of your goals in life must be *not* to be led by the nose, meaning, here, led by the belief that only that has value which is new. Your goal is to get at the truth, to find out what works, regardless of whether it is old or new. With the wide-awake application of this idea there is no telling what you can accomplish. You will be amazed at how much you can improve by keeping track of the mistakes and difficulties you have. Mark them on your exam papers, or on your homework separate page after each problem or in a separate notebook or file. Right now, do you know what your most common mistakes are? Failing to check your work? (Do it for just one day, even though it's boring. But if it gets you a good grade...) Copying the problems wrong? Not reading the problems carefully? Forgetting a rule, e.g., the rule for finding the partial derivative of a function. Errors in arithmetic? *There is no way you can improve unless you begin to get a handle on where most of your errors lie.* Get rid of those superstitions about what makes you a loser. Why bow your head to those who happen to have a facility which, at the moment, you don't have? Your business is not to knuckle under, your business is to win.

## Respect Your Own Questions!

Many students assume that any question that occurs to them about the math subject they are studying is probably either stupid or else a proof that they are way behind most of their fellow students. I often wonder how many beginning calculus students — even those getting good grades — have a nagging feeling that they don't really understand what "infinitesimal" means. "What does it mean for something to be 'arbitrarily small but not zero' or 'less than any quantity we can conceive, but not zero'? If $\Delta x$ is *not* zero, how come we can treat it as though it *is* zero when we derive formulas for derivatives? When we let $\Delta x$ 'approach' zero, how fast does it 'move'?" These students would no doubt feel much better if they knew that exactly such questions perplexed the best mathematicians (and a few philosophers) for over 100 years after the calculus was discovered by Newton and Leibniz.

"Bishop Berkeley (1685-1753) [wrote] a tract entitled *The Analyst: Or a Discourse Addressed to an Infidel* (generally believed to be [the physicist] Edmond Halley). The Bishop set out his objectives in the subtitle of his tract:

" 'Wherein it is examined whether the Object, Principles, and Inferences of the modern Analysis [i.e., calculus] are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.' "

Berkeley argued that the so-called truths of science are no more firmly based than those of theology; he used the calculus as a prime example of his thesis. His conclusion was that: 'He who can digest a second or third Fluxion [derivative with respect to time], a second or third Difference, need not, methinks, be squeamish about any Point in Divinity."

The questions about the meaning of infinitesimals and derivatives were not satisfactorily answered until the early 19th century, when the limit concept was made rigorous by Cauchy and others.

Another question about the calculus that I am sure occurs to many students, especially if they are also studying physics, is, *What exactly did Newton need the calculus for in developing his theory of gravitation?* Students are often told that he needed to be able to determine the instantaneous velocity of a moving object, but why did he need to determine that? We can write the equation for any ellipse, i.e., the orbit of any planet, without knowing anything about instantaneous rates of change. We can compute the gravitational force on any object without doing any calculus. What exactly — in his physics researches — motivated Newton to investigate the notion of the derivative and the integral?

Another set of questions which students often have but are afraid to ask is:

"How does the equals sign really 'work'? When we write $x = y$ does that mean that $x$ suddenly has the value that $y$ had before we wrote the equation, or is it the other way around, especially as $y$ is often used as the symbol for the dependent variable in equations like $y = f(x)$, or does it mean that both $x$ and $y$ both have had the same value all along, in which case why bother stating the obvious? How can we possibly say things like '$x = y$ is false'? Either the equals sign means equals or it doesn't."

A question that bothers even some of the brightest jr. high school students is: "How come it sometimes doesn't matter what letters you use for variables, and other times it does?" This question bothered a jr. high school student I was once tutoring. He didn't understand how I could say that the equation $y = 3x^2 + 4x + 1$ is the "same" equation as $z = 3v^2 + 4v + 1$ (provided $y$, $x$, z, and $v$ haven't previously been assigned specific values). When I told him that it's not the letters that matter, but what is done to them — what they are multiplied by or added to — he said, in exasperation, "You guys just make this stuff up as you go along!"

Another perfectly good question is, "Why do some people speak of *The* Calculus and others just talk of calculus? Is there just one calculus or more than one?" (Answer: there are many calculi. A calculus is simply a method of solving certain kinds of problems. The "The" may have arisen as an abbreviation for The Differential and Integral Calculus, of which there is only one, at least as taught in the schools.) The beleagured student may be amused to hear that I once wrote to a famous mathematician asking why he had called one of his systems a *calculus*. In a letter dripping with contempt he replied that this is a common practice, as any *mathematician* knows.

Which brings up an important warning: Beware of those teachers and professors who like to boast, in print, that, for them, "there is no such thing as a stupid question." Typically, this boast is made by professors who have achieved a certain amount of fame in their profession and are now basking in the glow of public admiration and their own benign regard for themselves. Those of us who have endured the withering contempt of more than one such professor — not to mention a number of similar-minded graduate students — for asking perfectly good questions, have learned to be suspicious of such pronouncements.

Young students are taught — or, rather, soon learn — to suppress their questions. Let me give you an example of a whole class of questions that the vast majority of math teachers, I have no doubt, would consider stupid, namely, "Why is ... called ...?" Every mathematically literate person knows that it doesn't matter what you call things provided you make clear your definitions and then use the defined terms consistently. For example, it would be perfectly legitimate (though unnecessarily difficult for your grader and, probably, for you) if, at the start of a set of homework problems, you stated, "In the following, '1' denotes the variable '*x*', '2' denotes the variable 'y', '≠' denotes '='" *provided* you then used these symbols consistently and got the correct results (as expressed in these symbols). A friend of mine, a jr. college teacher, once was tutoring a young woman student in first year physics. She had a very difficult time understanding a number of basic concepts, among them, the concept of the moment of intertia. (Informally, this is the "reluctance" of a mass to accelerate its rotation about an axis. Thus, e.g., "a heavy millstone is harder to start rotating than a roller-skate wheel; once it is rotating, it is also harder to stop."— ibid., Giancoli, p. 81.) After days of trying to determine where her difficulties lay, the teacher suddenly realized that the way she was attempting to understand this and all concepts in physics was by beginning with the *names of the concepts* themselves. Thus, since, in everyday language, a "moment" means a very brief period of time, she was trying to understand how a brief moment of time somehow caused a heavy wheel to be harder to turn than a light wheel. Now, in fact, her approach is precisely the way that concepts are explained and analyzed in some of the humanities, and especially in some schools of philosophy[1], where entire theories have been developed out of the etymology of ancient Greek words. A common form of argument in these fields is: "the term *x* is derived from *y*, and therefore what it really means is ..." (In passing, let me remark that, apart from technical terms which are consciously defined out of Greek roots, e.g., *cybernetics*, I think this whole approach is on very shaky ground, because in daily life, and especially in America, which has the gift of language creation as few other cultures do (consider merely the endless novelty of American slang) no one comes up with new words on the basis of their *etymology*, but rather on much more subtle bases, e.g., associations that are almost poetical (consider "cool", "hip", "hellafine").)

1. For example, that of the 20th century philosopher Martin Heidegger

"The other day I decided to talk to the other section [of a fifth grade class] about what happens when you don't understand what is going on. We had been chatting about something or other, and everyone seemed in a relaxed frame of mind, so I said, 'You know, there's something I'm curious about, and I wonder if you'd tell me.' They said, 'What?' I said, 'What do you think, what goes through your mind, when the teacher asks you a question and you don't know the answer?'

"It was a bombshell. Instantly a paralyzed silence fell on the room. Everyone stared at me with what I have learned to recognize as a tense expression. For a long time there wasn't a sound. Finally Ben, who is older than most, broke the tension, and also answered my question, by saying in a loud voice, 'Gulp!'

"He spoke for everyone. They all began to clamor, and all said the same thing, that when the teacher asked them a question and they didn't know the answer they were scared half to death. I was flabbergasted — to find this in a school which people think of as progressive; which does its best not to put pressure on little children; which does not give marks in the lower grades; which tries to keep children from feeling that they're in some kind of race.

"I asked them why they felt gulpish. They said they were afraid of failing, afraid of being kept back, afraid of being called stupid, afraid of feeling themselves stupid. Stupid. Why is it such a deadly insult to these children, almost the worst thing they can think of to call each other? Where do they learn this?

"Even in the kindest and gentlest of schools, children are afraid, many of them a great deal of the time, some of them almost all the time. This is a hard fact of life to deal with. What can we do about it?" — Holt, John, *How Children Fail*, Dell Publishing Co., Inc., N.Y., 1964, pp. 62-63.

Not only do math students learn at an early age that, in fact, you must constantly be on your guard against asking stupid questions, but among the stupidest questions are those relating to why you don't understand something. I have never heard a math teacher tell a class, "Now, as part of your work in this course, I would like you to develop the ability to know what you don't know. I would like you to start asking yourself questions like, "What makes this part of the subject so difficult for me?" (The answers might be, "I don't understand what good it is", "I keep forgetting all the formulas", "I never know what to substitute for what", "I hate fractions", "I hate equations.")

"...everything I learn about teaching I learn from the bad students..." Holt, John, *How Children Fail*, Dell Publishing Co., Inc., N.Y., 1964, p. 84.

The owner of a local business I patronize once remarked that her daughter was having difficulty with pre-calculus in high school. I asked the mother what the difficulties were but she wasn't sure. I then offered to help her daughter via the phone. She felt understandably reluctant to have a stranger — even a regular customer — do that, so I asked if her daughter would be willing to answer questions in a questionnaire I would make up. She could just fill in the answers and return the paper to me, via her mother. I made up two pages of questions and gave them to the mother. A couple of weeks later, the mother said that the daughter "didn't have time" to answer the questions. I sensed that what was going on here was a case of shame on the part of the daughter that she didn't understand the subject, and a determination not to reveal that lack of understanding to one more person than those who already knew about it. Clearly, the girl had never been encouraged to get in touch with her math difficulties, this in a culture which has made big business out of getting people to get in touch with their feelings, regardless what they may be.

There is no better way to gain confidence in your ability to improve your grades in math than by starting to respect your own questions. Good studentcraft will often dictate that you not ask the question in class, but you should ask it during your professor's (or another professor's) office hours, or ask it of another student you feel might know the answer. And I am not referring solely to questions about the content of the course. I also mean questions like "What good is this subject?", "Who thought up the ... theorem and why?" "How come this subject is so difficult?" "Why do you" [the professor] "find this subject interesting?" "Why is ... used as the symbol/term for ... ?" "Why is ... called ...?"

Questions are the knives of our thought. Write down your questions even if you can't answer them. Put a question mark in the textbook margin so that you will remember to ask the question later in class, or during the professor's office hours. When working under fatigue, a way to accomplish what you couldn't otherwise is simply to formulate questions. They are half the work — more than half. (See "When All Else Fails..." on page 163.) Trust your questions. In problem-solving, you are limited only by the questions you can ask.

And don't forget: you can ask questions about subjects you haven't studied! That is a perfectly legitimate thing to do. I once needed to know if certain theorems existed in a certain branch of number theory I had never studied. Since I had the (reluctant) ear of a good mathematician, I began flooding him with questions of the form, "Is there a theorem that says something along the lines of..."; "Is there a theorem that relates ... to ... ?" "Is there a theorem that specifies the minimum number of ... in the context...?" He replied that I couldn't really ask meaningful questions until I had studied the subject. Then he proceeded to answer my questions ("just this once ..."). I *think* (though he never admitted it) that I eventually convinced him that I could, in fact, formulate perfectly good questions about a subject I had not studied.

"The true test of intelligence is not how much we know how to do, but how we behave when we don't know what to do." — Holt, John, *How Children Fail*, Dell Publishing Co., Inc., N.Y., 1964, p. 205.

## Importance of the Question, "How Do You Know This?"

An extremely valuable question which students should learn to ask their teachers is, "How do you know this?", where "how" doesn't mean, "By what authority, by what logical or scientific argument", which we assume the subject makes clear, but rather, "In what way — how do you hold this subject in your mind? How do you think about it? What have you memorized and what do you still have to look up in the textbook? What were the first things you found yourself memorizing when you first studied the subject?"

The answers, of course, need not be precise. What they will provide, as long as they are honest, is a valuable guide to the student as to what the expert in the subject (the teacher) considers most important, and how he has organized this material for his own purposes.

## If One Learns, Why Not All?

To make the point in the previous section in a slightly different way: it is important that we attempt to separate, in any technical subject, what each student must do for him- or herself from what teachers and Environments (including the Environments that traditional textbooks are) can do for him or her. Of course, the very essence of the idea of "the advancement of learning" is that each successive generation does not have to repeat every discovery from scratch, but that it can take over, ready-made, the knowledge and new discoveries of the previous generation. But the prevailing assumption in technical courses is that each student must "learn for himself". On the basis of my own experience, I do not believe the assumption is valid as it stands. For example, in no traditional textbook that I have ever seen, or in any course that I have ever taken, has the structure of the subject, and of branches of the subject and of its principle concepts, been presented in the way it is in . (The typical table of contents is a very poor representation of these structures.) Yet I believe that an expert gradually fashions such structures — such global views — for him- or herself through continued work on the subject. Do most students benefit from being forced to discover for themselves — if some ever do — that these structures can be valuable aids to understanding? Do most students who do make the discovery benefit from having to then figure out the structures? I doubt it. Even if they do, can they afford to repeat this kind of labor when technical knowledge is increasing as rapidly as it is?

## Good and Bad Ways of Learning Logical Thinking

Teachers sometimes argue that one of the benefits of studying math is that it helps you to learn to think logically. This used to be the main argument for studying plane geometry, in which high school students learned to do proofs. Nowadays the argument is often applied to calculus. Yet neither plane geometry nor calculus are good ways to learn logic: in plane geometry, students often see logic as unnecessary — a means for proving the obvious; in calculus, logic is overwhelmed by the difficulty and strangeness of concepts like the instantaneous slope of a curve, the notion of quantities that are “arbitrarily small but not zero”.

The essence of logic is rule-based thinking. The purest (though perhaps not the best for high school students) way to teach rule-based thinking is via the languages generated by formal grammars. We begin with a set of symbols (an alphabet) and a set of rules (productions) for combining and replacing the symbols. We then prove that certain classes of strings are, or are not, in the language (set of strings of symbols) generated by the grammar. Here, at least, the student can see logic in its purest form, and thereafter have a model of what logical thinking ultimately boils down to. The students are already familiar with examples, namely, games like checkers and solitaire.

Computer programming is another way of learning logical thinking that is much better than plane geometry or calculus, because here the student also learns the concept of a variable and of a function, two of the most important concepts in mathematics (and in Environments), without at the same time having to master other, difficult concepts. The function concept is part of the larger concept of a procedure or subroutine, which is crucial to the understanding of problem-solving with Environments.

## Studying Math Horizontally vs. Vertically

The longer I study mathematics the more I find myself wanting to take "horizontal" slices through many different subjects at the same time (viewing the normal studying of one subject at a time starting at the beginning as studying "vertically"). These horizontal slices are centered on concepts which appear in different subjects, e.g., the concept of "quotient" as represented by the familiar quotient in arithmetic, quotient sets, quotient groups and rings, quotient topologies, etc., or the concept of repetition ("going-around-ness") as represented by loop and recursion in programming, the congruence relation, finite groups, periodic functions, certain operations on complex numbers, winding numbers, etc.

Thus, while in the vertical mode of studying, I ask of each new concept, "Does this appear in other branches of mathematics, and if so, how and where?" I have found few techniques of studying which deepen understanding more than this one.

Why don't universities offer courses in, say, the concept of a normal form, in which normal forms in various branches of mathematics would be studied? Why not a course in powers of *e*, the base of the natural logarithms, in which the occurence of these powers in various branches of mathematics would be studied?

## When All Else Fails...

Sometimes you will come upon parts of a math course that are so difficult that you can't even figure out how to enter the material into your Environment. You will feel that you can make no progress at all. Perhaps, in addition, the professor is intimidating, perhaps you are reluctant to reveal the depth of your ignorance to him or her during their office hours. Is there anything you can do? On the basis of my own hard experience, I think the answer is Yes. Here are a few techniques.

### Question/Answer

In the textbook, you begin with the first paragraph you are having difficulty with, and *write down and underline* the first question that comes to mind about the material. For example, "What do we seem to be trying to accomplish here?" ["we" being you and the textbook author]. You then try to answer that question by looking elsewhere in the book, by referring to your notes, by asking another student, by just thinking. When you have the answer, or part of the answer, you *write it down* immediately following the question. You don't underline the answers: the purpose of the underlines is to make the *questions* stand out the next time you review what you have written. In your answers, you use informal language. Talk to yourself as a friend. *Always* give page references. "Oh, yes, this is covered in the discussion of Theorem 12.5 on page 101." The *writing down* of question and answer is absolutely essential, because what you want to create is a record of your learning process, so that the next time you return to this material, you do not have to try to remember the questions and answers, or, still worse, repeat the process of generating the questions and finding the answers all over. You are "pre-learning" the material for the next time you come back to study it! You now read until you have another question. Perhaps it will be "What's the definition of ...?" or "How come he [the textbook author] can say ...?" You write the question down and proceed to try to answer it, and then write down the answer in a talking-to-yourself style. You keep repeating this process as long as necessary.

It is slow going, but I have found it a powerful tool for breaking through otherwise impenetrable material. It is, of course, a throwback to the old-fashioned learn-it-all-approach, but sometimes nothing else works. If you ever have a chance to try the technique outside of school, just because you want to teach yourself all or part of a difficult subject, I think you will find it a strangely satisfying experience — but only if you have time to take your time and can work without pressure.

## For the Really Smart Only...

If you have understood the Environment concept, and, in particular, understood that an Environment is a way of capturing — representing — a certain kind of *behavior*, namely, the behavior that results in solutions to certain large classes of problems in a subject, then you might have been thinking, "But this can be applied to subjects other than those in mathematics or the hard sciences!" And you would be exactly right. In particular, it can be applied to most industrial jobs, from the clerical level up to middle management — especially if these jobs rely heavily on the computer. This aspect of the Environment idea is covered in Schorer's *How to Create Zero-Search-Time Computer Documentation* (www.zsthelp.com). Unfortunately, it is a steep uphill battle getting professionals to accept such an idea, because it implies that their work is not as difficult as their salaries would imply. (It isn't.) But sooner or later, in the new kind of economy which the world is entering, the value of making workers much more flexible — much more capable of changing jobs rapidly — is going to be recognized by the powers that be, and then those who already understand the Environment concept will have a distinct advantage, both as users and creators of Environments.

## Speculations

In the final section of this chapter, I would like to set forth a few questions and speculations that have occurred to me in the course of developing and using Environments. It is entirely possible that none will lead to anything of interest.

*The concept of "nice"* — Often, when we are studying a subject, we come across theorems which, in essence, say that, in such-and-such circumstances, things go as we would like them to go; in other words, things exhibit "nice" behavior (sometimes the things are said to be "well-behaved", but the term is used in a more limited sense than I have in mind here). For example, if a function is commutative, e.g., if $f(a, b) = f(b, a)$ for all $a$, $b$, that might be considered a "nice" property of the function; if the product of a finite set of topological spaces retains a property which each of the spaces has, that might be considered a "nice" property of the product; if a function called a "sum" behaves in a way analogous to the way the arithmetic sum behaves — e.g., the "sum" of $a$, $b$ is always "at least as large" as the larger of $a$ and $b$ — then that might be considered a "nice" property of the sum.

Then we may ask of a given subject, how much of the entire subject is "nice", and how much isn't? If we could make this concept more precise, then, it seems, our job as students — as *users* — of the subject, would be made much easier, in that we would know a great deal more about the subject at the start, with much less effort.

*How do mathematicians know what theorems to prove?* — To the beginning student of college mathematics, theorems seem to come out of the air: "How, and *why*, did they think *this one* up?" But there are plausible explanations for how *some* theorems come to be thought up. As I have tried to show in this book, it is good to develop the habit, when you confront a new mathematical concept or entity, of asking: what are the basic *operations* on this entity? Mathematicians often proceed in the same way. One of the most important of these operations is that of determining equivalence between the entities, which, of course, requires that mathematicians first decide on a definition of equivalence. Then they may wonder if there is an algorithm (a computer program which is guaranteed to halt with a Yes or No answer) to determine if two entities are equivalent, and, if so, what is the maximum number of steps the program will require to compute the answer for any two entities (i.e., what is the computational complexity of the algorithm)? Once mathematicians have determined the basic operations (or while they are determining them), they want to know what the algebraic properties of these operations are (if that is an appropriate question — in some cases it will not be) — i.e., they want to know if the operations are closed, commutative, symmetric, transitive, associative, distributive. They may also want to know about other "common" or "useful" relationships between the entities, e.g., what kind of linear or partial orderings are possible among them.

Next, they may wonder about building blocks: can the entities be built out of simpler entities, and if so, what simpler entities? In the other direction, they may want to know how the entities can be combined to make "bigger" or more complex versions of the same entity.

All this the mathematicians will consider part of the "basic carpentry" of introducing a new mathematical subject. It is more a matter of hard work than of brilliant ideas. Mere competence will ordinarily suffice to accomplish the job.

The next stage is more difficult, namely, that of determining what properties, important in the subject, are transmitted under various functions, e.g., if something has property $x$, and we make a bigger version of the something, will it also have property $x$?

And, of course, there are other theorems which simply arise as "interesting questions".

*"Mapping" a subject* — Suppose you were asked to group the problems at the end of a textbook chapter into sets such that all the problems in a given set were more closely related to each other than they were to the problems in any of the other sets. Is this a reasonable request? You might reply, "Well, it depends on what you mean by 'related'". You might argue that, e.g., those problems that could be solved by substituting values into the same equation, or by applying the same theorem, belong in the same set. But is there any way to establish a more rigorous criterion for dividing up the problems? Our goal here is to "map" all the problems in the subject in the sense of showing "where" each problem belongs relative to the others, so that we can say things like, "Well, if you solve *this* problem, then it is only a 'distance' of such-and-such (computationally or heuristically speaking) to solving *that* problem."

"Every mathematician has the sense that there is a kind of metric between ideas in mathematics — that all of mathematics is a network of results between which there are enormously many connections. In that network, some ideas are very closely linked; others require more elaborate pathways to be joined. Sometimes two theorems in mathematics are close because one can be proven easily, given the other. Other times two ideas are close because they are analogous, or even isomorphic. These are two different senses of the word 'close' in the domain of mathematics. There are probably a number of others. Whether there is an objectivity or a universality to our sense of mathematical closeness, or whether it is largely an accident of historical development is hard to say. Some theorems of different branches of mathematics appear to us hard to link, and we might say that they are unrelated — but something might turn up later which forces us to change our minds. If we could instill our highly developed sense of mathematical closeness — a 'mathematician's mental metric', so to speak — into a program, we could perhaps produce a primitive 'artificial mathematician'." — Hofstadter, Douglas R., *Goedel, Escher, Bach: an Eternal Golden Braid*, Basic Books, Inc., N.Y., 1979, p. 614.

*How should integrals be classified?* — Suppose you want to devise a heuristic for doing integrals which is to be used by students through, say, second year calculus. Part of this heuristic would consist of one or more tables of already-computed integrals. What would be the criteria for placing integrals in one table rather than another, and how would you organize the integrals in each table? In answer to the second question, "alphabetically" seems to have certain problems, not the least being that it would depend on the letter used for variables, which is irrelevant. Remember that the goal of such a heuristic is the reasonably rapid computation of integrals assuming the student does *not* know the chapter in the textbook the integral was presented in.

*How do mathematicians know when a subject is "finished"?* — You would think there is no limit to the questions that can be asked in any mathematical subject, yet sometimes mathematicians will say that such-and-such a subject is "mature" — that it is unlikely that any really interesting theorems will be proved in the future. Three such subjects about which this has been said are: plane geometry, theory of complex numbers and point set topology.

So a question poses itself: assuming mathematicians are right in making these judgements, how do they make them? Is it purely a matter of intuition, or do they use a kind of checklist and if so, what is that checklist?

"In 1869, the Russian chemist Dmitri Ivanovich Mendeleev arranged the known elements in order of atomic weights and showed that a table could be prepared in which the elements, in this order, could be so placed as to make those with similar properties fall into neat columns.

"By 1900, this 'periodic table' was a deified adjunct of chemistry. Each element had its place in the table; almost each place had its element. To be sure, there were a few places without elements, but that bothered no one since everyone knew that the list of known elements was incomplete. Eventually, chemists felt certain, an element would be discovered for every empty place in the table. And they were right. The last hole was filled in 1948, and additional elements were discovered beyond the last known to Mendeleev." — Asimov, Isaac, *Asimov on Chemistry*, Anchor Books, Garden City, N.Y., 1975, pp. 109, 112.

The periodic table is an example of a structure which suggested pieces of knowledge that were missing. In other words, it is a structure which tells us something important about the subject. Is there a structure for mathematical subjects which will quickly reveal the parts of the subject which are still missing? Is it possible that the ways a subject can be represented in the computer, will in itself tell us important things about the subject?