CHAPTER 2

Mathematics in the University

(from William Curtis's *How to Improve Your Math Grades,* on occampress.com)[1]

1. This line has been added to the first page of the chapter because Google does not reveal this information when it makes the chapter accessible following a user's search.

Purpose of This Chapter

The purpose of this chapter is to explain why college mathematics courses (and textbooks) are as they currently are.

It is not necessary to read this chapter before you start applying the technique for improving your math grades that this book describes.

Academic Math Teaching — A Relic of the Middle Ages

Thoughts While Standing Outside a Calculus Lecture Hall

One day I was walking around Evans Hall, the home of the Mathematics Department at the University of California at Berkeley. I passed a lecture hall and heard a young professor or assistant professor or instructor lecturing on the subject of derivatives. It soon became clear this was a freshman calculus course. The students were all bent over their notepads and three-ring binders, taking notes. I suddenly thought to myself: *is this possible?* Here it is, the early 1990's, less than a decade before the 21st century, and I am witnessing a scene out of the Middle Ages. The professor stands at the front of the lecture hall and *transmits* the facts about derivatives. The students write down the knowledge he transmits. Thank God their parents have been able to afford tuition, because, otherwise, as everyone knows, they wouldn't be able to find out about derivatives! Some students seemed nervous; brows were furrowed ("Am I really understanding this? Why is it called a 'derivative'? They say he's a lenient grader, so maybe I'll be able to slip through...").

I thought of the anxiety of students about the professors who taught required, difficult courses. "Is he a tough grader? Does he take time to explain things? They say he has a sense of humor..." I thought of people I had met who told me that they had never learned a subject or concept under discussion because "we didn't have a good professor." And even sometimes, in the case of a particular topic, "...because I missed class that day."It suddenly struck me how bizarre it was that, in the students' minds, the personality of the professor was a major factor in determining if the student learned.

I thought of the huge, sliding blackboards in some lecture halls, so that the professor can *save* the equations and drawings he has just carefully put on one board by pushing it to one side, and get a fresh *new* board (the one behind the movable one) on which he can put more equations and drawings. And using *chalk*, an item of technology who knows how many hundreds, perhaps thousands, of years old? (The professor knows how to write so that the chalk doesn't squeak, he is an experienced teacher, but sometimes whoever is in charge of blackboard maintenance forgets to put enough pieces of chalk in the wooden runner beneath the blackboards, and then the professor has to spend minutes searching for a piece, just one piece, that will enable him to write the next equation (no chalk = no equation, no knowledge for this audience of young minds, eager to learn).)

I thought of the rumble that these blackboards made as they were moved, this ancient technology which the students unquestioningly assumed was necessary to convey the difficult subject that the brilliant man at the head of the lecture hall was expert in. (This is how difficult it is: it requires *movable blackboards* in order to be taught!) I thought of all the walking back and forth that the professor had to do to refer to things he had written on his various sliding boards. What a difficult job teaching is!

"The worthlessness of the lecture system is pointedly described by Alfred North Whitehead in *The Aims of Education*: 'So far as the mere imparting of information is concerned, no university has had any justification for existence since the popularization of printing in the fifteenth century.'"[1]

I thought: the students will go back to their dorms and rented rooms and turn on their computers and TVs and take for granted a collection of technologies that could make derivatives — much less the contents of numerous far more difficult courses — dance before their very eyes. A tiny fraction of the budget that goes into the TV ads they watch, or into the animations on the Internet, could reduce to minutes the hours they will spend struggling to understand these courses. Yet there they were, on a spring afternoon, hoping they would be able to write down fast enough everything that the professor was saying.

I thought of the textbook industry, of the professors and presumably university administrators who decide what shall be covered in this course, what in that other course: "Shall we put concept *x* in the Intermediate or the Advanced Course, which will only be offered every other year?" (When I need to know something, how do I know if it is an Intermediate or an Advanced concept? Why should I *have* to know?)

And I began thinking about the feelings that go along with this modern anachronism: the students' perpetual fear that they may not be able to please the tyrants who give them their grades and who hold their futures in the palm of their hands, particularly if the student aspires to an academic career; the institutionalized shame at not being able to understand the material rapidly, or at least more rapidly than most other students in the class — at not being always able to get an A or a B. (After the exam: "How did you do? Did you get them all? I stayed up all night... I don't think

1. Kline, Morris, *Why the Professor Can't Teach*, St. Martin's Press, N.Y., 1977, p. 98

I got the last one because I forgot the theorem on..." Think of it! These words represent the status of the students' knowledge of the subject in the hours immediately following the conclusion of the course (the status will deteriorate rapidly). This is called *learning*. This is what students go to college for. *Are you kidding me?*)

And then, as I watched the professor writing equations and drawing diagrams on the blackboard — slowly, carefully, clearly — I began thinking about all the professors all over the country writing the same equations on blackboards, drawing the same careful diagrams. All those presentations of the same concepts — some presentations good, some not so good, a few very good indeed — all this repetitive manual labor year after year, all the preparing for fall classes each summer by overworked professors who would much rather be doing research, all of this taking place as if the printing press had never been invented, much less offset lithography, or xerography, or movies, or video, or computer graphics. I thought: why not do the job once, *really well*, and then sell the result to everyone else? Why not have all the college and high-school teachers of calculus get together and (perhaps using some of the ideas in this book!) at the very least create a set of professionally drawn illustrations (lots of them!) of the basic concepts and save all that repetitive labor at all those blackboards? Then, *before* each class meeting, students could be merely assigned a reasonable amount of material to study from a carefully prepared text that used these illustrations, and assigned a reasonable set of problems to solve, and then, if they had difficulties or questions, they could ask the teacher at the next class session. The class would *begin* where the magnificently illustrated study materials *left off*; the transferral of knowledge, or most of it, would take place *outside* the class, not inside it. The purpose of the class would be to improve the study materials based on the students' success at solving the problems. Will the calculus teachers of the nation really ask us to believe that *no* standard presentation can be initially as good as *any* blackboard presentation? That's a hell of a claim! I don't believe it.

To summarize what we have tried to show in Chapter 1 and up to this point in Chapter 2: mathematics departments are citadels of inefficiency and intimidation.

How Do You Know That You Know?

If you plan to go into a field that requires mathematics, I urge you to make the experiment, at least once, of trying to teach yourself a little mathematics. It doesn't have to be an entire course. If you do this, you will probably be struck at some point by how naive the assumption of the universities is that you understand something in proportion to the grade that you get on an exam: "Well, at least at the end of the semester I understood the basics of vector calculus, because I got an A in the course."

When you start teaching yourself parts of subjects on your own, you will find questions arising that you might well have brushed aside or tried to answer some way, any way, that would get you a good grade. (Many of these questions might be due to the simple fact that the textbook was not designed to enable students to teach the subject to themselves.) But now that the only person witnessing your learning efforts is yourself, you may find questions arising that you cannot brush aside in shame and that you do not want to simply find a workaround that, were you taking a class, would enable you to get a good grade on the next exam. And you will start to see how naive it is to assume that a grade is an indication of understanding. Perhaps, because you are not surrounded by the buzz of other students studying the same subject, with information in the air much of which you (and they) are not conscious of, you will begin to wish that you were able to look up things rapidly in the textbook, and that you could write down procedures for doing even elementary problems because you know that the kind of memorized cleverness that works on exams is soon at least partially forgotten. And these two desires are what originally gave birth to the Environment idea, which you will learn about in subsequent chapters. (An Environment is an organization of your math notes that is optimized for problem-solving efficiency — specifically, for rapid look-up, like a dictionary or encyclopedia. But in addition, each concept and term has a template imposed on it that makes it easy to rapidly find information related to that concept or term, whether the information is your additional notes and explanations for yourself, or passages in a textbook.)

In the course of your self-teaching, you will also begin to realize that it is not a question of knowing or not knowing, understanding or not understanding, but that there are degrees of each. Sometimes, later in life, all you may want is an understanding of a subject at the highest level — at a prose and picture level, with few mathematical terms. At other times, you may want merely the statement of a lemma, and where a proof can be found. At other times, you may want to understand a proof that is given in the textbook. At still other times you may want to come up with on your own a proof of a statement that is not proved in the textbook. Each of these degrees of understanding is useful depending on what your goals are.

Of course, we can't blame the universities for using a simple criterion for assigning a grade in a course, namely, the grades the student has earned on homework and exams. But we must also not allow this criterion to replace our own judgement regarding what we know and understand. In the last analysis it is not the professor who decides if you know, it is — it must be — *you*!

Finally, another result of your self-teaching experience may be the realization of how *presumptious* it is for someone, e.g., a professor, to presume to tell you how to learn something and tell you how fast you should learn it. Even though it is extremely inconvenient for the universities to accept it, the fact is that people are different. They do not all learn best in the same way.

Perfect Student

A good exercise to carry out at least once in your student life is to sit down and try to describe the behavior of the Perfect Student of mathematics courses. At the very least, since most textbooks have inadequate indexes (to put it mildly) and since there is never any mention in lectures about the importance of indexing one's notes, much less any mention of mathematics encyclopedias where terms and concepts can be looked up quickly, you will probably say that such a student has a phenomenal memory. What he or she learned in a first-semester course is remembered throughout his or her undergraduate years. The theorems in Calculus 202 taken in the first semester of his or her sophomore year are available for instant recall when these theorems are used (often without the slightest indication of where the student can look up the statement and the proof) in the second semester of his or her junior year. Perfect Student remembers just about everything from a course he or she took over a year ago. Otherwise the textbook and the professor would give an explicit reference, right?

How does Perfect Student go through one of the Schaum's Outline texts? Does he or she work all the problems? That would require writing the solutions almost as fast as the problem statements can be read. If not all, then which ones? How does the Student know which problems are the most important to work on? Why should the answer to this question require going to an expert in the subject matter? You can see here the emergence of the idea of just-in-time learning: sooner or later (given the rate of increase of mathematical knowledge) all of us (including students) who use mathematics for one purpose or another will have to settle for learning how to solve many types of problem when we *need* to solve them. Sooner or later we will have to face the fact that there isn't enough time in anyone's life to learn everything in order to be able to do something. Sooner or later, as you start developing and using Environments, you will come to believe, as I do, that there is a far more efficient way of learning mathematics — no, not *learning* mathematics, but rather: solving the mathematical problems you need to solve, when you need to solve them — than sitting in a classroom and having the professor, in effect, read the dictionary to you.

What Do the Best Universities Offer That the Others Don't?

At this point, you may start asking yourself what exactly it is that the "best universities" offer that the others don't. The best universities, of course, have the best reputation, which means that if you get a degree from one of them, you will be looked on more favorably by prospective employers or graduate schools, than are graduates of the other schools. No question about that. But what exactly do the best universities offer as far as your access to — your *use* of — the subject matter is concerned? In mathematics, the hard sciences, and engineering, these schools certainly do not offer better teaching — you get to be a professor at one of the best schools by virtue of your research, not your teaching. These schools certainly can not offer anything different as far as basics are concerned — the substance of calculus is the same no matter where it is taught or how. What's left? Well, they probably offer more advanced courses than the other schools. Let us grant them that.

But many readers will say that the best universities are also "tougher", they flunk out more students. But all this means is that they serve as a certain kind of filter: "If you can get through the university of *x*, you must *really* be good!" They have taken one of their principal tasks to be the weeding out of all but a certain kind of student. To which we may legitimately reply, "**Why haven't they taken as one of their principal tasks, the making as easy as possible, rather than as difficult as possible, the solving of problems in each subject?**"

Why Is Math Teaching As It Is?

Indifference of Professors to Improving Their Teaching

Mention the improving of textbooks, the improving of the learning process, to graduate students or to recent PhDs and they clam up. That is how successful the brainwashing that is a graduate education, really is. And yet no branch of mathematics would tolerate such conservatism regarding the branch itself.

Every math student needs to know one fundamental *fact* about the teaching of mathematics: among the vast majority of math professors, teaching is regarded as a necessary evil, a nuisance that interferes with what is important in life, namely, research. I have known mathematicians who considered it a sign of their brilliance that they were inept teachers.

In many colleges and universities, the principal reward for outstanding performance in research is the privilege of teaching fewer courses, even though the fundamental reason for paying a professor a salary is that he or she — teaches! Of course, these same colleges and universities make sure that their brochures — aimed at unsuspecting parents who want above all that their kids get a good education — their brochures emphasize the number of outstanding professors on their faculties. As one renegade professor said in a *60 Minutes* program, "It's only a matter of time before parents bring a class action suit for fraud against a university."

You would think that professors who hated the burden of teaching so much would make every effort to enable students to do the job pretty much for themselves — as a complete Environment (to be explained later in this book) would help them to do — but nothing could be farther from the truth. Instead, they invoke psychotherapists' logic: If the student learns, that is because the professor is a good teacher. If the student doesn't learn, that's because the student didn't try hard enough.

Why should you give the benefit of the doubt — when it comes to learning mathematics — to people with such values? Why should you be forced to keep paying the Professor Tax? *The math professors can't have it both ways*: they can't say, on the one hand, "Teaching is beneath our contempt, so we spend as little time as we can thinking about it, much less studying how to improve it", but on the other hand, "Only we can judge how mathematics should be taught."

Questionable Value of the Education PhD

But this is *not* an argument for the application of education theory, which I think is all but worthless. As a teacher friend of mine remarked, "Why is it that the country with the most Education PhDs per capita has one of the worst public education systems?"

"[education f]oundations...seem ready at the drop of a hat to spend millions of dollars on grandiose projects which produce, in the main, only publicity and doctoral dissertations."[1]

I once car-pooled with a young woman who was studying for a PhD in Education at one of the nation's most prestigious universities. What struck me about the education she was receiving was how little it had to do with learning to think straight about the problems in her field, and how much it had to do with learning *to use words* —specifically, the words that were currently considered important in her field. When she was confused about a problem, or when I told her I had difficulty understanding something she had said, her solution was to use more words — to speak faster, pack more words into her sentences, invoke still more jargon. You sensed that she had no idea what it means to "quiet the mind" in an intellectual sense, meaning, to think with as *few* words as possible, to think conceptually, "geometrically", to ask those all-important questions, "What do we really mean by this?", "What am I really saying when I say this?".

"...whoever thinks *words* is an orator, not a thinker; he betrays that he does not think *things* but only in respect of things; he really thinks only of himself and his listeners." — Nietzsche, Friedrich, *The Genealogy of Morals*, sect. VIII.

---

1. Holt, John, *How Children Fail*, Dell Publishing Co., Inc., N.Y., 1964, p. 45.

Her curriculum included a course in statistics, and several problems required that she do proofs. She hadn't taken a course in mathematics since high school, nor had most of her fellow students. Few if any of the students had ever been taught what a proof is or some of the techniques for constructing proofs. Her professors knew this. Nevertheless, they clearly felt it was far better that the education profession be carried on by researchers and high-level government bureaucrats who could be counted on to be wide-eyed with respect and wonder for anything called a "proof", i.e., something that is hard to understand, something you never know if you've got right, but something that is *mathematical*, and therefore terribly important!

One course required that she use a statistics computer program — a program which, as you might expect, was virtually impossible to use by anyone but a computer programmer. Yet the professor seems to have made a point of not telling his intimidated charges that not everything difficult is equally important — that a program that is difficult to use by members of its intended class of users, is an object of contempt; that real intelligence consists in part in knowing what not to waste your time on — in this case, knowing when to demand, or ask for, or hire, someone to translate the tasks you want to perform, into the arcane language required by a program. But for a person who has been trained to believe in language, in words, in particular in *prestigious* words like those associated with computer programs, such an idea seems nothing but a sign of weakness.

The students, of course, wanted high-paying, high-prestige government careers in the state capital and in Washington, and so they dared not question the value of their ordeal. The woman struggled with trying to figure out how to get the program to read in the data (Escape back-slash in a command seemed to help some of the time but not always). The university, being strapped for money (it costs a fortune to pay all those professors), couldn't afford to have a computer technician on hand all the time, so she somehow had to find him (when he was available) and somehow cajole him into revealing the secrets about getting the program to do what she wanted it to do. All this, mind you, parading under the title of "statistics".

At the same university where the woman was pursuing her PhD in Education, a recent PhD who was considered brilliant was about to be offered a tenure track when it was discovered that he had spent time sitting in primary school classrooms in order to gather first-hand experience of the kind of education that was to be his specialty. He was warned that he had come within a whisker of losing the tenure track position because the business of Education professors is theory, not practice.

(The best education reformer I know of — John Holt[1] — developed many of his remarkable insights through years of teaching primary school. He did not have a PhD.)

On the Length of Courses and of Degree Programs

Every math student needs to remember that the university has a vested interest in extending the amount of time it takes to learn a subject. In an article on the life of Gauss, Bell speaks of "the easy Latin which sufficed for Euler and Gauss, and which any student can master in a few weeks"[2]. Not once in my life have I heard a teacher say, to me or anyone else, "Look, the basic ideas of this subject (or this part of the subject) you can learn in a few weeks. Thereafter it's just a matter of developing skills and applying them." I certainly never heard it in a language course, or a mathematics course. In every course I have taken, the message, explicit or implicit, has always been the opposite: "Look, this material is hard, and only the special ones among you will be able to get a good grade. In fact, if too many of you get a good grade, I will have been derelict in my duties." Meanwhile, the computer industry takes an entirely different view of intellectual work. See the section, "Work Smarter, Not Harder", in this chapter.

Brief quiz: Suppose that, for each subject in mathematics, a pill were invented that enabled the taker to know as much about the subject as his or her professor did. Would this make professors (a) happy (b) sad?

So one answer to the question, "Why does it take as long as it does to learn how to solve a given type of problem in a given mathematical subject?" — in other words, "Why is the study and practice of mathematics so inefficient?" — one answer may be, "Because there is no incentive for mathematics professors to shorten the time, and in fact a considerable incentive for them not to."

I believe that the use of pre-existing complete Environments of the type described in this book, would enable students cut *by half* the time to pass an undergraduate course.

"Why Can't I Learn Calculus in 15 Minutes?"

Perhaps, in a moment of exasperation, you have wondered to yourself, "Why can't I learn calculus in 15 minutes?" The question is a good one, because it leads to the

---

1. See, e.g., *How Children Fail*, *How Children Learn*.
2. Bell, Eric Temple, "The Prince of Mathematicians" in Newman, James R., *The World of Mathematics*, Vol. 1, Simon and Schuster, N.Y., p. 300.

question, "If I can't learn *all* of calculus in 15 minutes, how much *can* I learn in that time?"

Suppose you were being sent on a mission in which the fate of the human race depended on your ability to solve textbook calculus problems. Suppose you were allowed no more than a 15-minute briefing in that subject, but you were allowed to bring any books you wished along with you on the mission. Suppose you had had, say, within the past five years, high school plane geometry, trigonometry, algebra, plus courses in elementary mathematical logic and set theory. What should the experts briefing you concentrate on, and what might they reasonably expect you to figure out for yourself from a certain kind of book or computer program they would give you?

A briefing is an attempt to give one or more persons a global view of some subject or event or set of circumstances as quickly as possible. The aim is not to fill out an hour or a week or a semester with a course ("Work expands so as to fill the time available for its completion."[1]) but to get across enough information so that each person present can get the details he needs on his own.

It is revealing to observe what some technical experts do when *they* have to learn a new subject. In 1986, Richard Feynman, the Nobel Prize-winning physicist, was asked to be a member of the committee investigating the explosion of the space shuttle *Challenger* which killed the seven astronauts on board. Feynman's first task was to learn as much as he could about the design and assembly of the shuttle and its rocket-boosters. Feynman was an outstanding physics teacher and lecturer; *The Feynman Lectures on Physics*, a transcription of his lectures to undergraduates in the early sixties, is considered by some to be the best undergraduate text on modern physics.

So what did Prof. Feynman do when he had to learn this new subject of space shuttle technology? He asked for a *briefing* by various shuttle engineers.

"It's the only way I know to get technical information quickly: you don't just sit there while they go through what *they* think would be interesting; instead, you ask a lot of questions, you get quick answers, and soon you begin to understand the circumstances and learn just what to ask to get the next piece of information you need.

I got one hell of a good education that day, and I sucked up the information like a sponge."[2]

One of the functions that a good Environment performs is that of enabling any user to obtain a briefing on the technical subject it covers — to enable the user to rapidly

---

1. Parkinson, C. Northcote, *Parkinson's Law*, Ballantine Books, N.Y., 1957, p. 1
2. Feynman, Richard P., *What Do* You *Care What Other People Think?*, W. W. Norton & Co., N.Y., 1988, p. 122.

find answers to the typical questions he might ask of one or more professors who had consented to play the same role as the shuttle engineers did for Feynman.

Ask yourself: “Why can't I read math on the bus or the subway?” The immediate reply from experts will be, “Because you can only learn math by doing it — by working problems,” to which your reply should be, “Is it really true that I can learn *nothing* of mathematics merely by reading, and if the answer is no, what can I learn, and how might a new presentation of the subject enable me to learn more, merely by reading?”

Consider the amusing, and useful, *The Cartoon Guide to Physics*[1]. I have no doubt that the authors could produce *The Cartoon Guide to Calculus*[2]. The reason I say this is that virtually all the *concepts* in, say, the first four semesters of calculus are easily presented as drawings such as are found in the better textbooks, e.g., Stein’s *Calculus and Analytic Geometry*[3]. Among these concepts are: the area under a curve, the slope of a curve in two and three dimensions, the area of plane and three-dimensional surfaces, the volume of various three-dimensional objects, the length of a curve, the integral over a curve, speed and acceleration of a particle moving along a curve, work performed by a force moving along a curve, etc.

The concepts in much of vector calculus can likewise be presented as drawings. *And should be*, because most students take the difficulty in understanding the new *notation* — e.g., that for divergence, gradient, curl, dot product, vector product and integrals using this notation — to be the difficulty of the concepts. But here again we have an example of how the confusion of the What with the How can make matters more difficult than they need to be. A concept can be quite easy to understand even though the deductions involved in performing tasks associated with the concept, are difficult.

How Long “Should” It Take to Learn a Given Subject?

The question, How long “should” it take to learn a given subject? is a meaningless question if you understand Environments. The real question is, or should be, How long does it take a student, or user of this Environment, to learn to solve problems of the class *X*?

1. Gonick, Larry, and Huffman, Art, *The Cartoon Guide to Physics*, Harper Perennial, 1991.
2. I have heard a rumor that Gonick will publish a book with this title in 2012.
3. McGraw-Hill Book Company, N.Y., 1973 (there is at least one later edition)

"Knowledge Is Not Where It's At!"

"The tendency to keep things secret, to make a cult and mystery of them, and so to gain an advantage over the generality of men, has always been very strong in men's minds."[1]

Never forget that in the academic world, *all there is is knowledge*. "I know and you don't" is what careers are built on. Every graduate student learns to fear above all else (except rejection by the PhD Committee), the raised eyebrow and withering glance that inevitably follows his or her revealing of ignorance of something that "everyone knows". And, of course, out of sheer instinct for survival, the student learns to play the game him- or herself. Look for the opening, the brief uncertainty, and *blamo*! That's another one for me and another one against you. Are you sure you're graduate material?

But it is not merely knowledge that counts, it is knowledge that you acquired from the proper sources, namely, the university. Self-taught knowledge is worthless knowledge. "If you didn't learn it from us, you didn't learn it." No matter how inefficient, how plain *bad*, a classroom course might be, if you sat through it and got a good grade, then you have a proof that you can show to others that you had learned something.

But (to coin a phrase) *knowledge is not where it's at!* In a world in which there are "1,500 mathematical research journals publishing some 25,000 articles every year (in over a hundred languages)"[2] it matters less and less *what* you know — how many definitions, formulas, lemmas, theorems, proofs you have memorized — and it matters more and more how rapidly you are able to *access* the knowledge you *need* when you need it.

The purpose of research is not to demonstrate your knowledge or intelligence. It is to solve the problem.

A Mathematical Subject Is Just a Great Big Table

A slogan of the approach to improving your math grades that is set forth in this book is, "Whatever *can* be made look-up-able, *should* be made look-up-able".

Most of the content of most undergraduate math courses is look-up-able. In fact, the content is really just a great big table. Take Linear Algebra as an example. The Great Man who will teach the course has spent the month of August preparing his lectures: weighing the words in which he will present linear transformations and

---

1. Wells, H. G., *The Outline of History*, Doubleday & Company, Garden City, N.Y., 1971, p. 173.
2. Stewart, Ian, *The Problems of Mathematics*, 2nd ed., Oxford University Press, N.Y., 1992, p. 19.

matrices and determinants and vector spaces and ... The students will copy down what he says in the classroom, and then do their utmost to solve now this homework problem, now that one, and then solve now this problem, now that one, on exams. The professor never once states what the essence of the course is: to become able to solve, not problems, but *classes* of problems. What the students should get for their time and money are written out procedures for solving classes of problems. That is the essence of the matter. For example, they should be able to look up, in a set of notes they can buy, or in the complete index for the textbook they are using, "linear transformation" and then get a list of sub-headings including the following ones, with a page reference where the instructions for each task[1] can be found:

Linear transformation
- Definition of "linear transformation"
- Examples of linear transformations
- Ways of representing linear transformations
  - Doing this with matrices

- Common operations performed on or with linear transformations
  - Find the composition (product) of two linear transformations
  - Find the scalar product of a linear transformation
  - Find the sum of two linear transformations
  - Determine if a linear transformation has an inverse
  - Find the inverse of a linear transformation if one exists
  - Find it two linear transformations compute the same function
  - Find what type a given linear transformation is
  - Get a list of all the types of linear transformations in undergraduate math courses
  - Find the effect on a given linear transformation of a change of base of a vector space

  - Get a list of theorems pertaining to linear transformations

  - Get a list of closely related subjects and concepts
- ...

"Whatever *can* be made look-up-able, *should* be made look-up-able".

1. "Task orientation is the great vacuum cleaner of mathematical presentation."

On Textbooks

The main purpose of this book is to introduce a new way of organizing the notes you take during math courses — a way that is optimized for problem-solving efficiency. Such a set of notes is called an "Environment". The organization of an Environment is designed for rapid look-up, like a dictionary or encyclopedia. But in addition, each concept and term has a template imposed on it that makes it easy to rapidly find information related to that concept or term.

The organization of an Environment is clearly different from the linear, logical order of the standard textbook and classroom course. But I am not attempting to do away with the traditional textbook! The student has a perfect right to want to see the subject presented in logical order. It is just that this order is not the most efficient for problem solving, as will be explained.

There is no incentive in the teaching and academic community to make traditional textbooks efficient from this point of view and there is a great deal of incentive not to. (See Morris Kline's *Why the Professor Can't Teach*, in particular Chapter 10, "Follies of the Marketplace: A Tirade on Texts".) Anyone who looks at textbooks, particularly textbooks in higher mathematics, from the Environment point of view, can't help but come to this conclusion. See "Appendix C — Two Notably Bad Textbooks" on page 64.

The Textbook Industry

Textbooks are normally written by professors who teach quarter or semester courses. "Work expands so as to fill the time available for its completion."[1] Years ago, having watched many World War II movies, I began wondering how technical courses and textbooks would change if it suddenly became necessary to train a large number of persons as quickly as possible in the various technical subjects, including mathematics. Do subjects really come in quarter or semester hunks? How much could be taught in how little time or, far more important, what would have to be taught so that the student would be capable of digging out whatever else he or she needed to know by himself?

Someone once asked in a discussion why the Chinese were never inspired to develop a more easy-to-learn writing system than they did. The reply was made, if you had spent a lifetime learning to write, you probably wouldn't want to hear about a method that enabled almost anyone to do it by age 10 or so.

Similarly, an education system run by individuals who were required to put in at least nine years (undergraduate and graduate school) before they were deemed to have

1. Parkinson, C. Northcote, *Parkinson's Law*, Ballantine Books, N.Y., 1957, p. 1.

mastered their subject, will not be eager to contemplate alternate approaches to the task of problem-solving that suggests that some of this time may have been wasted.

One often hears, in discussions of educational reform, that there is no quick road to learning mathematics, and I agree with what I believe is the deeper meaning of the statement. But we must also realize that (a) it is not necessary to be an accomplished mathematician — an expert in one or more branches of the subject — in order to use mathematics to solve problems, be they engineering or mathematical (b) there may not be a quick road, but there may be a number of quick*er* roads: I am encouraged to think there are if for no other reason than that the problem of teaching — but as the reader now knows, I really mean, the problem of using various technical disciplines to solve technical problems — the problem of access to tools, in short — that this problem has received little attention so far.

The Shamefully Incomplete Indexes in Textbooks

The reprehensible attitude, "I know, you don't", is beyond criticism in mathematics. This attitude carries over into textbooks, so that, if the great man who has deigned to sacrifice some of his valuable research time to write a *text*book, doesn't want to include a complete index, well, then there must be some deep reason for this. Clearly we only reveal ourselves to be even more unworthy than we thought in even *wishing* for a crutch, a *loser*'s aid, like an index. If you start on page 1, and study every page, and work all the problems, you won't *need* an index. This attitude is even more detrimental to learning when it comes to proofs. It is considered a mark of the textbook author's genius if he or she unpredictably leaves out justifications for some assertions, i.e., references to other lemmas or theorems. The best mathematician I ever knew personally used to shake his head over the textbooks of one well-known and prolific author, remarking that "sometimes it takes you an hour to find out the basis for some of the statements in proofs." He didn't say this with the anger he should have, but with a tone that said, "*That's* how deep and important his textbooks are!" I'm sure that the author's reply to all criticisms such as I am making would be, "My book is complete, and there are no errors in it. Therefore there is nothing wrong with it." To which I reply, *If all it is is not wrong, then it's wrong!*

The author's words are not a measure of his depth, but a measure of his incompetence as a textbook author. They say to the reader, "I need to believe that mathematics and, in particular, this subject, is an all-or-nothing proposition. Learn it all or don't bother."

Something of the same attitude is present among computer programmers, because in the computer world, as in the academic world, knowledge is power. A programmer who has been working on the same program, or system of programs, for, say, two years, may know a great deal about the program or system. He or she doesn't need comments in the program text to explain what is going on. And so he or she believes (wrongly) that certainly a few words should suffice to explain the programs to anyone worthy to be called a programmer.

All of which time-wasting nonsense can be combatted by frequent applications of the question of questions, "How much of what this person knows could be made look-up-able?"

The reason that indexes are so bad in the vast majority of textbooks is that for most authors, an index is something added in the last minute as a gesture of scholarly kindness on the part of the author. Often, the creation of the index is sub-contracted to another person, possibly a mere functionary of the publisher. This person, who may have only the most superficial understanding of the subject, and who may have long forgotten his own struggles to solve problems quickly in mathematics courses in the past, then simply "goes through" the text and picks words that seem to belong in an index. Hence the numerous serious omissions from these indexes.

For example, the index in Herstein's classic text, *Topics in Algebra*, has no entries for such basic terms as "algebraic over", "embed", "generator of ideal", "regular matrix", "singular matrix" (not even under "matrix, regular" or "matrix, singular"), "torsion", etc.

The masterful survey, *The Princeton Companion to Mathematics*, has no entries for such basic terms as "finite field", "Peano axioms", "quotient map", "space-time distance" (or "distance, space-time"), "ruler-and-compass" and many others.

Neither of these books has an index of symbols.

Another classic text, Hardy and Wright's *An Introduction to the Theory of Numbers* (Fifth Edition), has no index at all!

These inadequate (or missing) indexes cost the student an enormous amount of time over the course of his study, and force him or her to believe that, ultimately, you memorize everything, or it's hopeless. Which of course is not true.

Why I Don't Give Opinions On Textbooks Until I Use Them

Typeface, layout, and other characteristics of visual design, and yes, even prose style, are *as nothing* as far as problem-solving is concerned, compared to how fast the typical user can find out what he or she wants to find out. Yet speed of access cannot be measured by a mere few minutes' inspection of a textbook.

When someone shows me a new textbook, I try to be as honest as I can in my response: "It looks very nice, but I hope you will understand that I can't say anything more about it until I use it."

The final arbiter in all arguments about textbooks is, or should be, the student — or, more precisely, is, or should be, a test on randomly selected students. But at present the student has no say about the textbooks he or she is forced to use. I recall a physics professor who decided (commendably!) to write a physics text for undergraduate liberal arts students. Wherever possible, he used everyday objects to describe physical ideas. A student who had seen a pre-publication copy of the book made several suggestions, among them, that it might be a good idea if the book were tested on a representative sample of students. The professor became noticeably angry, apparently feeling that, after all the work he had put into the book, he was not about to go about making changes. His attitude is typical.

Nevertheless, it is the the height of absurdity for the worth of a textbook to be judged by professors in the field. In advanced subjects, not one textbook author in ten understands the difference between a clear presentation and a good tutorial presentation. A clear presentation is addressed to those who already know the subject; a good tutorial presentation is addressed to those who don't.

On the Shameful Lack of Adequate Drawings in Most Textbooks

One thing that self-teaching makes you realize is how much time you spend doing drawings. Let me begin by assuming you have elected to teach yourself a course in, or related to, the calculus, which has a strong geometric content. The professor normally does the drawings in class and then you copy them into your notes. But I put it to you: why shouldn't the textbook have these drawings? Why should it be necessary for you to go through registration, pay your tuition, endure the ordeal of signing up for the course, trudge across campus several times a week, and sit in a classroom just so you can get adequate illustrations for the material you are studying?

You can easily find examples of the negligence of textbook authors in connection with drawings. Two of the best examples can be found in R. Courant's *Differential and Integral Calculus*[1] on p. 59, concerning the total differential in the case of functions of one variable, and on p. 114, concerning the derivative of implicit functions of two variables. There is no question but that it is difficult to make the drawings for these pages show what they are supposed to show. So what? It is still the obligation of competent textbook authors, i.e., their publishers, to find draughtsmen who can make such drawings, and thus save the countless hours of students and professors having to struggle to make them anew in each course.

(Not only the lack of sufficiently many drawings, and the infuriating lack of justifications for many statements, but also the incomplete index, make it impossible

1. Wiley Interscience, two volumes, 1936, ..., 1968.

not to say: If ever there was a textbook that was job security for mathematics professors, it is Courant's text.)

Other examples of the irresponsible omission of drawings is Oliver Dimon Kellogg's *Foundations of Potential Theory*[1], also regarded as a classic.

At this point, let me hasten to mention the two best calculus textbooks I know of, namely, Morris Kline's *Calculus: An Intuitive and Physical Approach*[2], and Sherman Stein's *Calculus and Analytic Geometry*[3], the latter covering considerably more material than the former. Each book could use more illustrations, but even as they are, they far surpass the Courant and Kellogg texts.

Drawings can serve two purposes in a mathematics text: first, in a geometric subject, like the calculus, their use and value is obvious. But second, they also have a major use in courses whose subject matter is not at all geometric, e.g., group theory. Here, the drawings can be used, in the form of closed curves and arrows, to show domains and sub-domains of functions and how these are related to the range of the function. They can also be used to show relationship of sets. As in geometric courses, this use of drawings can greatly increase the student's speed of learning.

On the Shameful Lack of a Catalog of Graphs of Functions

It would be interesting to know how many hours the average math major spends, over the course of his or her undergraduate years, trying to find (or figure out) the answer to questions such as: "What does the graph of the function $y = f(x)$ (or $z = f(x, y)$) look like?" (What does the graph of each type of elliptic integral, and elliptic function look like? These are single-variable functions.) "What does the graph of the function named *u* look like?", where *u* might be, e.g., "catenary" or "folium of Descartes", or "hyperboloid", etc. Surely such graphs help enormously in the student's intuiton — her or her understanding — of the function.

What we might call the inverse questions are equally valid: "What is the general equation for the curve represented by $y = f(x)$ (or $z = f(x, y)$) — i.e., what is the equation that represents the curve when it is positioned at any location and at any rotation relative to the coordinates? E.g., what is the general equation for the ellipse with center at (x, y) and rotated 60 degrees from the horizontal?

There is absolutely no reason why there should not be a catalog of graphs of functions that provide immediate answers to these questions. Let the mathematics community devote a web site to this purpose. I am well aware that the vast majority of

1. Dover Publications, Inc., N.Y., 1954 (the book was first published in 1929)
2. Dover Publications, Inc., Mineola, N.Y., 1998
3. McGraw-Hill Book Company, N.Y., 1973 (there is at least one later edition)

mathematics professors will say that the student should either have memorized the answers to these questions from previous courses, or else be able to figure them out. Yet this same professor, if he writes a textbook, (usually) thinks it only common decency to provide a table of integrals in his book.

Another Reason Why Calculus Is Difficult

I know of no textbook that sets forth the *algebraic rules* governing dy, dx, dz, etc., and δy, δx, δz, etc. Instead, we get conflicting statements about the nature of these terms:

"...the derivative...is the quotient of the differentials *dy* and *dx*...[1]

"Leibniz's notation [dy/dx, etc.] suggests that the limit [that defines dy/dx] is a quotient, whereas the limit ... is *not* a quotient. Leibniz's symbol ... must be taken in its entirety."[2]

"...in the notation...dy/dx, the symbols *dy* and *dx* have no meaning by themselves. The symbol dy/dx should be thought of as a single entity, just like the numeral 8, which we do not think of as formed of two 0's."[3]

"...in many calculations and formal transformations, we can deal with the symbols *dy* and *dx* in exactly the same way as if they were ordinary numbers."[4]

The authors who proclaim that dy/dx is a "single entity" have no hesitation in later chapters that deal with lengths, areas and volumes, of writing, e.g., $(ds)^2 = (dx)^2 + (dy)^2$. But far more disconcerting to the student is that he or she is never told what is allowed and what is not allowed algebraically in the use of these terms. For example, if $2dx = 3\ dy$, can we cube both sides of the equation, and write $(2dx)^3 = (3\ dy)^3$. If not, why not?

If e is the base of the natural logarithms, can we write $e^{(dx)^4}$? If not, why not?

Can we write $\sqrt[5]{dy}$, and if not, why not?

---

1. Courant, R., *Differential and Integral Calculus*, John Wiley & Sons, Inc., 1970, p. 107.
2. *Kline, Morris, Calculus: An Intuitive and Physical Approach*, Dover Publications, Inc., Mineola, N.Y., 1998, p. 26.
3. Stein, Sherman, *Calculus and Analytical Geometry*, McGraw-Hill Book Company, N.Y., 1973 , p. 90
4. Courant, op. cit., p. 101.

Is it legitimate to write, e.g., $x^2\delta y = y\delta x$ and if not, why not?

In first-semester calculus we learn that if $dy/dx = x$, then we are allowed to write $dy = xdx$, and then to integrate both sides, that is, to write

$$\int dy = \int x dx$$

We learn how to evaluate such an integral.

Fine. But suppose we have

$$\frac{\partial^2 y}{\partial x^2} = x$$

Can we then write $\delta^2 y = (\delta x^2)x$ and then integrate both sides:

$$\int \partial^2 y = \int (\partial x^2) x$$

If not, why not, and if so, then how do we evaluate that integral?

If

$$U = r^2 \dot{\theta}^2$$

where

$$\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

what is

$$\frac{\partial U}{\partial \dot{r}}$$

where

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t}$$

Most — I would say nearly all — beginning students are plagued by these and similar uncertainties and yet dare not admit the fact even to themselves, much less raise the question in class. (An amusing and sympathetic description of one student's struggles in this area can be found in the chapter, "RPI", in John Franklin's excellent autobiography *Genius Without Genius* on thoughtsandvisions.com.) As a result, the students' nagging doubts about their ability to learn mathematics are only reinforced — "If the textbook doesn't give precise rules governing the use of these terms, that must be because the rules are *obvious to those who are meant to study mathematics*. Clearly, I am not one of those people." And so, once again, perfectly legitimate questions are suppressed by shame and fear.

Calculus Without Math

I have always been struck by how relatively easy the *concepts* in the calculus are, and how difficult the problem solving usually is. Let me list a few of these concepts. If you have had any calculus, you will be able to visualize the drawing that good textbooks provide to illustrate each concept.

• the concept of the slope of a curve at a point. (It is incredible that at this late date there is not a standard drawing, used in all textbooks, to communicate what is all-too-often expressed primarily or solely in symbols: I am referring to the approximations associated with the slope, all the $\in$ terms, the difference between df and Δf, the business about "linear approximations" and the rest of it. One good drawing, tested on new students, would save countless hours of struggle. That drawing will appear soon in this book.).

• the concept of the change in area under a curve as one moves from left to right, being related to the slope of the curve (rapid increase in area means steep slope). (The reciprocal relationship between change of area and slope of curve is the fundamental theorem of the calculus.)

• the concept that the maximum and minimum points on a curve are the points where the slope is zero;

• the concept of breaking down the area under a curve, and between upper and lower points on the x axis, into a sequence of narrow rectangles, then computing the area of all the rectangles, then increasing the number of rectangles to get a closer approximation to the area, etc.

• the concept of a volume being the sum of the volumes of thin slices through the volume;

• the concept of the length of a curve being the sum of the lengths of short straight lines connecting points along the curve;

• the concept of the area of a surface being the sum of little tangent planes all over the surface;

etc.

In advanced calculus and vector calculus:

• the concept of the sum of the "rays" (flux) emerging from each point inside a plane region, or a three-dimensonal region, being equal to the sum of the flow directly perpendicular to and away from the boundary of the region (Gauss' Theorem, Green's Theorem);

• the concept of the sum of the "circles" (curl) around each point of the surface of a cap being equal to the circulation around the edge of the cap (Stokes' Theorem).

etc.

I think that most students can accept these concepts as plausible, but once the students have to learn how to manipulate the daunting symbolism associated with divergence, gradient, and curl, things suddenly become very difficult. *Yet it is of fundamental importance in all technical subjects to separate the What from the How.*

The experiment should be made at least once of attempting to communicate as many of the concepts of the calculus solely with labeled drawings containing few or no mathematical symbols. Of course, the exercise would be largely a waste of effort unless a set of questions accomplanied each drawing. If students had difficulty answering the questions, then the drawing would have to be changed.accordingly. Since I detest the modern university practice of allowing liberal arts students to earn a bachelor's degree without having the slightest idea of the concepts of modern mathematics and physics, I would be very interested in seeing if such a set of labeled drawings might help these students acquire some idea of what the dread subject of the calculus is all about.

Why Are There So Many Textbooks in Each Subject?

Why are there so many textbooks in each subject, or at least in the basic college math subjects like calculus and linear algebra and statistics?

"As might be expected, publishers do what they can to undermine the used book market, principally by coming out with new editions every three or four years. To be sure, in rapidly changing fields like biology and physics, the new editions may be academically defensible. But in areas like algebra and calculus, they are nothing more than a transparent attempt to ensure premature textbook obsolescence." — Granof, Michael, "Course Requirement: Extortion", "Week in Review", *The New York Times*, Aug. 12, 2007, p. 10.

After all, it is not that the content is so much different in each of these books. What is the difference between them? The authors would probably reply, "the presentation", "the treatment", meaning not only that the explanations of concepts and proofs are different, but also that the drawings, and even the typefaces and margins are different. In the late eighties and nineties a reason for publishing new editions of some of the standard texts was the major breakthrough of — more color in the drawings and even in the text! Just think!

"...I came in, carrying the elementary physics textbook that they used in the first year of college. They thought this book was especially good because it had different kinds of typeface — bold black for the most important things to remember, lighter for less important things, and so on."[1]

The "they" Feynman is referring to were the students and professors at a Brazilian university, but the same superficial criteria for judging textbooks are no less common in American universities.

The question, "why are there so many textbooks in each subject", was one of the major inspirations for the Environment concept described in this book, because it led naturally to questions such as, "How do textbooks in a given subject differ?" "Why are these differences important, if indeed they are?" "What is the *content* of a textbook?" "How can the content of various textbooks in a given subject be compared?" "How much of the prose in a given textbook can be eliminated via better formats for presenting the material, e.g., structured proofs?" And so on.

A very worthwhile research project — though one that is unlikely ever to be carried out — would be to take the half-dozen best-selling calculus textbooks and compare them on a topic-by-topic, definition-by-definition, lemma-by-lemma,

1. Feynman, Richard, *Surely You're Joking, Mr. Feynman!*, W. W. Norton & Company, N.Y., 1988, p. 215.

theorem-by-theorem, proof-by-proof, exercise-by-exercise, sentence-by-sentence basis. Ideally, such a project would include student rankings of each textbook, but these could not be taken at face value, since the rankings would be affected by the quality of professors' teaching and by student abilities.

The Breakthrough Answer to Why There Are So Many Textbooks

The "Gestalt" Presentation of a Mathematical Subject

It took me years to realize that the *real* reason there are so many textbooks in each subject is not merely the greed of textbook publishers, or the greed and vanity of professors, but something much deeper: it has to do with what I will call the gestalt concept of a technical subject. The word "gestalt" is a psychological one derived from the German word for shape or form. It means, very roughly, the emphasis on "configurational wholes" instead of on the elemental analysis of stimulus, percept, and response. I use the word here because it seems to me to represent the "all-or-nothing" view that professors (and students) have of technical subjects ("you can't tell the time unless you know how the watch is built"). The professor wants the student to believe that by working hard, the student will eventually "understand" the subject, he will be able to create an understanding (a gestalt) in his head that will enable him to answer most or all of the questions on the final exam. All his studying should be directed toward acquiring this understanding, the professor believes. *Different treatments of the subject in all those textbooks are different attempts on the part of professors to enable students to build this gestalt of understanding.*

Gestalt texts are discursive. "Let f be a continuous function such that first and second derivatives exist. Assume that ... Then we know that ... and therefore it follows that... and so we have proved that ..." In a gestalt text ideas and theorems are developed in a prose presentation. The formal statement of the theorem typically comes *after* the development even though it is usually much more difficult to understand an argument if you don't know what the goal is. Statements are often made without justification, because the reader is assumed to have learned all the material up to that point, and thus is assumed to remember a statement, and its proof, that was given a hundred or two hundred pages previous. The textbook author is attempting to build an Understanding of the subject in the reader's mind, this Understanding being a Whole that the reader has within his or her mind and can call on during exams.

A perfect example of a gestalt text is R. Courant's *Differential and Integral Calculus*[1].

---

1. Wiley-Interscience, two volumes, 1936,..., 1968.

The Top-Down, Task-Oriented Presentation of a Mathematical Subject

You might ask: How else can it possibly be? The answer is: the subject can be presented as an Environment. An Environment is a top-down, task-oriented presentation of a mathematical subject. *Task-orientation is the great vacuum-cleaner of the presentation of mathematics*. Let me explain what this means using two subjects that students justifiably regard as difficult, namely, Galois theory.and algebraic topology. The Environment for each begins with the question, "What is this subject *for*?" , meaning, here, "What are the principal task(s) that this subject makes it possible to perform?" In the case of Galois theory the principal task is that of determining if a given polynomial $p(x)$ of degree ≥ 5 is solvable by radicals. In the case of algebraic topology, the principal task is that of determining if two topological spaces are equivalent, that is, are homeomorphic.

And so each Environment is organized as a procedure for carrying out the principal task. We regard the principal task(s) as being at the top level, which we can call level 0. Each level i (where $i > 1$) contains steps (tasks) for implementing the tasks at level $i - 1$.

In the case of Galois theory, level 1 consists of the task of finding a certain finite sequence of fields such that all the roots of $p(x)$ lie in the last field of the sequence. If no such sequence exists, then $p(x)$ is not solvable by radicals. Lower levels then introduce, when they are needed in the procedure, the familiar concepts of Galois theory: irreducible polynomials, extension fields, Galois groups, etc.

In the case of algebraic topology, level 1 consists of the top-level steps for determining homology and cohomology groups for each space, and then determining if the groups are isomorphic. If they are not, then the spaces are not equivalent (not homeomorphic). Lower levels then introduce, when they are needed in the procedure, simplexes, complexes, and the varieties of each type of group — with tree diagrams to show the relationships of all these groups!

The structure of an Environment is similar to that of a structured computer program. Proofs (analogous to sub-programs that implement functions in a program) are written in top-down, step-by-step form, as described in the chapter, "Proofs", and not in the paragraph form of the typical textbook and academic mathematics paper. In an Environment it is *not* necessary to know everything that constitutes the logical building blocks for something, in order to understand the something (this is the "bottom-up" approach of the traditional textbook[1]).

Furthermore, the statement of what is to be proved always comes *first*. You may have reservations about this, arguing that the theorem-proof format is an intimidating way of presenting math, but I ask you to compare your experiences. Also, the Environment format certainly does not forbid informal prose passages that, e.g., explain the history of the theorem or lemma, explain why mathematicians wanted to prove the theorem in the first place, and it certainly does not forbid brief, informal descriptions of the intuition underlying the proof.

The Environment concept is also based on the realization that all, or most, of a technical subject, is representable most concisely, most conveniently for problem solving, *in tables*. I am using "table" here in a broad sense. The Universal Template for Mathematical Entities that is used in an Environment, is a table. A structured proof is a table (see chapter, "Proofs"). A table is a picture of a technical concept, or of a relationship between related technical concepts. One argument for the validity of this view, I argue, is the crib sheets that students sometimes carry around with them, or even attempt to sneak into exams. (I can remember a time when professors would allow such sheets as long as they were less than a certain maximum physical size! I hope and pray that this kind of nonsense is a thing of the past.) In any case, crib sheets typically contain tables of one kind or another.

Don't reject the idea of the fundamentally *tabular* nature of technical subjects because it is so simple. It provides one of the answers to the fundamentally important question, "Why is mathematics difficult?." I said in the chapter with that title that one of the reasons is the difficulty of being able to rapidly look things up. And the reason why textbook authors are not concerned to make it easy to rapidly look things up is that these authors present their subject in prose aimed at helping the reader to build a *gestalt* understanding of the subject.

But let us take the next step, and recognize that a table, in the broad sense I am using the term here, *is a data base*! *There should be a data base for each undergraduate subject.* If you have difficulty accepting this, just consider undergraduate courses in Linear Algebra, and ask yourself what, in such courses, could not be represented in, and made accessible by, a data base.

More Than One Textbook in a Course? Yes!

Many students believe that their only hope of learning a given subject is to take the right course and then use only the required textbook (which sometimes happens to be one that was written by the professor). These students believe that if they are any good — if they are to have a future in any course of study that is mathematical — then they

1. James R. Munkres's *Elements of Algebraic Topology* (Addison-Wesley Publishing Company, Menlo Park, CA, 1984) is an excellent example of the traditional bottom-up approach to the subject — you have to learn everything about a great many difficult things before you can even begin to start to figure out the procedure for using these things to determine if two topological spaces are equivalent. (But, let me not fail to say that Munkres is among the best of the authors of traditional textbooks.)

will be able to get good grades solely by attending class, taking careful notes, and using the one textbook.

I am sure that one source of this belief is the whole notion of "the textbook" for the course. Understandably, the professor wants to use just one set of notations, and wants the textbook to be coordinated with his lectures, and wants to assign problems that he is familiar with. *However*, the truth is that explanations in different textbooks are often easier to understand than the ones in the course text. It is simply a fact of life. Yet some students feel that it is only a sign of weakness if they start wanting to look into other texts.

Not so! Whatever works, works. A student should never be reluctant to look at other texts, and to buy the ones he or she finds most helpful. Let me mention again what I consider are far and away the two best calculus texts, and to unhesitatingly recommend that any student who plans to study, or is already studying, the calculus, go out and buy them. They are Morris Kline's *Calculus: An Intuitive and Physical Approach*[1], and Sherman Stein's *Calculus and Analytic Geometry*[2]. They are available through www.abebooks.com.

Concise presentations of all the most important subjects in modern mathematics are given in the *Encyclopedic Dictionary of Mathematics*, which, along with the *Princeton Companion to Mathematics*, I would argue are must-haves for anyone intending to make a career in any branch of mathematics.

Popularizations and Histories? Yes!

The belief mentioned in the previous section — that if you're any good, all you should need to do is to attend classes and use the assigned textbook — in some students carries over into a prejudice against popularizations ("they're not *real* math"). And yet a good popularization can shed light where only darkness prevailed. There are several outstanding writers of popularizations of mathematics, among them Ian Stewart. You can also find brief introductions to subjects on Google, e.g., on Wikipedia.

A good history can also be a great help. The best one I know — and it far surpasses any of the others — is Morris Kline's *Mathematical Thought from Ancient to Modern Times*[3]. It is available through www.abebooks.com. Unlike other histories, which amount to little more than a collection of names, dates, and theorem statements — "who proved what when"— Kline's makes clear what *motivated* the various important subjects, concepts, and theorems he covers. In particular he makes clear

1. Dover Publications, Inc., Mineola, N.Y., 1998
2. McGraw-Hill Book Company, N.Y., 1973 (there is at least one later edition)
3. Oxford University Press, N.Y., 1972.

how much mathematics, at least up to the 20th century, was motivated by questions arising in physics and engineering. He has none of the 20th century mathematician's shameful prejudice against applied mathematics.

And since we are on the subject of history, let me state in no uncertain terms that I think it is *disgraceful* that elementary calculus texts do not explain why Newton invented (discovered) the calculus! The students know that Newton set forth the theory of gravity, and some of them may know that he did so in order to explain Kepler's three laws governing the orbits of the planets. But the students are never told what made him realize that he needed a mathematical mechanism that could deal with instantaneous rates of change. In my opinion the students would benefit enormously by being given a few quotes from Newton's masterpiece, *Philosophiæ Naturalis Principia Mathematica*, with, of course, explanations of the text in modern terms.

It is equally disgraceful that elementary texts do not explain why the relationship between the derivative and the integral makes sense. The texts give the definitions and derivations, but I doubt if any students could respond correctly to the challenge, "Show, using diagrams of curves, what the derivative is in terms of area increments." (This is shown in "Appendix A — The Derivative and the Area Under a Curve" on page 57.)

A Note to Middle- and High-School Teachers

Even though this book is aimed primarily at students of college math, I want to emphasize that the Environment concept can be taught to middle- (jr. high-) and high school students, also. In particular, the writing down of the steps needed to solve a problem can be made a practice from the start. It takes training to learn to think procedurally. "I think, in fact, that a generation of children is learning at least some of the art of programming, a mental discipline far richer and more useful than all the algorithms of traditional school arithmetic." — Gleick, James, "Books: Net Losses" in *The New Yorker*, May 22, 1995, p. 88.

Students can, in this way, learn the rudiments of programming. They can be told, truthfully, that this is how programmers make computers work. So they kill two birds with one stone. It is not an either/or proposition. Students can be taught how to keep notes alphabetically (as in an Environment), so they don't have to spend so much time searching.

To explain the concept of operations, or tasks, on things, the teacher can use cars, bikes, various food items, as things. What are the common operations we perform on a bike? Ride it, park it (and lock it), clean it, repair it. What are the common operations we perform on fractions? Reduce them to lowest terms, add them, subtract them, multiply them, divide them. What are the common operations we perform on eggs? Fry them, scramble them, poach them, beat them so we can put them in cakes, etc. What are the common operations we perform on equations? Convert them to standard form, multiply through them, divide through them, move terms from one side to the other, solve them.

As far as the types of problems used in class are concerned, I believe that whatever works, works. I sometimes see the students in the high school a few blocks from my home playing craps on the sidewalk. I think: are the math teachers aware of what a great opportunity this gives them to teach probability? Begin by saying to the class, "Two games. In one, the player's point is six. In the other, the player's point is four. Which player is more likely to win, and why?" (It would provide a valuable opportunity to see how accurate the students' intuitions about probability really are.) In the mid-nineties, a set of high school exam questions circulated on the Internet in which drugs and guns were the subjects of the arithmetic. I believe that the teacher who attempted to use these in class was sternly reprimanded. I think that's a shame.

Another goal of jr. high and high school math teachers should be to teach students to know what they don't know. In my occasional tutoring of students who are in these years of school, I am always amazed at their uneasiness at being asked questions about what exactly they are finding difficult within a given part of the subject. Clearly, learning how to clearly identify what you don't know is one subject about which no self-respecting student ever speaks. Students are invariably ashamed that they don't understand, and that is the beginning and end of the matter.

Finally, let me propose a technique for generating interest in learning math. Why not have one class challenge another? Whichever class does best will get, say, free pizza. The winner is decided on the basis of the *total point score* for all members of the class over the entire semester, so that, e.g., if the entire class gets a C, that would result in a higher score than if a few get As and the rest Ds and Fs. This might then motivate the good students to help the less good ones.

"Work Smarter, Not Harder"

There is a peculiar myth that the best students are those who spend the most hours on their homework: a student who does 20 problems is twice as good as a student who does only ten. A student who stays up half the night doing homework is better than one who doesn't. A famous computer scientist increased the awe in which he was held by many in the computer science field when he remarked that, in his student days, he had worked every problem in a well-known undergraduate math text.

But if the world is to make any progress in improving the problem-solving ability of mathematics students, this prestige of sheer quantity of work must be called into question. In the 1970s, several companies in the electronics industry promoted the slogan, “Work Smarter, Not Harder”, a slogan that should also be applied to the study of mathematics and other technical subjects. The Environment idea developed, in part, out of the belief that what a student learns — namely, one or more successful heuristics — after working all, or many, of a given type of problem can and should be committed to paper (or word-processor file); that the writing down of these heuristics, for later use, is the main goal of the activity of solving problems!

Here as elsewhere in mathematics, the question is not “Should students (and mathematicians) exercise their brains or not?”, the question is, “Should students (and mathematicians) have a choice?” Jogging is great exercise; it promotes vigor and long life, but that doesn’t mean we should jog to work and back and to all the other places we want to go. Similarly, doing calculations in your head and trying to solve problems without pencil and paper is good exercise, but that doesn’t mean we should always limit ourselves to these forms of problem solving. “...not all hard problems are good...”[1] Sometimes, many times, it is simply more efficient to use a calculator. The argument that most great mathematicians could do extraordinarily complex calculations in their head and that *therefore* this ability is a reliable indicator of mathematical talent is not under discussion here. In fact, the argument is invalid.

“Such a talent is, in reality, distinct from mathematical ability. Very few known mathematicians are said to have possessed it: one knows the case of Gauss and Ampère and also in the seventeenth century, Wallis. Poincaré confesses that he is a rather poor numerical calculator, and so am I.” — Hadamard, Jacques, *The Psychology of Invention in the Mathematical Field*, Dover Publications, Inc., N.Y., 1954, p. 58.

Nor are we discussing how great mathematicians go about studying their subject.

If you believe that universities have the slightest concern with promoting the problem-solving efficiency of students, imagine that the nation became involved in a war in which victory depended on the number of correctly solved math problems that high school and college students produced each year, with partial credit being given.

---

1. Stewart, Ian, *The Problems of Mathematics*, 2nd ed., Oxford University Press, N.Y., 1992, p. 12.

In other words, just for the moment, imagine that the nation's students suddenly needed to become a kind of giant problem-solving factory.

Do you think that courses would remain as they are? Do you think that teachers would continue to promote the idea that each student must learn "for him- or herself" — in other words, that there is nothing in common between what two different students in the same class do in the process of attempting to solve the same problem, or problems in the same class of problems? Or do you think that perhaps this idea would come to be recognized as, among other things, job security for teachers?

At the very least, don't you think that teachers would start teaching students how to organize their work, start teaching them about priorities, about how to decide what to work on when it is clearly impossible to do everything?

The Proper Place of Speed in Solving Mathematics Problems

Mathematics teachers, and, through them, their students, believe that *getting the answer fast* (which means getting the answer faster than anyone else in the class) is what separates the students with a future from those without one. The same belief operates in intelligence tests, and I have often wondered what the difference in intelligence is between a person who works intelligence test problems rapidly, and someone who works these problems slowly but then devises a computer program or a heuristic for working *all* of one or more *classes* of such problems faster than any human can.

I think that encouraging (or forcing) students to learn to work math problems rapidly without at the same time helping them see the necessity of *first* writing down the heuristic or algorithm they use to solve those problems, does far more harm than good. It is very close to being a waste of time in the long run. What is important is the heuristic, the method — what you can look up days, weeks, months, years from now and thereby very quickly regain your ability to solve that class of problem. *That* is what is important. Not the somehow-acquired, soon-to-be-forgotten, ability to knock out problem solutions quickly — an ability which is, after all, really just the unconscious, or semi-conscious, acquiring of a heuristic that exists (briefly) in the student's mind only. A well-known computer scientist once remarked in an interview that as an undergraduate, he had worked all the problems in a certain difficult calculus textbook. Most persons who read the interview, especially students, were awed by such an accomplishment. They shouldn't have been *if* all that he did was work the problems and get the right answers. If, in addition, he had kept track of the heuristics and algorithms he used, and then published them, thus possibly saving countless other students the labor of having to re-discover them for themselves — well, now, *that* would have been something.

Importance of a Written Procedure for Doing Integrals

The calculus is a particularly good example of how the emphasis on speed in problem solving makes a subject more difficult than it needs to be. You can prove this for yourself by attempting to write down a heuristic for just about any commonly performed task in the calculus: evaluating an integral, for example. (It is an absolute disgrace that the development of such an explicit heuristic is never even mentioned in calculus texts[1].) When you undertake this task, you will see just how much is swept under the rug by textbooks *and* by professors in their classes. The justification is the perennial "The student is expected to figure it out for him- or herself." The trouble is that the vast majority of students have no clear idea what "it" is. They usually think "it" is some combination of intuition, memory, cleverness, speed at doing algebraic manipulations, and, most important, luck. The idea of writing down a sequence of steps which most of the time will produce the right answer, seems, if anything, like something that losers — students who "don't get it" — do. In fact, I have come to believe that the main reason why an explicit heuristic for doing integrals is all but ignored is that skill in doing integrals is one way that the mathematics Establishment can separate the winners from the losers. So why do away with this useful tool?

I used to believe that devising such a heuristic is a challenging intellectual task in itself for students, and worth the time spent. But having made repeated attempts to come up with such a heuristic, I now believe that the task is too difficult and time consuming to impose on each student. I now believe that the calculus professoriat should work toward coming up with a reliable heuristic, *extensively tested on students*, which would then be incorporated in all new texts.

I say without hesitation that *one way* a person can become really good at mathematics is by learning to develop and write down heuristics and algorithms — by learning how to do *as little* as possible the *next* time he or she comes back to solving that type of problem. How do you know when you have worked enough problems of a

---

1. Algorithms — computer programs that always yield an answer — have been proved to exist for finding derivatives. However it has also been proved that no algorithm exists that will be able to evaluate any integral. But heuristics s do exist for evaluating certain sub-sets of integrals. By a "heuristic" I mean a procedure to be carried out by a human — a procedure that will help the human evaluate an integral most of the time. The only one I have ever come across in a calculus textbook is given in the section "What to Do in the Face of an Integral", p. 362 ff, in Sherman Stein's *Calculus and Analytic Geometry* , Fourth Editition, McGraw Hill, N.Y., 1987. This textbook, incidentally, is the best calculus text I have ever come across.

given type? When you have discovered, and written down, a heuristic for solving that type of problem, which, you have satisfied yourself, really works. (A good slogan to keep in mind as you develop Environments is, "Not problems but *classes* of problems!")

How many students or, for that matter, professors, do you know who could give you a procedure for evaluating integrals, a procedure which, though it may not be the most efficient such procedure, will, nevertheless, guarantee that you will be able to compute any integral that occurs in a homework or exam problem in, say, the first four semesters of calculus? Such a procedure begins with breaking the given integral into a sum of simpler integrals (if possible), then gives you a rule for going through various tables in order to solve the individual integrals. However, make no mistake: the procedure I have in mind is far more than simply the rule, "Use the tables in the back of the book." In particular, it tells you when, e.g., it is time to use partial fractions, when it is time to do integration by parts, in fact, when it is time to use each of the techniques you learn. (There may be many such times, of course.) Developing and (then saving!) such a procedure is what "learning to do integrals" really means. Ask yourself: in the long run, which is likely to be more lasting: (a) my memory of how to do integrals based on all the exercises I did, plus, of course, the textbook, or (b) the procedure I developed as I used it, plus the textbook?

"If you can't describe what you are doing as a process, you don't know what you're doing." — W. Edwards Deming

(Let me caution you not to be discouraged by the difficulty of the task of writing down a heuristic for evaluating integrals. It *is* difficult. Furthermore, you will never be finished with the task, because new kinds of integrals will always be occurring in your continuing studies of calculus. But the attempt is definitely worthwhile, and no matter how incomplete your heuristic is, it will improve your ability at evaluating integrals.)

If you are inclined to reply, "Well, yes, a procedure for finding integrals might be useful, but a motivated student shouldn't need procedures for the many elementary tasks that are performed in the calculus."

To which I reply, "OK, let's see how long it takes you to write down a procedure for determining maxima and minima of curves $y = f(x)$ — specifically, for each interval (range of x) specified in each problem, a procedure to:

find if the curve has critical points (one or more relative maxima and minima) in the interval, and, if so, what their coordinates (x, y) are;

find if the curve has an absolute maximum in the interval and if so what its coordinates (x', y') are;.

find if the curve has an absolute minimum in the interval and if so what its coordinates (x'', y'') are;.

find where (if anywhere) in the interval the curve is increasing, and where (if anywhere) the curve is decreasing;

find the inflection points (if any) of the curve in the interval.

Needless to say, your procedure will need to contain a procedure for curve tracing.

But the calculus is not the only subject in which procedures are worth developing. For example, write down the most efficient procedure you can discover for determining if a permutation over a finite set is even or odd. (Obviously, the procedure should be more than a mere quoting of the definition of even and odd permutation.) Your procedure should explain how to determine the sign of a term in the expansion of a determinant.

So what does this have to do with the question of speed in solving mathematics problems? The answer comes from computer programming. In the early years of programming, programmers worried primarily about the speed of their programs; they considered their primary task to be that of writing a program that would produce answers as fast as possible.

Gradually, however, it was found that a much better approach — one that produced fewer errors — was to worry about correctness *first* and only *then* worry about speed of execution. (Separation of the What from the How.)

Many, perhaps most, math students, are in the same state of mind as those early programmers: they believe that speed and getting the correct answer are somehow one. Every time they try to solve a problem, they actually try to solve two problems simultaneously. An Environment separates these two problems. Developing Environments will help you once and for all get rid of the idea that to know is to be fast and that not to know is to be slow, and replace it with the much better idea that first comes knowing how to solve a type of problem and then, if it is important to you, comes speed.

Against Competitive Mathematics Exams

This will be a controversial section, I know, but my opinion has some precedent in the opinions of at least one great logician and one great mathematician. My opinion is simply this: that competitive exams such as the United States of America Mathematics

Olympiad (USAMO) should not be regarded as the main criterion by which a hopeful mathematican judges his or her ability.

"I was encouraged in my transition to philosophy by a certain disgust with mathematics, resulting from too much concentration and too much absorption in the sort of skill that is needed in examinations. The attempt to acquire examination technique had led me to think of mathematics as consisting of artful dodges and ingenious devices and as altogether too much like a cross-word puzzle." — Russell, Bertrand, *The Basic Writings of Bertrand Russell*, Simon and Schuster, N.Y., 1961, pp. 57-58.

G. H. Hardy, one of the great mathematicians of the 20th century, sought to change the Tripos system of difficult mathematics exams at Cambridge University.

"He was a secretary of the committee [at Cambridge University] which forced the abolition of the order of merit in the Mathematical Tripos through a reluctant Senate in 1910, and many years later he fought hard for the abolition (not reform!) of the Mathematical Tripos itself, which he considered to be harmful to mathematics in the United Kingdom." — *The Princeton Companion to Mathematics*, ed. Gowers, Timothy, Princeton University Press, Princeton, N.J., 2008, p. 798.

There have been other top-rank mathematicians (e.g., J. E. Littlewood, who ranks with Hardy) and logicians who did not think competitive exams should have the prestige that is usually accorded them — or, at least, who did not think that the huge effort required to perform well on these exams was justified.

Littlewood's Miscellany[1], a delightful book, contains many pages about the exam system in Cambridge University in the early and mid-20th century. When we read these pages, we cannot escape the impression that the major concern of the mathematical establishment is to make sure that only those who fit a narrow and demanding set of criteria are allowed to do mathematics.

But suppose you like mathematics and want to spend as much of your life as possible working on difficult problems and, hopefully, in the process discovering at least one or two beautiful new ideas. What do you need? (1) rapid access to the knowledge that you need or think you need; (2) someone to talk to about your work; (3) a way to register your discoveries with the mathematical community (normally, this is done by publication in refereed journals — journals that, needless to say, must

1. Cambridge University Press, N.Y., 1986.

be willing to give a fair hearing to your work when you submit it!). That's all! Whether you are gifted or not will be determined by the work you produce. The opinions of experts (who have their own prejudices) as to whether or not you should be *allowed* to do mathematical research is irrelevant.

My objection to the exams is also motivated by the fact that I do not believe that, in a world in which more than 200,000 new theorems are published each year, the kinds of skills required to perform well on the above competitive exams, are the most important mathematical skills. Far more important are the skills involved in creating and using Environments (which are described in this book), and, though far more difficult to measure than exam grades, the skills involved in thinking creatively about unsolved mathematics problems. I certainly have no wish to see the exams abolished, and in fact I spend a fair amount of time each year trying to hire those who have performed well on such exams, to read my papers.

If you read the pages about exams, and indeed about doing mathematics in general, in Littlewood's book, and then spend a little time understanding the main ideas in this book, I think you will begin to feel, "What an enormous amount of *manual labor*!" By "manual labor" I mean the brutal courses-plus-exams process and the huge memorization demands these entail —

"Einstein claimed never to memorize anything which could be looked up in less than two minutes." — "Albert Einstein Anecdotes", Google, 9/22/18.

— the struggling through mathematics books and papers that are written in the same format as Euclid used 2300 years ago (start on page 1, learn it, then go to page 2, learn it, ... — the "all-or-nothing" approached described in the next section), the shamefully inadequate indexes and missing tables of symbols that are a direct consequence of the all-or-nothing approach, the presenting of proofs in paragraph form, so that you are forced to understand everything up to each point in the proof before you can proceed to the next point (unlike structured proofs as described in the chapter, "Proofs"), the medieval approach to teaching mathematics (see "Academic Math Teaching — A Relic of the Middle Ages" on page 13), and, finally, the neglect of the computer to aid in *rapid access to mathematical knowledge*.[1]

The words of David Brooks are appropriate here:

---

1. Having said that, I must also add that I think it would be worthwhile for the Artificial Intelligence community to attempt to write programs to solve at least some of the problems on competitive exams. (In this book we call attention to the importance of *classes* of problems, not merely problems.)

"Think about the traits that creative people possess. Creative people don't follow the crowds; they seek out the blank spots on the map... Instead of being fastest around the tracks everybody knows, creative people move adaptively through wildernesses nobody knows. [One thinks of Galois here.]

"Now think about the competitive environment that confronts the most fortunate people today and how it undermines those mind-sets.

"First, students have to jump through ever-more demanding, pre-assigned academic hoops. Instead of developing a passion for one subject, they're rewarded for becoming professional students, getting great grades across all subjects, regardless of their intrinsic interests...

"Then they move into a ranking system in which the most competitive college, program, and employment opportunity is deemed to be the best..." — Brooks, David, "The Creative Monolpoly", in *The New York Times*, Apr. 24, 2012, p. A21.

The "All-or-Nothing" Approach to Learning Mathematics

The previous section prompts us to ask, "How do outstanding students in a subject hold the subject in their minds?" At present, I believe that the answer is, "They hold it as though every frequently-occuring symbol, term, expression, plus every named concept, plus the statement of every important theorem and lemma, and at least the outline of the proofs of these, were thoroughly *cross-indexed* in their mind." I think that if you ask a star performer in the calculus what he or she thinks of when you say "π/2", he or she will not merely reply that it is a radian measure equivalent to 90 degrees, but will also cite various *locations* in which the term appears, e.g., as the value of the Wallis product,

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2m}{(2m-1)}\frac{2m}{(2m+1)}\cdots$$

and as the value of the Dirichlet integral,

$$\int_0^\infty \frac{\sin x}{x}\ dx = \frac{\pi}{2}$$

and numerous others.

And similarly for frequently-occurring terms like $1 - x^2$, $1 + x^2$, $\frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}$, and $\int_0^\infty \ldots$, among many, many others.

As far as the star performer is concerned, the subject is “flat” — things are all at the same “distance” from one another, the subject is “all one”. If you ask such a student about the structure of topics in the subject, he or she will be surprised at the question, and in the case of the calculus will probably reply with words to the effect, “Well, let’s see: it’s broken down into stuff about derivatives and stuff about integrals — in two dimensions and three dimensions — and then about applications of integrals, e.g., in computing areas and volumes.”

For such a student — and for the professors who teach the courses he or she takes and write the textbooks he or she studies — learning a subject is an all-or-nothing proposition. The student starts on page 1, studies it until he or she understands it and has memorized the important initial concepts and definitions, then moves to page 2, does the same, as well as studying and memorizing the statements of lemmas and theorems and their proofs, as they occur, etc., then works all the assigned problems at the end of the chapter, then moves on to chapter 2.

This explains why textbook authors often do not give references to justifications for statements they make and why the authors typically devote such little time and effort to indexes. If the student doesn’t remember the statement that he or she certainly must have proved in an exercise 100 pages previous, then that is the student’s problem.

There is a similarity here to learning a language in school. The student is taught that he or she must start “thinking in the language”, and so what the successful student does is memorize not only rules of grammar but also the meanings of numerous words and phrases until these are all “flat”, in the sense that he or she less and less has to figure out how to say what he or she wants to say. It is simply “there”.

In a mathematical subject, “learning the language” means learning how to rapidly solve problems in the subject. To do a proof is to “say” something in the language that is the subject. Students who have mastered a subject in this way often reply, when asked how they were able to rapidly solve a given problem, or type of problem, “You just have to know.”

Now as far as mathematics is concerned, all this might be well and good if unlimited time were available to the student and to the professional mathematician. But at present something like 200,000 new theorems are being published each year, and so we must ask of the “all-or-nothing” approach, “How much time does the typical star performer have to spend to achieve his mastery of the subject?” “How much time does the all-or-nothing approach *cost*?” This means not only classroom time, but homework time including time spent trying to figure out solutions to problems, whether or not that time is spent sitting at a desk. Is such an expenditure of time reasonable in an age of such burgeoning mathematical knowledge?

This book argues that the answer is No, and then presents a method of organizing your notes that, I believe, will shorten the time you require to solve problems — not necessarily to become a star performer, but at least to get a good grade.

Living and Working in a World of "Too Much" Knowledge

The "too much" in the title of this sub-section means "too much for any one human being to absorb, to have at his or her fingertips."

"An acquaintance of mine once told me that, during a conversation with a scientist at a leading university, the scientist remarked that it is sometimes easier to redo an experiment than to look it up in the literature. And yet certainly one of the reasons why scientists publish the results of their work is to save others from having to duplicate that work.

"We are inclined to say that there are information systems so complicated that they begin to 'fold over on themselves', meaning ... that, for their users, [the systems] begin to look the same as they were at an earlier stage of their history. (E.g., scientific knowledge at a stage when it was extremely unlikely to find the answer to your question in the literature.)" [1]

"Consider the evolution of structures such as bureaucracies, bodies of knowledge, telephone and road systems. Initially, the amount of communication relative to the capacity of the connecting lines is small. You can accomplish what you want any time at the maximum transmission speed of the lines. You seldom have to wait for someone else to finish using the lines. Thus, in the case of bodies of knowledge, it is easy to find out what has already been done: it would be foolish *not* to try to find out, since the cost of the search is so much less than the cost of duplicating the experiment.

"As the system evolves, it grows in size and number of users. You must plan for the use of the system; maps are required..." — ibid., p. 165.

An Environment, with its Big Picture(s) of the subject domain, is an example of such a map.

The explosion of knowledge forces us to think about new ways to access — *to organize* — knowledge, even the most prestigious forms of knowledge, such as mathematics.

1. Schorer, Peter, *Shaving With Occam's Razor*, Occam Press, San Jose, CA, 1985, p. 175.

Although I think that the gross inefficiency of classroom courses must eventually dawn on one or more mathematicians, I will not say that the old world of classroom courses is becoming obsolete, because these relics of a bygone age are the only way that universities can continue to sell credentials, and credentials are the only means the professions have of judging who is eligible to join them and who isn't. Nevertheless, at least in industry, engineers and programmers often need to apply *parts* of a subject which they have never studied before. Initially, they don't even know how much of the subject they will need to know. Clearly, Environments are needed.

What will universities be like in an age of Environments? Students will buy an Environment for each subject and improve it during the course. They will emerge from their undergraduate and graduate studies with a collection of Environments which they will keep throughout their lives, thus keeping what they have learned at their fingertips at all times.

Theoretical papers *could* be miniature Environments, designed for maximum speed of understanding of the central ideas and their application. But I am afraid that it will be a long time before such an innovation in the format of theoretical papers will begin to take hold: there is still too much prestige and job security to be gained by keeping papers difficult and in fact understandable only by specialists.

"You Don't Have to Learn (Memorize) What You Can Rapidly Look Up!"

I was well along in my math studies before the question occurred to me — before I allowed the question to occur to me — "Why do I have to learn (memorize) all this stuff?" The answer was immediate: "So I can solve problems!" But I then thought to myself, "It is true that having all the material in your head is one way of having it accessible when you need it to solve a problem. But another way is to have alphabetically-arrange notes (called an *Environment* in this book) that would approximate a complete index so that you could rapidly look up almost anything you needed!" Such notes would contain every term belonging to the subject, plus every symbol, that was not in the textbook index. But it would also contain, e.g., in the subject, Linear Algebra, entries like

equations, linear systems of, how to solve

Immediately following this heading in the Environment would be the *procedure* for solving such linear systems of equations that the student had written down. This would be in keeping with one of the mottoes for Environments, "Not [solutions to] problems, but [solutions to] *classes* of problems!"

"Life Is an Open-Book Exam"

I believe that, wherever possible, one's education should reflect the world in which one will use that education. It should therefore be clear from the previous sections that I do not consider closed-book exams to be reflective of the world in which graduates will use the mathematics they have learned. On the job, say, in industry, it would be absurd for an employee to say to him- or herself, when required to perform a certain integration, "I know we covered that kind of integration in that sophomore calculus course I took ten years ago, but I am not exactly sure now of all the details involved. Well, I guess I have no choice but to go to the boss and tell him that, since I am not certain of being able to solve the problem correctly from what I remember, honesty compels me to leave the company."

Life is an open-book exam. What school should first and foremost tell us is where we can find what we need to know to solve various problems. (That is precisely what the Environments described in this book do.)

A Forbidden Thought About the PhD Process

If you are planning to get a PhD in mathematics, or if you are already doing so, and if you read just the next chapter or two of this book, sooner or later, no matter how hard you fight it, the question will obtrude itself regarding the PhD process, "How much of the knowledge I am told I need to acquire through courses and seminars is, or could be, made look-up-able?"

You might then find yourself asking, "What is a PhD?" Answer: It is proof of original work in a subject, in this case, mathematics. And then you might ask, "How does the current PhD process dictate that this goal be achieved?" Answer: It dictates that you take all the undergraduate courses required for an undergraduate degree in mathematics, then that you take all the graduate courses deemed to provide necessary preparation for the PhD research.

You might now begin to reflect on the assumption underlying the claim that you need to take courses, namely, the assumption that (1) you can only get the knowledge you need by sitting in classrooms or seminar rooms; (2) once you have acquired the knowledge, you will have it accessible for use in your thesis and, possibly, for use in courses you will later give, or act as T.A. for. You load up on the knowledge so you will have it on hand when needed. You are a vessel that needs to be filled.

But now you may reflect on reality, namely, that you don't literally remember everything you were ever taught in every math course you took. (The vessel has leaks.) What you have forgotten, you look up in the appropriate text.

But then you will almost certainly ask yourself if, once you have reached the mathematical maturity represented by a bachelor's degree in mathematics, you really are incapable of finding the knowledge you need on your own. For a PhD thesis, you certainly do not need everything you have been taught. Far more important than having done lots of homework exercises and taken lots of exams, is *knowing where the ideas are*. Imagine a directory of mathematical ideas — ideas like discrete vs. continuous, limits, the ideas underlying the fundamental theorem of the calculus, and Galois theory, and the techniques for finding lengths of lines, areas, volumes as taught in elementary calculus; ideas like those underlying topology (e.g., "nearness" in a sense more abstract than that contained in concepts of metric distance), ideas underlying group theory, modern algebra, set theory, logic, proof, elementary number theory, probability, and those in at least some of the more advanced subjects, e.g., tensor calculus. Not lemmas and theorems and proofs, just *ideas*. Then, in your thinking about your research topic, you might say, "I seem to be wrestling with a concept of *X*; now I wonder what subjects deal with *X*." And you would turn to your directory and find out.

I bring all this up because the difference between concept and development of concept (an example of the What vs. the How) is of fundamental importance in this book. I also bring it up in order to urge you to question, *but not openly rebel against!*, the current PhD process.

Never forget that *it is in the interest of the university to make the PhD process as difficult as possible*, because difficulty means prestige. If your thesis is so difficult, so full of specialized terms and arcane concepts, that it can only be understood by one or two members of your PhD committee, then it must be important. And yet if truth be told, many, perhaps even most, PhD theses, not only in mathematics but in other disciplines, are mountains made out of molehills — modest insights expanded into the work of years and culminating in an intimidating document of unknown real worth.

The Truth About the University

After many years of observing and thinking about the university, I cannot avoid the conclusion that the university is largely a prestige racket. I put it that bluntly because someone has to. The way things are done in the university — the way students are taught, what they must do to get a degree (bachelor's, master's, or PhD) — has one main purpose, namely, the maintenance and enhancement of the university's, hence the faculty's, prestige. Everything else is secondary.

If you doubt this, consider the importance of a student's getting into the "right" school (*U.S. News & World Report* publishes an annual ranking of U. S. colleges and universities). Ask yourself how significant the difference can be between, say, a school ranked sixth and one ranked only ... seventh. If you go to the seventh-ranked school, do you only learn 93% of everything you could learn, but if you go to the sixth-ranked school, do you learn *94%*? What do they leave out in the calculus courses at the seventh-ranked school?

And now I am hoping the question will occur to you (although you will be a rare student if it does): but I can always look up what they leave out when I need it! College courses are not the only way of finding out things!

Always remember that those huge tuitions (a major portion of which goes to pay the excessive salaries of bloated administrative bureaucracies), and that half-a-lifetime of debt that you will take on in order to afford them, are not because some schools have knowledge to sell that other schools do not. You are paying for the prestige of having gone to the school.

It has been said that if you can read, you can give yourself a liberal arts education. As far as a mathematics education is concerned, I say that if you had available the kind of presentation of each undergraduate subject that is called, in this book, a "complete Environment", you could get a higher grade in each subject than you would otherwise, and could do so without having to attend most lectures.

Consider elementary calculus and physics courses. The lecturer gives an explanation of each topic (often assuming, because he has taught the course so often, a level of sophistication that many students simply do not have), then students scramble to solve the assigned problems. The student is rewarded in proportion to the number of correct answers he turns in. He crams for exams, and months and certainly years later, forgets most of the tricks he has mastered. That doesn't matter: get the grade and move on! So powerful is the academic stranglehold on faculty thinking that no professor is bothered by this.

No professor asks if this cram-and-forget culture is really the best for the student. No professor recognizes — dares to recognize — that *what is important is not problems but classes of problems* — that the most important question for each student should be, "Can I write down a procedure for solving any problem *in this class* of problem?" What the student should hand in, in these elementary courses, is not this problem solution, then that problem solution, then this other, each solution having been achieved by a frantic scramble through lecture notes, pages in the textbook, memory, and trial-and-error, but a written procedure, with a convincing demonstration that if one follows the procedure, then one has a good chance of solving the assigned problems.

In higher level courses, one typically has to do proofs, and that means, proofs in the format given in textbooks and by professors in the classroom — a format that is some 2,300 years old!, and consists of a succession of paragraphs that requires the reader to hold everything in his mind as he proceeds. There is a much better format, one that makes it easier to understand proofs in the textbook or on the blackboard, and also makes it easier for students to develop their own. This format, which is called "structured proof", is described in chapter 5, "Proofs".

As this book will make clear, there is a *much* better way to present the course material than via lectures and textbooks. Once again, the point is to concentrate on classes of problems and then to write down guidelines for doing each type, with the ability to *rapidly* find definitions, theorems and lemmas that you might need.

Why hasn't this approach to mathematics been instituted long ago? Could it be because nothing enhances prestige like difficulty?

Appendix A — The Derivative and the Area Under a Curve

The Fundamental Theorem of the Calculus establishes a basic relationship between a curve (which we will call the *lower curve*[1]) and a curve (which we will call the *upper curve*[1]) that plots the area under the lower curve. See Figs. 1 and 2. One way of expressing the theorem is as follows:

If $f'(x)$ is a continuous curve (the lower curve) between the points $x = 0$ and $x = b$, and $f(x)$ is the curve (the upper curve) that plots the area under $f(x)$ over that range of x, then

$$\int_0^b f'(x)dx = f(b)$$

Most calculus students more or less learn the theorem by applying it repeatedly in calculating derivatives and integrals.

And yet some students, while doing what they are told, wonder, if only for a few moments, why the slope of a curve at each point of a curve should have anything at all to do with the area under some other curve. Sadly, most of these students assume that if they were smarter — if they had any real talent for mathematics — it would be crystal clear why there is such a relationship.

Actually, the existence of that relationship, and the fact that it is of fundamental importance, eluded all but a few of the best mathematicians in Europe and England for more than a century, namely, during the 1600s, even as they were struggling to find (1) ways to calculate the slope of a curve at any point, and (2) ways to calculate the area under a curve between two points. It was only Newton and Leibniz, toward the end of the century, who grasped that the two problems were intimately related.

Calculus courses do not give students any idea of the history of the discovery of the calculus — which is in accordance with the general contempt for history of mathematics that prevails among academic mathematicians. The only generally available history of the calculus that I know of is Boyer's *The History of the Calculus and Its Conceptual Development*[2]. Unfortunately, it is filled with tedious, repetitive prose, and not a single subtitle in 309 pages of text: we get the titles of eight chapters

---

1. Calculus students will immediately recognize that the lower curve is the derivative curve, and the upper curve is the integral curve. I am deliberately not using these terms so that I can use the more concrete terms "lower curve" (the curve defining the area that is being tallied in the "upper curve") and slope of a curve (the upper one) at a point.
2. Dover Publications, Inc., N.Y., 1959

and that's it! There are a few illustrations (there should be many more), and sometimes it is only with considerable effort that the reader can correlate the text with the illustration. There are no summaries, no tables, no lists, e.g., no list, in roughly chronological order, of the names of the mathematicians who worked on one or both of the above problems, along with a brief description of the major accomplishments of each mathematician in connection with the problems.

My purpose in this Appendix is to try to make clearer, using a geometric presentation, why the slope of a curve and the area under another curve are so closely related. I cannot claim that the thinking I present here is similar to that of the few mathematicians who glimpsed the relationship. My sole purpose is clarity for the student.

Let us begin by putting ourselves in the middle of the 1600s and let us imagine that we are one of the few who are wondering if there could be any relationship between slopes of curves and areas under other curves. We might start by drawing the two curves in Fig. 1. The upper curve plots the area under the lower curve. But let the reader be sure he or she understands what this means.

(<u>*Note*</u>: *The reader might find it helpful to* ***think of the upper curve as plotting the total amount of money in a bank account as of time x****, and* ***the lower curve as plotting the amount of money added to or subtracted from***[1] ***the account in each moment****.*)

If we draw a vertical line from a point x on the x axis in the lower curve, to the curve itself, then the area under the curve from $x = 0$ to x is the *number* that is plotted as the value of the function at x in the upper curve. That number is, e.g., the number of square inches, or square centimeters, or whatever square unit we are using, in that area.

So, we might ask, do the two curves make sense? We might tentatively answer Yes, reasoning as follows:

When we first start moving, in the lower curve, toward the right, starting at $x = 0$, we have no area under the lower curve.

But then that area starts to increase, as shown in the upper curve.

It keeps increasing until we get to the point where the lower curve crosses the x axis. We can assume that areas below the x axis in the lower curve are negative. So at the point where it crosses the x axis, the upper curve should be flat, and then it should start to decrease, because now we are subtracting area in the lower curve.

---

1. Of course, the amount of money added or subtracted might be zero.

This should continue until we reach a point in the lower curve where the total area — positive plus negative — is zero. At that point, the upper curve's value should be zero.

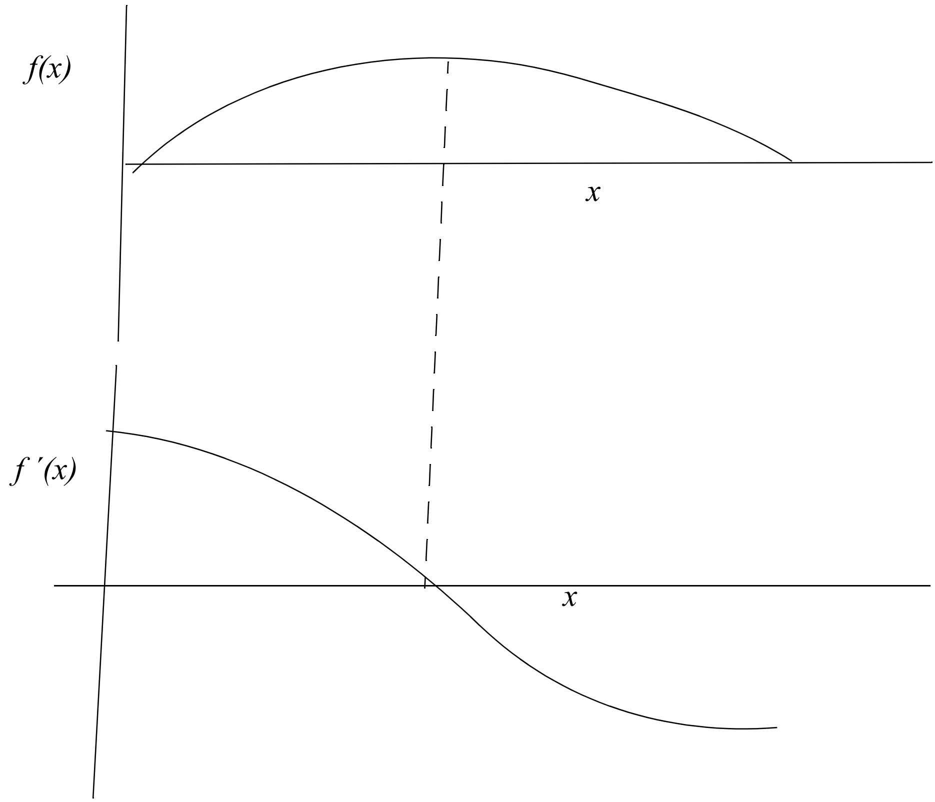

Fig. 1. The upper curve plots the area beneath the lower curve starting from x = 0 in the lower set of axes to any x to the right of 0..

Now suppose that we, back in the mid-1600s, understand that the area under a curve can be approximated by a sequence of rectangles under the curve. Suppose (see Fig. 2) that we draw a different lower curve — a straight line — and suppose we draw the upper curve that is the area under the lower curve up to each *x* for the lower curve.

Let's let *y* stand for *f*(*x*), just as an abbreviation. Suppose we know (obviously, here the reader and the author of this Appendix are leaping ahead from our mid-1600s time) that the slope of a curve at a given point can be represented as *dy/dx*, where *dy*/*dx* is the tangent line to the curve. So $dy/dx = tan(\theta) = f'(x)$, as indicated in Fig. 2.

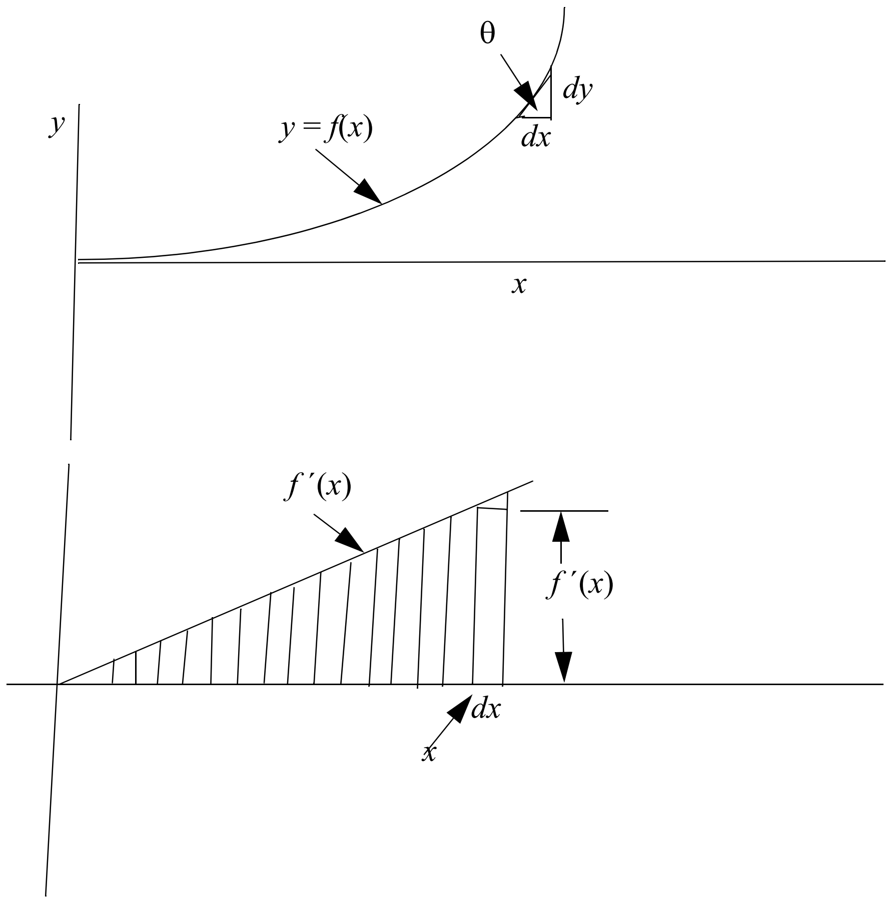

Fig. 2 Relationship between slope of upper curve and area underneath lower curve

But multiplying through the equation $dy/dx = f\,'(x)$ by dx we get $f\,'(x)dx = dy$. And so we conclude:

dy represents the change in f(x) as x goes from x to x + dx.
But this change is the ***change in area under f ′(x)*** *as f ′(x) goes from f ′(x) to f’(x + dx).*
What is that change in area? It is f ′(x)dx — <u>*the area of the rightmost rectangle below the lower curve*</u>*!* (*See Fig. 2.*)

These two facts, I hope, will make the close relationship between slope of a curve, and the area under another curve, a little clearer.

It appears[1] that precisely this line of thought is how Newton arrived at his version of the fundamental theorem of the calculus.

1. See Boyer, op. cit., p. 191.

Appendix B — Geometric Representation of the Total Differential

Sooner or later, the calculus student runs into equations like:
(1)

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

where $z = F(x, y)$ is a continuous function.

The student is told that dz is the "total differential". But I have never seen a geometric representation of the equation. Here is one. Common decency and one of the principle rules of this book demand that I provide a diagram to accompany my explanation, but at present, I do not have the graphic tools to present a diagram that is not crude and difficult to understand.

For a given x and y, imagine a staircase with a finite number of steps rising above the x axis . Label the projection of the staircase on the x axis, dx. The angle of ascent of the stairs is, say, θ. Let:

$$\tan\theta = \frac{\partial z}{\partial x}$$

Then, since (recalling the rule from trigonometry) tan = opposite/adjacent, we see that

$$\frac{\partial z}{\partial x}dx$$

is the vertical distance from the x axis to the top of the staircase.

Now imagine another staircase, parallel to the y axis, one whose lowest step begins at the top of the first staircase. The staircase ascends some finite distance (actually, up to $z = F(x, y)$). Label the projection of the staircase on the xy plane, dy. Apply the same reasoning as for the x staircase, call the angle of ascent of the stairs γ, and let

$$\tan\gamma = \frac{\partial z}{\partial y}$$

We see that

$$\frac{\partial z}{\partial y}dy$$

is the vertical distance from the *top of the first staircase* to the top of the second staircase.

And so the vertical distance dz from the xy-plane to $z = F(x, y)$ is the sum of the two vertical distances (see (1)) we have just determined.

Appendix C — Two Notably Bad Textbooks

No textbooks that I have studied — apart from those on the tensor calculus (see "Appendix D —The Worst Textbooks of All" on page 73 — have revealed so clearly the dreadful inefficiency of the modern textbook as have James R. Munkres's *Elements of Algebraic Topology*[1] and Allen Hatcher's *Algebraic Topology*[2]. These books have all the faults that are described in the chapter, "Follies of the Marketplace: A Tirade on Texts", in Morris Kline's *Why the Professor Can't Teach*[3]. (It is strange that Munkres's algebraic topology text is so bad, since his earlier text on point-set topology[4] is certainly adequate.)

Let me emphasize at the outset that the following are in no way criticisms of the mathematical creativity of the authors, or of their knowledge of the subject matter. The following are solely criticisms of the textbooks as means of making it possible for students to become able to solve problems in the subject. The criticisms were answers to a fundamental question that students are most certainly not encouraged to ask themselves, namely, *Why is this difficult?*

Critique of Munkres's Text

Inadequate Index

In a subject with as many terms and symbols peculiar to it as algebraic topology, an index is of special importance. There are more than 900 terms in Munkres's index, and yet, as usual with mathematics textbooks, it is disgracefully inadequate. I emailed a mathematics professor who had taught the subject from Munkres's text, how many terms he felt an *A* student would have had to memorize. No answer.

Since so many terms and symbols are missing from the index, a student who has forgotten a definition will in most cases have to do a linear search for it starting on page 1. (To make matters worse, some symbols are not even defined *in the text* — for example, "≈", and no, the symbol does *not* mean, " is approximately equal to".)

There are far too many omitted terms to list here. I will merely give a few examples. Each of these is a term that is basic to the subject of algebraic topology:

"Bd"; "bounds" [the verb]; "chain"; "component"; "homology, theory of"; "homology class"; "natural exact sequence"; "naturality"; "Pasting Lemma"; "*p*-chain"; "topological pair", ...

1. Addison-Wesley Publishing Company, Menlo Park, California, 1984.
2. Cambridge University Press, Cambridge, England, 2002.
3. St. Martin's Press, N.Y., 1977.
4. Munkres, James R., *Topology: A First Course*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1975.

I am sure that it never occurred to Munkres to ask, “What does the number of terms and symbols in a subject say about the nature of the subject?” And yet I think this is a fundamentally important question — one that may have a connection with Greg Chaitin’s *algorithmic information theory.* See ““What Is the Structure of This Subject?”” on page 66

Inadequate Presentation of Fundamentally-Important Topics

One of the best examples of the inadequate presentation of fundamentally-important topics is the presentation of the computing of homology groups, in particular, the homology groups of certain simplexes. In a span of four pages (pp. 30-33), the reader is presented with a few tricks and, naturally, assumes that mastery of these will enable him to compute these groups in general. Nothing could be farther from the truth. For an idea of what is involved in computing these groups, the reader need only look at the lectures on homology groups in Wildberger’s excellent series of introductory lectures on algebraic topology. (In Google, enter “UNSW - The Algebraic Topology: A Beginner’s Course”.)

Inadequately-Labeled Figures

The reader can easily find examples in which points of the figure identified by letters in the captions and in the text, are not in the figures!

Inept Presentations of Proofs

I could give numerous examples, but as a start, the reader should take a look at the proof of “Theorem 65.1 (Poincaré duality — first version)” (pp. 383-385). I encourage the reader to try to create a structured proof[1] of the Theorem by working from Munkres’s version, and to keep track of the amount of time he or she spends.

The reader’s proof will of course *not* require a reading of most of the proof in order to find out the top-level steps in the proof, as Munkres’s proof does! The reader’s proof will also provide justifications *for every statement in the proof* when the justification is given elsewhere in the textbook, and is not part of the minimum knowledge assumed of every reader. Each justification, of course, must give the page number(s) where details can be found. The absence of these justifications, along with the inadequate index, is the chief reason why the book is so difficult. See under “What Would Have Made the Above Two Texts Enormously More Useful” on page 69.

Here are some questions that the reader will have to answer:

1. as described in the chapter, “Proofs”, in this book

• What symbol (if any) is used for the isomorphisms in the statement of the Theorem? Is it φ? If so, this must be stated explicitly!

• Why does the author define φ for all dimensions n, and then for all dimensions p less than n?

• Why exactly does the commutativity of the function diagram in the proof imply the existence of the isomorphism claimed by the Theorem?

"What Is the Structure of This Subject?"

A straightforward answer to the question, "What is the structure of this subject" in the case of Munkres's text, is a set of graphic trees, each tree presenting the set of subdivisions of the root entity. A good place to begin is probably with the root "group". Each entity must appear as the root or a node in some tree, and must appear only once in the set of trees.

The no. of entities is then a measure, but certainly not the only one, of the complexity of the structure.

It seems to me that it would be interesting to compare these sets of trees for different subjects. [1] It would be equally interesting to try to find an answer to the question, "What makes a subject have the number of terms and symbols it has?"

A beginning toward answering this question is to ask, "What are the principal tasks that this subject enables the user to perform?" In the case of Munkres's text, there is, by his statement on p. 1, one principal task, namely, the determining, where possible, if two topological spaces are homeomorphic or not.

A competent author would then have set forth a procedure (top-down, step by step, as in computer programming) for carrying out this task, assuming, say, that the student had at least one year of calculus and one semester of group theory. But there is no such procedure in the text. In fact, there are fewer than five pages in the entire book that could be considered part of such a procedure.

But, assuming that the author had provided such a procedure, the terms and symbols occurring in it would have provided an answer to the question, "What makes the procedure have the number of terms and symbols it has?"

---

1. I emailed the mathematics professor mentioned in a previous footnote, who had taught courses in algebraic topology from Munkres's text, what he thought might be a good answer to the question, "What is the structure of this subject?" No reply. I then sent him a draft of this section. No reply.

Had the author been competent, he would next have provided a procedure for the second principal task, and we could have added the terms and symbols new to that procedure, to those that were used for the first principal task and are used in the second, and come up with an answer to the question, “What makes the procedure for the second principal task have the number of terms and symbols it has?”

Etc.

One thing we observe in Munkres’s text in the course of thinking of procedures, is that there are sub-procedures based on the same concept. For example, part of the procedure for the main task would involve sub-procedures involving each of the types of building blocks that are used to establish the algebraic structures that in turn are used to determine, if possible, whether the two spaces are homeomorphic. The simplest of these building blocks are triangles — actually, one type of triangle for each dimension, where a point is a zero-dimensional triangle, a line segment is a one-dimensional triangle, a two-dimensional triangle is the familiar type, a three-dimensional triangle is a tetrahedron, etc.

But there are at least three types of building blocks: simplicial (n-dimensional triangles), singular, and CW. Each has its own terminology and symbols, some of which differs little from that for the other types.

A way to deal with this tedious near-duplication of terminology and symbols is to present the presentations of the three types of building blocks “side by side”, so that students can see “at a glance” what is equivalent to what.

Critique of Hatcher’s Text

I suspect that this text has destroyed the hopes of more graduate students than any other. Comments by students on the Internet make me think of what an interviewee says about Doug Piranha of the notorious Piranha brothers[1]: “I was terrified of him. Everyone was terrified of Doug. I’ve seen grown men pull their heads off rather than see Doug.” The tenor of the students’ comments can be paraphrased, “I was terrified of Hatcher’s course. We were all terrified of it. I’ve seen graduate students pull their heads off rather than take that course.”

And yet, year after year, students trained not to question their masters’ performance as teachers or textbook writers, trudge to their fate, resigned to what they imagine is the inexorable truth: if you can’t understand it, then you are not intended to be a mathematician. Which, of course, is the most appalling, most unforgivable rubbish, since there is no question but that the material could be made readily accessible to any motivated graduate student, and the reason it is not can only be that

1. *The Complete Monty Python’s Flying Circus*, Vol. 1, Pantheon Books, N.Y., p. 189.

Hatcher cares far more about his prestige as a professor of a formidably difficult subject (how brilliant he must be!) than he does about the success of his students in understanding the subject.

Dreadfully Inadequate Index

Relatively speaking, the 5½-page index in Hatcher's text is far more incomplete than that in Munkres. It contains about 440 entries. The reader who owns a copy of the book, which is 532 pages long, will, I think, agree, that at least three new terms and symbols are introduced on each page, so that there are over 1,500 terms and symbols peculiar to algebraic topology in the entire book. Therefore, about 2/3 of the time, the reader who wants to look up the definition of a term or symbol, must do a linear search through the text. So, at a minimum the index should be at least 15 pages long, not 5½! And, if the author *really* cared about his readers, he would include an index of frequently-occuring expressions.

There is no doubt about the enormous range and depth of the content of the text. But I simply cannot understand the brain of a man who would expect his readers to do linear searches 2/3 of the time for definitions of all symbols and technical terms they have temporarily forgotten, or never have read in the first place.

I am sure that it never occurred to Hatcher to ask, "What does the number of terms and symbols in a subject say about the nature of the subject?" And yet I think this is a fundamentally important question — one that may have a connection with Greg Chaitin's *algorithmic information theory.*

No Figure Numbers!

I also wrack my brain as to why the author decided to do away with figure numbers! Figures are placed in the related text, but there is no way to succinctly refer to them from other parts of the text. What can explain this affectation on the part of the author?

Page Upon Page of Dense Mathematical Prose

The prose contains numerous facts and their proofs, but if you don't memorize them, you will have a devil of a time finding them. (By a "fact" here I mean a statement that is of lesser importance, at least for the author, than a theorem or lemma.) If each such fact and proof were given separately, with a numbered heading, e.g., "Fact 37..." that would make understanding easier in itself.

Inept Presentations of Proofs

See the subsection with this title in the above critique of Munkres's text.

Subsequent editions of this Appendix will contain examples of very inadequate presentations of the subject matter.

False Advertising — The Titles of Algebraic Topology Textbooks

If you buy a book whose title is the name of a subject, you have a right to expect that it will contain at least an introduction to the subject as a whole. In the case of algebraic topology you will be sadly disappointed.

Algebraic topology consists of three major parts: homotopy theory, homology theory, and cohomology theory. Here are some titles and contents:

Czes Kosniowski's *A First Course in Algebraic Topology*[1] is almost entirely devoted to homotopy theory.

William S. Massey's *Algebraic Topology: An Introduction*[2] is likewise almost entirely devoted to homotopy theory.

James Munkres's *Elements of Algebraic Topology*[3], on the other hand, is almost entirely devoted to the homology and cohomology theories.

Only Allen Hatcher's *Algebraic Topology*[4] devotes a roughly equal amount of space to the three major parts of the subject.

My guess is that the reason for the predominance of homotopy theory is that it is the most popular algebraic topology subject in the universities, and the reason for that is that it is conceptually much simpler than the other two theories.

What Would Have Made the Above Two Texts Enormously More Useful

The texts would have been enormously more useful, and less intimidating, to students if they had the following. I will begin with the most controversial item:

Task-Oriented Presentation of the Entire Subject

A motto of the method of math study and presentation described in this book is, "Task-orientation is the great vacuum cleaner of mathematical exposition."

---

1. Cambridge University Press, Cambridge, England, 1980.
2. Harcourt, Brace & World, Inc., New York, 1967.
3. Addison-Wesley Publishing Company, Menlo Park, California, 1984.
4. Cambridge University Press, Cambridge, England, 2002

Certainly one of the major tasks in algebraic topology is that of determining if two topological spaces, X and Y, are homeomorphic or not. And so a top-down procedure for performing that task should be a major part of the text. ("Top-down" means analogous to a top-down program structure in computer science.) The procedure will have to be divided up into sections, each section dealing with different pairs of properties in X and Y. Which pairs of properties suggest a homotopy approach? Which a homology approach? Which a cohomology approach? Under what conditions is a short exact sequence useful? Or a universal coefficient theorem? Or ... ? The student will thus see that the esoteric definitions and facts that are thrown at him and her from nowhere in the typical text, actually have a purpose related to a basic task in algebraic topology.

Some sections, of course, might conclude "If unsuccessful then go to sub-procedure ..."

The text can provide procedures for other tasks, as long as they are commonly performed tasks in algebraic topology.

All the tasks covered in the text must be listed on the first page of the text, with page references.

Other Aspects of the Presentation

• A complete index, including sub- and sub-sub- entries, and a complete index of symbols, and an index of frequently-occurring expressions;

• Each non-theorem, non-lemma fact in the text starting on a new line, and labeled, in boldface type, and numbered, for quick reference from other pages, e.g., " **Fact 37** on page ...":, then followed by a statement of the fact, and then the proof. No burying facts in long paragraphs.

• One standard form for statements of theorems, lemmas, and facts, with "if", "and", "or", "then", "iff", in bold-face type and always flush left.

• All examples that follow definitions stating explicitly what in the example corresponds to what in the definition. The examples following Munkres's definition of *quotient space* are typical of his, and Hatcher's, ignoring of this important rule. Apparently the authors believe that the clever students — the future members of the Club — will quickly be able, on their own, to figure out how the definition is reflected in the example. But a textbook should *not* be a means for separating the Winners (future mathematicians) from the Losers (everyone else), it should be as efficient a means as possible for enabling students (and others) to solve certain classes of problems in a subject. *It is important to understand my point here: I am not concerned with the speed at which students solve problems in a certain class of problems; I am concerned with how long it takes the students to arrive at the ability to solve such problems, and that is very much dependent on the textbook (and, of course, classroom lectures).*

• Each proof in structured form, as described in the chapter "Proofs", with justification (or reference to justification) for each step if the justification is given in the text. Ask yourself, recalling your own experience as a student, ***What can possibly excuse a textbook author's not giving such justifications always?*** Is it that the author wants to provide a sufficient challenge to students? But he can do that via the exercises. Is it that the author wants to force students to memorize everything as they proceed? But why introduce this unnecessary difficulty? Or is it rather that the author feels that one of his prime duties is to separate the Winners from the Losers? But surely that is done far better via the exercises, and exams.

• Answers to questions that students are afraid to ask — questions like:

Do vertices in a complex have to be labeled in a certain way? If so, in what way (in accordance with what rule), and why?

Why is it legitimate to have the number of repetitions of an element of a chain be different from the number of repetitions of the preceding element? For example, it is obviously legitimate to speak of any number of cycles around a triangle in the clockwise or counterclockwise direction. But why is it legitimate to speak of m repetitions of one edge, followed by n repetitions of the next edge, when $m \neq n$? How can that make sense geometrically? The answer, "Because that allows us to have a nice group structure on chains," is not satisfactory.

How exactly does one compute the homology group $H_p = Z_p/B_p$? That is, what is the step-by-step procedure (in top-down format) for computing Z_p, and for computing B_p, and then for computing Z_p/B_p? In other words, what is the step-by-step (top-down) procedure underlying N. Wildberger's online lecture on computing H_p (in Google, enter "UNSW The Algebraic Topology: A Beginner's Course").

Similarly for the cohomology group H^p.

Each homology group $H_p = Z_p/B_p$ is a quotient group. How does one find all the cosets of this group? (If two chains c, c', are in such a coset, then they are said to be *homologous*.)

What, if any, is the geometrical interpretation of a cohomology chain? An element of such a chain is, by definition, a homomorphism from a group of homology chains, to another group. A boundary operator is defined on cohomology chains. What is the geometric meaning, if any, of the boundary of a homomorphism?

Why does the cohomology boundary operator map in the opposite direction from the homology boundary operator, that is, why does the cohomology boundary operator map in the direction of increasing dimension, while the homology boundary operator maps in the direction of decreasing dimension?

(Munkres gives no explanation; Hatcher does, but without making clear that the operator definition he uses is a standard one for homomorphisms between sets of dual vector spaces. Homology chain groups are vector spaces, cohomology chain groups are dual vector spaces.)

• All figures numbered, all figures labeled, including with numbers indicating the order in which the parts of the figure are to be looked at, so that it is easy to understand them from the text.

• And need I say it again? The book must be tested on students, and revised appropriately, *before* it is published. By "tested" I mean via the student reading the text on his own, and then answering questions the correct answers to which are deemed by the author to be indicative of the student's understanding of the text. (I am hoping that authors will have the intelligence and the intellectual integrity *not* to include questions like, "Did you understand the text?")

Appendix D —The Worst Textbooks of All

Without question, the worst textbooks of all are those for the tensor calculus[1]. I have been told, by more than one mathematician, that tensor calculus can only be learned in the classroom. It apparently never occurs to these mathematicians to ask why this is so, and whether it is necessarily so.

There is no question about the importance of the subject. It is, among many other things, the mathematics for Einstein's General Theory of Relativity. (A way to make General Relativity much easier to understand is given under "How Einstein's General Theory of Relativity Should Be Presented" on page 79.) Einstein himself found the tensor calculus a very difficult subject. One would think that it would therefore be a prime candidate for an investigation as to why some mathematical subjects are more difficult than others. But, as the reader of Chapter 1 of this book knows, making things less difficult is not a goal that mathematicians regard as important. Subjects that can only be learned in the classroom guarantee the ongoing job security of mathematics professors.

I urge the student who is forced to study tensor calculus, or who simply wants to study it out of interest in the subject, to keep a record of all the questions he or she had but that were not answered in textbooks (or Google articles). Then I urge the student to ask him or herself: "Is it really true that answers to these questions could not have been written down in a book or article?"

Critique of Tensor Calculus Texts

In my opinion, tensor calculus textbooks, and introductions such as are found in Google, are the work of academic hacks who believe that the subject consists merely of a bunch of definitions, and occasional statements, often without proofs. Furthermore, these authors believe that they can pick and choose among the topics they want to call "basics". These basics differ in the minds of the authors to the extent

---

1. The worst presentation of a logical argument I have ever come across is the derivation of the transformation law of the Christoffel symbols that is given in Synge, J.L., and Schild, A., *Tensor Calculus*, Dover Publications, Inc., N.Y., 1978, pp. 39-40. The job done right (or almost right (the derivation of a partial derivative term is not given)) can be found in Hay, G. E., *Vector and Tensor Analysis*, Dover Publications, Inc., N.Y., 1953, p. 173. The strategy is elegant and easy to understand. I challenge the mathematics professor who is inclined to say anything positive about the Synge/Schild presentation, to randomly select two qualified students who have never seen a presentation of any derivation of the law, give one the Synge/Schild version, and the other the Hay version, and time how long it takes each student to come to an understanding of the presentation. Sadly, the Hay text has major shortcomings, beginning with the complete absence of an index!

that, after the student sweats and strains to master one introduction, he or she finds that, apart from a few terms, the next introduction might as well be an introduction to an entirely different subject.

In my experience, tensor calculus textbooks and introductions are fodder for test takers, not material for students who above all want to understand concepts, and hence who want in-depth explanations of the meaning and need for things like covariance, contravariance, and the transformations of tensors, not to mention tensors themselves.

The Major Fault: Dreadfully Inadequate — Or Absent! — Indexes

The single worst quality of tensor calculus textbooks and introductions, is the dreadfully inadequate, or absent, indexes.

"Einstein claimed never to memorize anything which could be looked up in less than two minutes." — "Albert Einstein Anecdotes", Google, 9/22/18.

If you have spent any time at all looking through tensor calculus texts, ask yourself how many technical terms, peculiar to the subject, there are in the tensor calculus. I think you will agree that a conservative estimate is at least 50.

Yet some tensor calculus textbooks and introductions have no indexes at all! Others have indexes that give only a smattering of technical terms, and no symbols! The message from the textbook authors is clear: you either memorize the definitions of all these terms and symbols as you proceed, or you build your own index as you proceed, or else by God you do a linear search for every definition you are not sure of.

Outrageous.

What Should Be In Every Introductory Textbook on the Tensor Calculus

I claim that if a tensor calculus textbook were written in accordance with the recommendations in this section, and then tested against randomly-selected students with the minimum requisite abilities, and the results compared with tests against randomly-selected students using any of the usual tensor calculus introductory textbooks, students using the textbook described here would require *less than a third the time* to achieve similar grades.

As I have said throughout this book, the important first step for a mathematician thinking about writing a textbook is to decide on the minimum abilities that will be assumed of all readers. In the following, I am assuming that these abilities will be those of a student who has successfully completed the first two semesters of undergraduate calculus.

Yes, The Textbook Should Be a Computer Data Base!

Ideally, the textbook should be a computer data base, but not just a bunch of programs to do calculations. It should be able to carry out, on command, the long, tedious algebraic manipulations that fill so much of Lieber's book (see below). It should be able to immediately present tables summarizing the properties of various kinds of tensors.

If it is not possible to construct such a data base, then an Environment, as described in this book, will constitute a major improvement over what currently exists. (At present, I consider a tensor calculus Environment the ultimate test of the Environment concept.) Data base or Environment should include the following information.

Basic meaning of "*invariance*"

- What is meant by the *invariance of vectors*
- What is meant by the *invariance of tensors*
- Where the term came from

Covariance and contravariance

- Basic meaning of the two terms
- Examples
- Where the terms came from

Definition of "*tensor*"

- A tensor is often described as a generalization of a vector.
  - How would you [the student] generalize a vector?
    - Why not several vectors coming from each point in a space?
- The main types of tensor (there are at least three types: multi-dimensional arrays, mult-linear maps, and tensor products)
  - For each type:
    - The name of the type
    - Formal definition of the type
    - Standard representations of the type
    - Basic idea — generic diagram — of any tensor of the type
      - (See "On the Generic Diagram for Any Tensor" on page 76.)
    - Examples of the type
      - If one of the examples is the stress tensor, then explain clearly what it means to have a force per unit area entirely in a plane (see chapter XIV in the Lieber text referenced below).

Important properties of the type (the following might not apply for all types of tensor)

No. of components vs. rank (no. of indices) of the type

No. of equations defined by the type

Difference between a component and the "contents" of the component

How do covariant and contravariant indices get decided on in the making of this type of tensor?

Mixed tensors of the type.

Proof that the type is equivalent to each of the other types

Full explanation of the transformation of coordinates; why exactly must there be a product of partial derivatives, and why just these partial derivatives?

Formal definition of transformation, with full expansion of symbols

How exactly does the transformation prove that what is transformed is or is not a tensor? How exactly is the invariance of tensors demonstrated?

Explanation of fact: If transformation is applied to a tensor, then the result is a tensor

If transformation is applied to something, and the result is a tensor, does that prove the something is a tensor?

What changes with each new definition of coordinates?

Basic operations on or with the tensor type

[Description of each operation including what led to that operation]

What made mathematicians think of the contraction operation?

On the Generic Diagram for Any Tensor

After a course in linear algebra, a student knows a generic diagram for a vector: a vector is an arrow of a certain finite length pointing in a certain direction; this arrow can be represented by a n-tuple of numbers (its coordinates), where n is the dimension of the space in which the arrow appears.

A vector is a one-dimensional array and, in fact, a one-dimensional tensor. What about a two-dimensional tensor? Well, textbooks tell us that a two-dimensional tensor is an array of numbers. We ask: is that true for all the types of tensors? Including tensors derived from tensor products?

Some textbooks then tell us that three-dimensional tensors are three-dimensional arrays, and so on, for multi-dimensional arrays. But what is a multi-dimensional array, when the dimension of the array is greater than three? How can we picture it, or at least conceive of it?

The textbooks tell us that the number of components of a tensor is n^r, where n is the dimension of the tensor and r is the rank. How can we picture an array containing that many components? One answer might be the following. Let us assume we have a 10-dimensional tensor of rank 3. Then we can present the components of this tensor as

Table 1: Numbers Identifying the 10^3 Components of a Tensor

000	020	...	100	120	...	200	220	...	990
001	021	...	101	121	...	201	221	...	991
...	...	...	...	...	...	...	...	...	...
019	029	...	119	129	...	219	229	...	999

Normally, the numbering of indices begins with 1, not 0, but that is a triviality, easily overcome.

A question immediately presents itself: how, and on what basis, do we determine the components that are covariant and those that are contravariant?

The next question, and it might be an important one, is: if we are given, or supposed to create, a tensor of dimension n and rank r, why not *first* create a table such as the above, *then* give appropriate labels ("contents") for each number in accordance with what the tensor is supposed to represent?

A Reluctant Criticism of Lillian Lieber's Text, "*The Einstein Theory of Relativity*"

I have nothing but contempt for all the tensor calculus textbooks and Google articles I have read — with one exception, namely, the textbook[1] cited in the title of this sub-section. Prof. Lieber set out to bring the subject of the tensor calculus to a much larger audience than it had had (the book was first published in 1945). Unfortunately, she took for granted that the academic student environment was a good guide for the creation of textbooks, which it certainly is not.

In particular, she assumed that to make mathematics accessible to a wide audience, one should use more words, a mistake that virtually all popularizers make[2].

1. Lieber, Lillian, *The Einstein Theory of Relativity*, Paul Dry Books, Philadelphia, PA, 2008.
2. As any observant self-teacher knows, what makes mathematics more accessible is fewer words, a more table-like presentation of concepts, bold-faced sub-titles to distinguish each fact or definition of importance, structured proof, and complete indexes.

But sensing that the number of words might also be a problem for her readers, she used a kind of “stacked prose”, in which each line contained only one thought. Thus, for example,

“It will be noticed,
in equations (9),
that
x depends on BOTH *x*’ and *y*’,” — p. 144

Having put all her faith in her stacked prose, and in her friendly writing style, *she omitted all titles and sub-titles in her 350-page book and entirely omitted an index!* The result is a sprawl that makes understanding the text, and just finding things in it, at the least very tedious.

Why I Gave Up Trying to Understand General Relativity From Lieber’s Text

The first part of her book covers Special Relativity and then shows how coordinates of a given vector change when the basis of the vector space changes. All well and good. We are told that the tensor calculus is the mathematics underlying General Relativity, and that we want tensors not to change value when the basis of vector space changes, just as vectors don’t. We want this because the laws of the universe are expressed in tensors; the laws are the same for all observers.

Fine. But starting at chapter XXIII “The Curvature Tensor At Last”, p. 200, we enter a formidable wilderness of deductions without knowing why they are there. This is the traditional, utterly-misguided, walk-in-the-woods so beloved by math professors. Every student has confronted these walks many times. “Let us assume that ... and furthermore that ... This implies that ... and therefore... Now assume that ...” and on and on until, suddenly an important fact is shown to have been deduced! The fact was never mentioned at the start, or anywhere along the way. The What did not precede the How. Mathematics professors, with their colossal indifference to ways of improving their teaching, ignore the studies that reveal that students don’t like walks-in-the-woods; they far prefer knowing at the start what the goal is. Which is precisely the case with a normal proof: one states the fact (for example, lemma or theorem), and then one gives the proof.

In this case, the goal is apparently the curvature tensor. But there is no indication what this tensor actually is.

The road to the curvature tensor begins with the covariant derivative of any covariant tensor, A_σ. Just any one? Apparently. We are not told why this is a good idea.

Then the second covariant derivative follows. Then two of the indices in the resulting tensor are reversed. No explanation why. Then the tensor with the indices reversed is subtracted from the first tensor (why?) and the hard, tedious, unmotivated labor really begins.

Eventually we wind up with a symbol we are told represents the Riemann-Christoffel tensor.

More pages of tedious deductions until we arrive at "the new Law of Gravitation consisting now of only the THREE equations:

$G_{11} = 0$; $G_{33} = 0$; and $G_{44} = 0$"

I must be honest: if, before I knew much about mathematics, someone had told me that this is "real mathematics", I would have sought my future in sociology or literary criticism.

For the student who has already learned to avoid asking the Wrong Questions, let me ask a few for him.

In Lieber's book we have been told, before chapter XXIII, that Einstein's General Relativity replaces the notion of the Newtonian force of gravity with the notion of the curvature of space-time. Gravity curves space-time.

Fine. Perhaps we can imagine a bent metal bar, with lines running its length. And suppose at some place in the bar there is a lump that causes the lines to curve around it. Let us say the lump is gravity.

Fine. We now need the mathematics to express situations like this. How should we begin? Is our task to come up with a mathematics to describe the curvature of space-time? And then from there figure out how to make it behave properly in the vicinity of gravity. But how shall we do that?

None of this is explained in the mind-numbing algebraic manipulations in Lieber's text.

If someone asked me where, in General Relativity, the figure(s) defining the particular source of gravity are inserted, then after all the time I have spent struggling with these pages, I could not answer.

And how did Newton arrive at his famous formula for the force of gravity? How did Einstein arrive at his new formula (set of equations)?

How Einstein's General Theory of Relativity Should Be Presented

In the late 1960s, in order to deal with the growing size and complexity of computer programs, computer scientists developed what was called *structured programming* (now also known as *top-down programming*).

The basic idea is to divide a program into a hierarchy of levels. The top level contains a few — say, less than seven — steps that, if each is correctly implemented, yield a correct program.

Then each step in turn is treated in the same way: it is implemented by a few steps that, if each is correctly implemented, yield a correct step. Etc., recursively.

The same idea can be applied to mathematical proofs, though I have never seen it done. But I can say from personal experience that it vastly decreases their complexity.

An obvious place in physics where the idea could be applied is in the presentation of Einstein's General Theory of Relativity, which is built on the tensor calculus. At the top level would be a single equation expressing the curvature of space in the vicinity of a gravitational object. At the next lower level would be a sequence of steps that, if each is correct, would imply that the single equation is correct. Etc., recursively.

All the presentations I have seen are bottom-up, meaning that the reader is expected to master all the facts and proofs *before* arriving at the single equation. (Mathematicians continue to believe that you can't tell the time unless you know how the watch is built.) But in the top-down approach, the reader learns details only when he needs them.

Since the General Theory is built on the tensor calculus, such a presentation would be an excellent means of teaching that subject.

A variation on it for use in popularizations of General Relativity, would begin, *first and foremost*, with the author writing down the minimum knowledge he will assume of any reader. Then the author would ask himself, "Suppose the reader only had one minute. What would be the best summary of the subject I could give to him?" The author might then come up with something like the following:

"Einstein's General Theory of Relativity describes the curvature of four-dimensional spacetime, which is where our universe exists. There are three dimensions of space, one dimension of time.

"The curvature is caused by mass. Thus, if, during a solar eclipse, you look for a certain star whose location you know is near the edge of the solar disk, you will find it appears to be in the wrong place. The reason is that the light from the star has been curved, in spacetime, by the sun's mass."

The author might then ask himself, "Suppose the reader had a few minutes more. What would be the best extension of this summary I could give him? The author might then come up with:

"As you might have guessed, one of the most important tasks in General Relativity is to find the curvature of spacetime in a particular region of spacetime where there is a large mass. The major steps to do this are ...

Etc.

A slogan that we use in creating Environments, as described in this book, is "*Task-orientation is the great vacuum cleaner of mathematical exposition*." This is exemplified in the above two levels of a popularization. In my experience, it tends to greatly simplify and focus the presentation. It is simply not necessary to know how the watch is built before you can tell the time!