Linear Algebra:

A Partial KWIC Index

of Definitions, Theorems and Lemmas

The main goal of an Environment is to increase the speed of problem-solving. Typically, in solving a problem, there is a need to find all theorems or lemmas that might be useful in proving the statement one is trying to prove. At present the student's only way of doing this is a by a combination of memory and linear searches through class notes and textbook.

Is there a better way? In an Environment, ideally all lemmas and theorems pertaining to a given entity are listed in the "*entity*, theorems" section for that entity. But the typical lemma and theorem contains other entities in its statement, and so we cannot be sure we will have, in that section, all the lemmas and theorems needed to prove a statement involving that entity.

A solution to this problem might be what is called a "Key Word in Conext" (KWIC) index. In this type of index, each definition, lemma, and theorem is reproduced under each key term it contains. Thus a given definition, lemma or theorem appears several times in the index, one for each key term. Obviously, symbols and terms can also be included in such an index.

This Appendix contains an example of a KWIC index, in this case for linear algebra. But let me hasten to make clear that the Appendix contains only a few of the definitions, lemmas, and theorems in the subject, since my purpose at this point is to solicit comments from students as to the usefulness of such an index. I am hoping that students will not feel that, because the KWIC index in this Appendix does not contain all relevant definitions, lemmas and theorems in the typical linear algebra course, the KWIC idea is worthless! The index is being expanded on a regular basis. My belief at present is that KWIC indexes for undergraduate subjects would not only be very helpful to students, but would also be profitable. I welcome hearing from readers who would like to join me in creating a complete index for an undergraduate subject (but probably not calculus at the start, because of the quantity of material). The one disadvantage of a KWIC index in problem-solving is that the student has to check logical priority, and not use a lemma or theorem that is, in effect, the statement that he or she is trying to prove.

In the following:

"(Def.)" indicates that the item is a definition.

"B/M *n*" means that the index item is from page **n** of Birkhoff, Garrett and Mac Lane, Saunders, *A Survey of Modern Algebra*, A. K. Peters, Natick, MA, 1997.

"J/B *n*" means that the index items is from page *n* of Jacob, Henry G., and Bailey, Duane W., *Linear Algebra*, Houghton Mifflin Company, Boston, MA, 1971.

Web site addresses are also the sources of index items.

Symbols that are not obviously alphabetical are at the end of the index.

A

Common symbol for a matrix

|*A*|

|A| is a symbol for the determinant of the $n \times n$ matrix *A*.

array

(Def.) A rectangular array of elements of a field *F*, having *m* rows and *n* columns, is called an $m \times n$ matrix over *F*. (B/M 180)

B

Common symbol for a matrix

basis, bases

(Def.) A basis of a vector space is a linearly independent subset which generates (spans) the whole space. A vector space is finite-dimensional iff it has a finite basis. (B/M 178)

A vector space can have many different bases. (B/M 194)

All bases of any finite-dimesional vector space have the same number of elements. (B/M 178)

An $n \times n$ matrix over a field *F* is non-singular iff its rows are linearly independent — or, equivalently, iff they form a basis of F^n. (B/M237)

The vector space **C** has the dimension 2, for 1 and *i* form a basis. (B/M 195)

If the finite-dimensional vector space *V* is the direct sum of its subspaces *S* and *T*, then the union of any basis of *S* with any basis of *T* is a basis of *V*. (B/M 196)

column

(Def.) A rectangular array of elements of a field *F*, having *m* rows and *n* columns, is called an $m \times n$ matrix over F. (B/M 180)

det (see also "determinant")

Symbol for determinant.

The vectors x, y, z in $\mathbf{R}^3$ are linearly dependent iff det (x, y, z) = 0 (J/B 254)

If *A* is an $n \times n$ matrix with two rows which are equal, then det $A = 0$. (J/B 259)

If *B* is the matrix obtained from *A* by multiplying a row of *A* by α, then det $B = \alpha$ det *A*. (J/B 259)

If $\mathbf{A}_i$ is the *i*th row of the matrix *A*, then det $(\mathbf{A}_1, ..., \mathbf{A}_i + \mathbf{A}_i', ..., \mathbf{A}_n) =$ det $(\mathbf{A}_1, ..., \mathbf{A}_i, ..., \mathbf{A}_n)$ + det $(\mathbf{A}_1, ..., \mathbf{A}_i', ..., \mathbf{A}_n)$. (J/B 259)

det $E_n = 1$, where E_n is the identity matrix. (J/B 259)

determinant (see also "det")

|*A*| is a symbol for the determinant of the $n \times n$ matrix *A*.

det is a symbol for the determinant of the $n \times n$ matrix *A*.

Each square matrix A over any field has a determinant. (B/M 318)

If an $n \times n$ matrix A has two rows alike, then the determinant of $A = 0$. (B/M 320)

A homogeneous system of n equations in n unknowns has a solution iff the determinant of the coefficient matrix is 0. (home.scarlet.be/~ping1339/stels2.htm)

If the determinant of a square matrix is 0, the matrix is *singular.* Otherwise it is *regular.* (home.scarlet.be/~ping1339/stels2.htm)

dimension (see also "finite-dimension...", "*n*-dimensional")

If a vector space V has dimension n, then (i) any $n + 1$ elements of V are linearly dependent, and (ii) no set of $n - 1$ elements can span V. (B/M 178)

The vector space **C** has the dimension 2, for 1 and i form a basis. (B/M 195)

direct sum

(Def.) A vector space V is the direct sum of two subspaces S and T if every vector ξ of V has one and only one expression $\xi = \sigma + \tau$, $\sigma \ \varepsilon \ S$, $\tau \ \varepsilon \ T$. (B/M 195)

elementary matrix

If E is an elementary matrix, then $|EA| = |E|\,|A| = |AE|$. (B/M 321)

elementary row operations in a matrix

(Def.) The elementary row operations are (i) Interchange any two rows; (ii) Multiplication of a row by any nonzero constant in F; (3) Addition of any one row to any other row. (B/M 181)

The inverse of any elementary row operation is itself a row operation. (B/M 181)

F

Common symbol for a field.

F^n

(Def.) F^n is the set of all n-tuples $(c_1, c_2, \ldots, c_n)$, where c_i is an element of the field F.

Any finite-dimensional vector space over a field F is isomorphic to one and only one space F^n. (M/N 194)

field

Any finite-dimensional vector space over a field F is isomorphic to one and only one space F^n. (M/N 194)

(Def.) A rectangular array of elements of a field F, having m rows and n columns, is called an $m \times n$ matrix over F. (B/M 180)

The $m \times n$ matrices A, B, C over any field F form an mn-dimensional vector space. (B/M 180)

An $n \times n$ matrix over a field F is non-singular iff its rows are linearly independent — or, equivalently, iff they form a basis of F^n. (B/M237)

Each square matrix A over any field has a determinant. (B/M 318)

finite-dimension...

(Def.) A basis of a vector space is a linearly independent subset which generates (spans) the whole space. A vector space is finite-dimensional iff it has a finite basis. (B/M 178)

All bases of any finite-dimesional vector space have the same number of elements. (B/M 178)

Any linearly independent set of elements of a finite-dimensional vector space is part of a basis. (B/M 179)

Any finite-dimensional vector space over a field F is isomorphic to one and only one space F^n. (M/N 194)

If the finite-dimensional vector space V is the direct sum of its subspaces S and T, then the union of any basis of S with any basis of T is a basis of V. (B/M 196)

homogeneous system of linear equations

A homogeneous system of n equations in n unknowns has a solution iff the determinant of the coefficient matrix is 0. (home.scarlet.be/~ping1339/stels2.htm)

The only solution of a system of n linearly independent homogenous linear equations in n unknowns $x_1, ..., x_n$ is $x_1 = x_2 = ... = x_n = 0$. (B/M 191)

I_n.

Symbol for the $n \times n$ identify matrix (B/M 185)

inverse of a linear transformation

If the linear transformation T:V → W is a one-one transformation of V onto W, its inverse is linear. (B/M 239)

left-inverse of a square matrix

Every left-inverse of a square matrix is also a right-inverse. (B/M 238)

linear combination

The set of all linear combinations of given vectors $\alpha_1, ..., \alpha_m$ in a vector space V is a subspace of V. (B/M 173-174)

linear equations, system of

The set of solution vectors of a non-homogenous system of linear equations with cofficients in a field F is not changed under each of the following operations:

(i) the interchange of any two equations;

(ii) multiplication of an equation by any nonzero constant c in F;

(iii) addition of any multiple of one equation to any other equation. (B/M 182) linearly dependent

A set of vectors is linearly dependent iff it contains a proper subset spanning the same subspace. (B/M 177)

The nonzero vectors $\alpha_1, ..., \alpha_m$, are linearly dependent iff some one of the vectors α_k is a linear combination of the preceding ones. (B/M 176)

If a vector space V has dimension n, then (i) any $n + 1$ elements of V are linearly dependent, and (ii) no set of $n - 1$ elements can span V. (B/M 178)

The vectors x, y, z in $\mathbf{R}^3$ are linearly dependent iff det (x, y, z) = 0 (J/B 254)

linearly independent

(Def.) A set of vectors, $\alpha_1, ..., \alpha_m$ are linearly independent over a field F iff for all scalars c_i in F, $c_1\alpha_1 + c_2\alpha_2 + ... + c_m\alpha_m = \mathbf{0}$ implies $c_1 = c_2 = ... = c_m = 0$. (B/M 176)

Any subset of a linearly independent set is linearly independent. (B/M 176)

A square matrix is non-singular iff its columns are linearly independent (B/M 238)

An $n \times n$ matrix over a field F is non-singular iff its rows are linearly independent — or, equivalently, iff they form a basis of F^n. (B/M237)

Any finite set of vectors contains a linearly independent suset which spans the same subspace. (B/M 177)

Any subset of a linearly independent set is linearly independent. (B/M 176)

Any linearly independent set of elements of a finite-dimensional vector space is part of a basis. (B/M 179)

For n vectors $\alpha_1, ..., \alpha_n$ of an n-dimensional vector space to be a basis, it is sufficient that they span the vector space or that they be linearly independent. (B/M 179)

If $p(x) = 0$, where the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$ and the a_i are elements of a field F, then the powers $1, x, x^2, x^3, ..., x^n$ are linearly independent over F. (B?M 195)

The non-zero rows of a row-reduced matrix are linearly independent. (B/M 184)

The only solution of a system of n linearly independent homogenous linear equations in n unknowns $x_1, ..., x_n$ is $x_1 = x_2 = ... = x_n = 0$. (B/M 191)

linear transformation

If the linear transformation T: V → W is a one-one transformation of V onto W, its inverse is linear. (B/M 239)

matrix (see $m \times n$ matrix, $m \times n$ matrix)

$m \times n$ matrix

(Def.) A rectangular array of elements of a field F, having m rows and n columns, is called an $m \times n$ matrix over F. (B/M 180)

The $m \times n$ matrices A, B, C over any field F form an mn-dimensional vector space. (B/M 180)

Two $m \times n$ matrices A and B are row-equivalent iff they have the same row space.

Every $m \times n$ matrix A is row-equivalent to one and only one reduced echelon matrix. (B/M 187)

$n \times n$ matrix (see also "square matrix")

|A| is a symbol for the determinant of the $n \times n$ matrix A.

An $n \times n$ matrix over a field F is non-singular iff its rows are linearly independent — or, equivalently, iff they form a basis of F^n. (B/M237)

If an $n \times n$ matrix A has two rows alike, then the determinant of $A = 0$. (B/M 320)

An $n \times n$ matrix A has rank n iff it is row-equivalent to the $n \times n$ identity matrix I_n.

mn-dimensional

The $m \times n$ matrices A, B, C over any field F form an mn-dimensional vector space. (B/M 180)

n-dimensional

For n vectors $\alpha_1, ..., \alpha_n$ of an n-dimensional vector space to be a basis, it is sufficient that they span the vector space or that they be linearly independent. (B/M 179)

non-singular

A square matrix is non-singular iff its columns are linearly independent (B/M 238)

An $n \times n$ matrix over a field F is non-singular iff its rows are linearly independent — or, equivalently, iff they form a basis of F^n. (B/M237)

A square matrix is non-singular if its determinant $\neq 0$.

one-one transformation

If the linear transformation T:V $\rightarrow$ W is a one-one transformation of V onto W, its inverse is linear. (B/M 239)

polynomial

If $p(x) = 0$, where the polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$ and the a_i are elements of a field F, then the powers $1, x, x^2, x^3, ..., x^n$ are linearly independent over F. (B/M 195)

The set of polynomials of degree at most seven is a subspace of the vector space of all polynomials — whether the base field is real or not. (B/M 173)

rank

The rank of any matrix A is the number of nonzero rows in any echelon matrix row-equivalent to A. (B/M 186) reduced echelon matrix

Any matrix is row-equivalent to a reduced echelon matrix. (B/M 184)

regular matrix

If the determinant of a square matrix is 0, the matrix is *singular*. Otherwise it is *regular*. (home.scarlet.be/~ping1339/stels2.htm)

right-inverse

Every left-inverse of a square matrix is also a right-inverse. (B/M 238)

row

(Def.) A rectangular array of elements of a field F, having m rows and n columns, is called an $m \times n$ matrix over F. (B/M 180)

The non-zero rows of a row-reduced matrix are linearly independent. (B/M 184)

row-equivalent

Two $m \times n$ matrices A and B are row-equivalent iff they have the same row space.

Any matrix A is row-equivalent to a row-reduced matrix, by elementary row operations of typles (ii) and (iii):

(ii) multiplication of an equation by any nonzero constant c in F;

(iii) addition of any multiple of one equation to any other equation. (B/M 182)

Any matrix is row-equivalent to a reduced echelon matrix. (B/M 184)

Let the $m \times n$ matrix A be row-equivalent to a row-reduced matrix R. Then the nonzero rows of R form a basis of the row space of A. (B/M 185)

row space

(Def.) The *row space* of a matrix A is that subspace of F^n which is spanned by the rows of A, regarded as vectors in F^n.

singular matrix

If the determinant of a square matrix is 0, the matrix is *singular*. Otherwise it is *regular*. (home.scarlet.be/~ping1339/stels2.htm)

solution

The only solution of a system of n linearly independent homogenous linear equations in n unknowns $x_1, ..., x_n$ is $x_1 = x_2 = ... = x_n = 0$. (B/M 191)

solution vector

The set of solution vectors of a non-homogenous system of linear equations with cofficients in

a field F is not changed under each of the following operations:
(i) the interchange of any two equations;
(ii) multiplication of an equation by any nonzero constant c in F;
(iii) addition of any multiple of one equation to any other equation. (B/M 183)

span...

(Def.) A basis of a vector space is a linearly independent subset which generates (spans) the whole space. A vector space is finite-dimensional iff it has a finite basis. (B/M 178)

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For n vectors $\alpha_1, ..., \alpha_n$ of an n-dimensional vector space to be a basis, it is sufficient that they span the vector space or that they be linearly independent. (B/M 179)

square matrix (see also $n \times n$ matrix")

A square matrix is non-singular iff its columns are linearly independent (B/M 238)
Every left-inverse of a square matrix is also a right-inverse. (B/M 238)
Each square matrix A over any field has a determinant. (B/M 318)

subspace

A subspace S of a vector space V is a subset of V which is itself a vector space with respect to its operations of addition and scalar multiplication in V. (B/M 173)

The set of all linear combinations of any set of vectors in a vector space V is a subspace of V. (B/M 174)

The intersection $S \cap T$ of any two subspaces of a vector space V is itself a subspace of V. (B/M 174)

A nonvoid subset S is a subspace iff the sum of any two vectors in S lies in S and any product of any vector of S by a scalar lies in S. (B/M 173)

Geometrically, a "subspace" is simply a linear subspace (line, plane, etc.) through the origin O. (B/M 173)

The null vector **0** alone is a supspace of any vector space. (B/M 173)

The set of polynomials of degree at most seven is a subspace of the vector space of all polynomials — whether the base field is real or not. (B/M 173)

The set of all linear combinations of given vectors $\alpha_1, ..., \alpha_m$ in a vector space V is a subspace of V. (B/M 173-174)

(Def.) The *row space* of a matrix A is that subspace of F^n which is spanned by the rows of A, regarded as vectors in F^n.

V

Common symbol for a vector space

<u>*W*</u>

Common symbol for a vector space

<u>vector space</u>

(Def.) A basis of a <u>vector space</u> is a linearly independent subset which generates (spans) the whole space. A <u>vector space</u> is finite-dimensional iff it has a finite basis. (B/M 178)

A <u>vector space</u> can have many different bases. (B/M 194)

All bases of any finite-dimesional <u>vector space</u> have the same number of elements. (B/M 178)

Any linearly independent set of elements of a finite-dimensional <u>vector space</u> is part of a basis. (B/M 179)

For n vectors $\alpha_1, ..., \alpha_n$ of an n-dimensional <u>vector space</u> to be a basis, it is sufficient that they span the <u>vector space</u> or that they be linearly independent. (B/M 179)

The $m \times n$ matrices A, B, C over any field F form an mn-dimensional <u>vector space</u>. (B/M 180)

Any finite-dimensional <u>vector space</u> over a field F is isomorphic to one and only one space F^n. (M/N 194)

The field **C** of all complex numbers may be considered as a vector space over the field **R** if one ignores all the algebraic operations in **C** except except the addition of complex numbers and scalar multiplication of complex numbers by reals. This space has the dimension 2, for 1 and *i* form a basis. (B/M 195)

A subspace S of a <u>vector space</u> V is a subset of V which is itself a vector space with respect to its operations of addition and scalar multiplication in V.

The set of all linear combinations of any set of vectors in a <u>vector space</u> V is a subspace of V.

The intersection $S \cap T$ of any two subspaces of a <u>vector space</u> V is itself a subspace of V.

(Def.) A <u>vector space</u> V is the direct sum of two subspaces S and T if every vector ξ of V has one and only one expression $\xi = \sigma + \tau,\ \sigma\ \varepsilon\ S, \tau\ \varepsilon\ T$. (B/M 195)

If the finite-dimensional <u>vector space</u> V is the direct sum of its subspaces S and T, then the union of any basis of S with any basis of T is a basis of V. (B/M 196)