Is It Legitimate to Begin a Sentence With “If Counterexamples Exist, Then...”?

by

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Sept. 10, 2015

Motivation

Earlier versions of our proposed proof(s) of the $3x + 1$ Conjecture, in the paper, "A Solution to the $3x + 1$ Problem", on occampress.com, received several repeated criticisms. The purpose of this paper is to respond to those criticisms.

Summary of Repeated Criticisms

Most of the criticisms boiled down to readers' attempts to dismiss our proof(s) by inappropriate applications of the truth table for material implication. By "inappropriate" we mean, "contrary to normal practice."

For example, the statement, or its equivalent, "If counterexamples exist, then there is a minimum counterexample", has no doubt been uttered many times, and accepted as obviously true, by $3x + 1$ researchers. If someone were to say, in response to the statement, "But if in fact, counterexamples do not exist, then the antecedent of the statement is false, and since false implies anything, the statement is ambiguous", we are sure the vast majority of $3x + 1$ researchers, and of mathematicians in general, would reply that such a response is inappropriate, perhaps even absurd, because the purpose of the statement is to state a fact, and, indeed a fact from which one might try to prove the $3x + 1$ Conjecture.

In normal practice, an implication, p implies q, is used to convey information, which means that p and q are both assumed true. Consider, for example, Fermat's Little Theorem, which states that if p is a prime positive integer, and a is a positive integer relatively prime to p, then $a^{p-1} \equiv 1 \bmod p$. In normal practice, this Theorem is not meant to be an opportunity to apply the implication truth-table to show that the statement is ambiguous. It is meant to convey the information that, if p is a prime positive integer, then if you choose another positive integer a relatively prime to p, and raise it to the $p - 1$ power, you will find that the result is congruent to 1 mod p.

The inappropriate applications of the truth table for implication remind us of an item in *Littlewood's Miscellany*:

"Schoolmaster: 'Suppose x is the number of sheep in the problem.' Pupil: 'But, Sir, suppose x is not the number of sheep.' "[1]

The specific criticisms that have been made are as follows:

- The word "counterexample" is not legitimate in a lemma or in a proof connected with the $3x + 1$ Problem

- It is illegitimate to have "If counterexamples exist..." and "If counterexamples do not exist..." in a proof;

- The phrase "Whether or not counterexamples exist..." is meaningless or at best ambiguous; therefore any proof that contains this phrase is fallacious.

- A sentence beginning, "If counterexamples exist..." is ambiguous;

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1. Littlewood, J. E., *Littlewood's Miscellany*, ed. Béla Bollobás, Cambridge University Press, N.Y., 1990, p. 59.

• The phrase "If counterexamples exist" can only be used in a proof by contradiction

• The sentence, "The set *X* is the same regardless whether counterexamples exist or not" is meaningless.

• No comparison of the two cases, counterexamples exist and counterexamples do not exist, is legitimate because we live in only one universe and in that universe only one of the cases holds. Our only problem is that we do not know which universe we are living in, the one in which counterexamples exist or the one in which counterexamples do not exist.

We will respond to each of these criticisms below. We invite reader comments — especially comments that are supported by authoritative sources.

Criticism: The Word "Counterexample" Is Not Legitimate in a Lemma Or In a Proof Connected With The 3*x* + 1 Problem

Only one reader has made this objection. The reader asserted that one can only show directly, without proof by contradiction, that each odd, positive integer maps to 1, or does not map to 1.

Reply

We believe that most researchers working on the $3x + 1$ Problem would regard this objection as extreme.

Criticism: It Is Illegitimate To Have "If Counterexamples Exist..." And "If Counterexamples Do Not Exist..." in a Proof

This in essence is a claim that it is illegitimate to *compare* the two cases, counterexamples exist, and counterexamples do not exist, because such a comparison implies that counterexamples exist and counterexamples do not exist, which, of course, is absurd.

Reply 1

Those readers who objected to our making the comparison in essence claimed that "comparison implies simultaneous existence", and, obviously, counterexamples cannot simultaneously exist and not exist. But a simple truth-table argument refutes the readers' claim.

Let p denote "Counterexamples to the 3x + 1 Conjecture exist". Now consider:

(1)
$(p \Rightarrow r)$ and $(\sim p \Rightarrow s) \Rightarrow (p$ and $\sim p)$,

where "$\Rightarrow$" denotes "implies",
"$\sim$" denotes "not",
r is a true statement describing properties that exist if p is true, and
s is a true statement describing properties that exist if $\sim p$ is true.

The truth table for (1) yields (true ⇒ false), which is a false implication. So it is false that the comparison of the two cases, *p* and *~p,* implies that both exist simultaneously. We now regard "comparison implies simultaneous existence" to be a logical fallacy.

Reply 2

The truth of the following two statements confirms the validity of our comparison strategy.

(1) If a mathematician writes, on a sheet of paper, "If *p*, then *X*" , where *p* is a statement, and then below this, on the same sheet of paper, he writes "If not-*p*, then *Y*" he has *not* thereby written a contradiction, in particular, the contradiction "*p* and not-*p* ..."!

(2) Furthermore, if in *X* he then shows that the integer *w* has the property *U*, and in *Y* he shows that the integer *w* has the property *not-U*, he has *not* thereby asserted that *w* has both the properties *U* and *not-U*.

Our Comparison Strategy compares *X* and *Y* and draws conclusions from that comparison. Neither *X* nor *Y* concludes with the equivalent of the statement, "Therefore the $3x + 1$ Conjecture is true" or of the statement "Therefore the $3x + 1$ Conjecture is false."

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However, we must hasten to add that, for our proofs of the $3x + 1$ Conjecture, the existence of the large Fixed-Set (see "Definition of "Fixed-Set"" on page 8) is essential, and, in particular the large subset of the Fixed-Set that consists of all odd, positive integers $\leq 10^{15} + 1$, these having all been determined by computer test, to be non-counterexamples. Our proofs most certainly are *not* based on the invalid argument that *because* this large subset exists, *therefore* all odd, positive integers are non-counterexamples. The large subset is just the point of departure for our proofs.

Reply 3

The phrase form, "If *p* then *X* but if not-*p* then *Y*" is routinely used, without objection, in everyday discourse in mathematics and the hard sciences. For example:

"If an odd perfect number exists, then ... but if an odd perfect number does not exist, then...", and

"If a counterexample to the $3x + 1$ Conjecture exists then there is a computer program that, in principle, will find the counterexample and halt. But if a counterexample does not exist, then the program will run forever," and

(Prior to the confirmation of the existence of the Higgs boson), "If the Higgs boson exists, then ... but if it doesn't exist, then ..."

An authority on mathematical logic has written us as follows:

"In Set Theory we frequently compare the situation where the Continuum Hypothesis [CH] holds with the situation where it fails, in the ongoing attempt to understand if the problem of CH has an answer. There are deep insights in each case and it is entirely possible that the problem of CH can only be 'solved' by this kind of analysis and comparison — i.e. the argument that CH is true may only be persuasive based on an analysis of the 'universes' where CH fails; or vice-versa."

Reply 4

Our Lemma 8.7 in the paper, "Are We Near a Solution to the $3x + 1$ Problem?" on occampress.com uses the two phrases in its statement, and thus far we know of no objections from readers.

Reply 5

Readers who believe that the two cases, counterexamples exist and counterexamples do not exist, cannot be compared, are obligated to give a rigorous description of the language that, as a result, must be prohibited — not only in our paper(s) but in all discourse about the $3x + 1$ Problem. We suspect that such a prohibition would be strongly resisted by $3x + 1$ researchers.

Criticism: The Phrase "Whether Or Not Counterexamples Exist..." Is Meaningless Or At Best Ambiguous; Therefore Any Proof That Contains This Phrase Is Fallacious

Reply

This phrase is equivalent to the two phrases in the previous Objection. We can only point to examples in which the phrase is neither meaningless nor ambiguous. Among them:

"Whether or not odd perfect numbers exist, 6 and 28 are even perfect numbers."
"Whether or not the Riemann Conjecture is true, Fermat's Last Theorem is true."
"Whether or not counterexamples to the $3x + 1$ Conjecture exist, 13 maps to 1."

These are not meaningless. Nor are they merely trivially true because they are equivalent to *(if p then q) and (if not-p then q)*, which, if we assume that *q* is true, is a true statement by the associated truth table. They are instead *statements of fact* of the form *whether or not p, q*. They assert that the truth or falsity of *p* has no effect on the truth of *q*. The reader can no doubt add numerous other examples involving statements of fact.

Criticism: A Sentence Beginning, "If Counterexamples Exist..." Is Ambiguous

The claim is that the sentence is ambiguous because we do not in fact know at present if counterexamples exist or not.

Reply 1

The claim has been restricted to sentences in our earlier proposed proof of the $3x + 1$ Conjecture, which seems strange, because readers have voiced no objections to a similar sentence in the statement of Lemma 5.0, which is referenced in our earlier proof, and which states:

If a counterexample exists, then for all i ≥ 2, each i-level tuple-set contains an infinity of i-level counterexample tuples and an infinity of i-level non-counterexample tuples.

Nor have there been any objections to our proofs by contradiction, which in effect begin with the sentence, "If counterexamples exist, then ... "

So, at the least, the readers seem to have been inconsistent in their claim.

Reply 2

Readers making the claim have also applied it (in email communications) to implications such as

If counterexamples exist, then there are odd, positive integers that do not map to 1 via the 3x + 1 function.

These readers seem not to be aware of what logicians call "Definitional Implication". In this type of implication, the antecedent is assumed to be true. The consequent then provides information *about the state of affairs that exists if the antecedent is true*. For example, consider:

If unicorns exist, then there are white horses with a goat's beard and a large, pointed, spiraling horn projecting from their forehead.

This implication is true because the consequent is the definition of "unicorn". The implication is not rendered ambiguous by the fact that we do not know for certain if unicorns exist. In fact, the implication is true even if we decide that unicorns do not exist. The truth follows from the definition of "unicorn".

But Definitional Implication is not merely implication in which both the antecedent and consequent are true. For example, "If counterexamples exist, then 2 + 2 = 4" is not an example of Definitional Implication, because the consequent gives no information about the antecedent.

Reply 3

Other readers have applied the claim to implications such as:

If counterexamples exist, then we may not be able to discover this fact using the computer because the length of infinite cycles might be too long to detect under the limitations of computation time or storage space, or because, even if there are no cycles, there is no way of detecting, with a computer, if the iterations of the 3x + 1 function on the argument x will never yield a 1.

Such an implication may be called an Informational Implication. In this type of implication, the antecedent is assumed true, and the implication is true if the consequent provides true information about the subject of the antecedent. But this true information is not necessarily a direct consequence of the definition of the antecedent, as is the case in Definitional Implication.

Here again, it is not valid to call the implication ambiguous on the grounds that we do not know if the antecedent is true. For, consider the equivalent of a statement that was certainly made in the literature prior to Wiles' proof of Fermat's Last Theorem (FLT), namely, *If a counterexample to FLT exists, then it will not be possible to discover that counterexample by computer test, because the numbers involved are too large.*

Again: it is not valid to call the implication ambiguous on the grounds that no one knew if the antecedent was true. The journal editors would certainly not have published the statement if they thought it was ambiguous.

Furthermore, the statement remains true even after Wiles' proof, provided we make appropriate indications of the time of its utterance, for example,

If a counterexample to FLT had existed, then it would not have been possible to discover that counterexample by computer test, because the numbers involved were too large.

Both Definitional Implication and Informational Implication (the latter is our term) are examples of Formal Implication, "in which a certain formal connection between antecedent and consequent is an indispensable condition of the meaningfulness and truth of the implication."[1]

Criticism: The Phrase "If Counterexamples Exist" Can Only Be Used In a Proof By Contradiction

Reply

No reader that we know of has objected to Lemma 5.0 in the above-cited paper, which begins with a sentence that is the equivalent of "If counterexamples exist" , namely, "Assume a counterexample exists." No reader that we know of has objected to Lemma 8.7 in the above-cited paper, which uses the phrase explicitly.

In any case, it would seem that these objections are nullified if the lemmas or other statements they appear in, are part of a proof by contradiction, which must begin with, "Assume counterexamples exist."

Criticism: "The Set *X* Is the Same Regardless Whether Counterexamples Exist or Not" Is Meaningless

This criticism was applied in two cases: (1) to the initial wording of Lemma 3.0[2], and (2) to the statement in one of our earlier proofs of the $3x + 1$ Conjecture, "The contents of tuple-sets are the same regardless whether counterexamples exist or not".

Reply to the Criticism as Applied to the Wording of Lemma 3.0:

The set J of odd, positive integers maps to 1, regardless whether counterexamples exist or not.

The criticism was retracted by most readers when we changed the wording to

1. Tarski, Alfred, *Introduction to Logic and to the Metholdology of Deductive Sciences*, Oxford University Press, N.Y., 1970, p. 26

2. Lemma 8.8 in earlier versions of our paper.

Exactly one set J of odd, positive integers maps to 1, regardless whether counterexamples exist or not,

and then added:

In other words:

If counterexamples exist, then the set of odd, positive integers that map to 1 is J.

If counterexamples do not exist, then the set of odd, positive integers that map to 1 is J.

However the criticism has persisted in other contexts. For example, let "Conjecture *Y*" denote any conjecture that is unresolved at the time of reading the following. Then it seems to us perfectly clear that

The set P of positive primes is the same regardless whether counterexamples to Conjecture Y exist or not,

simply means

There is exactly one set P of positive primes, regardless whether counterexamples to Conjecture Y exist or not.

In other words:

If counterexamples to Conjecture Y exist, then the set of positive primes is P = {2, 3, 5, 7, ...}.

If counterexamples to Conjecture Y do not exist, then the set of positive primes is P = {2, 3, 5, 7, ...}.

Reply to the Criticism as Applied to a Statement in a Proof of the $3x + 1$ Conjecture

In one of our earlier proofs of the $3x + 1$ Conjecture, a step concluded with the sentence, "The contents of tuple-sets are the same regardless whether counterexamples exist or not". Many readers rejected the sentence, arguing that the sentence implies that counterexamples exist and do not exist at the same time, which is absurd. (In subsequent proofs we were able to eliminate the sentence.) None of these readers found an error in any of the sentences leading up to the one in question, which as far as we were concerned meant that these readers considered the sentence as logically valid but false. Their failure to recognize the significance of such an occurrence was, we felt, an indication of their lack of mathematical knowledge and insight.

That the sentence is perfectly valid is easily shown by the following argument. Pick any odd, positive integer in any tuple in any tuple-set. If it is a counterexample, then, by the sentence, its behavior must be the same as that of a non-counterexample. But then it is not a counterexample, a contradiction. Therefore we can never pick a counterexample, hence counterexamples do not exist.

Criticism: No Comparison of the Two Cases, Counterexamples Exist and Counterexamples Do Not Exist, Is Legitimate Because We Live in Only One Universe

Reply

We can make the error in this criticism clear by imagining that there is a black box that contains either a white marble or a black marble. The Marble Conjecture states that the marble is white. Now here, it is correct to argue that any line of reasoning that considers both the case, The marble is white, and the case, The marble is black, is illegitimate, because we live in only one universe. Either the marble is white or it is black. Therefore, to write, "If the marble is white, then ..." leaves open the criticism that if the marble is in fact black, then we have a false antecedent, and false implies anything.

However, the $3x + 1$ Problem is fundamentally different, because there are odd, positive integers (in fact an infinity, as is easily demonstrated) that are known to map to 1, and these integers will map to 1 no matter if the $3x + 1$ Conjecture is one day proved or disproved — in other words, whether or not counterexamples exist. (The proof is in Lemma 3.0 in our paper, "A Solution to the 3x + 1 Problem", on occampress.com.) This fact is crucial to understanding our proofs of the Conjecture.

A Word on "False Implies Anything"

Readers who are impatient to convince themselves that our proposed solution to the $3x + 1$ Problem is without merit, often find a statement of ours that (because they do not understand it) they deem to be false , and then immediately point out that "false implies anything", meaning that the rest of our proof cannot be taken seriously.

These readers seem to be unclear on the nature of material implication, that is, on the meaning of the truth table for p implies q. This truth table says that:

if p is false, and q is true, then the *implication* is true, and

if p is false, and q is false, then likewise the *implication* is true.

But most emphatically, the truth table does *not* say that, in the second case, the falsity of p somehow "proves" anything we want. Consider proof by contradiction. We begin by assuming that what we want to prove is false. In other words, we begin with a false p. Using correct laws of deduction, we arrive at a contradiction. From that we infer that our original assumption was false, and that p is really true.

Our original assumption did not allow us to prove "anything". It compelled us to deduce a contradiction, and from that to conclude that our original assumption was false.

In fact, whoever says "false implies anything" must be prepared to answer the question, "Implies anything in accordance with correct laws of logic, or no?" If the answer is no, then not only a false antecedent but a true antecedent implies anything.

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