Appendix A — Statements of Lemmas

This section contains statements of all results that we have obtained to date. A few of the results are already known in the literature, but are included for ease of reference. *The reader is encouraged to use the"Table of Symbols and Terms" on page 103 to look up definitions of terms*, and, of course, in any of our papers, to use the Search facility that is available with all .pdf files on a web site.

Proofs that are not given in this paper are given in the papers, "The Structure of the 3*x* + 1 Function", "Are We Near a Solution to the 3x + 1 Problem?" and "A Solution to the 3x + 1 Problem" on the web site www.occampress.com.

The term "[so]" following a lemma number means that the statement and proof of the lemma will be found in the paper, "A Solution to the 3*x* + 1 Problem" on the web site www.occampress.com.

The term "[ar]" following a lemma number means that the statement and proof of the lemma will be found in the paper, "Are We Near a Solution to the 3*x* + 1 Problem?" on the web site www.occampress.com.

The term "[st]" following a lemma number means that the statement and proof of the lemma will be found in the paper, "The Structure of the 3*x* + 1 Function" on the web site www.occampress.com.

In some cases, a given lemma has a different number in one or more of our other papers. We have indicated this. Sometimes we have inserted a lemma from another paper into a group of lemmas dealing with the same subject, even though the number of that lemma is not in proper numerical order.

Note: The vast majority of the following results apply equally to counterexamples to the 3*x* + 1 Conjecture. Many of them also appear to apply to other 3*x* + 1- like functions (the definition of this type of function is given in [so]), in particular to those for which counterexamples to the corresponding conjectures are known to exist.. This is one of the major reasons why the 3*x* + 1 Conjecture is so difficult to prove. At present we believe that the best hope for a proof lies in the "Filling-in" Strategy" that is described in the first file of this paper.

Tuple-sets

Lemma 0.0 *Given a range element y, there exists a range element x such that C(x) = y.*

Lemma 0.2. (Lemma 9.0 in [so], Lemma 10.0 in [ar]) *No multiple of 3 is a range element.*

Lemma 0.4 (Lemma 10.0 in [so], Lemma 11.0 in [ar]) *Each odd, positive integer (except a multiple of 3) is generated by a multiple of 3 in one iteration of the 3x + 1 function.*

Lemma 1.0 *(a) Let* $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, *be a sequence of exponents, and let* t_k, t_m *be tuples consecutive at level i in* T_A. *Then d(i, i), the distance between* t_k *and* t_m *at level i, is defined to be the absolute value of the difference between the level i elements of* t_k *and* t_m, *i.e., is defined to be* $|t_{k(i)} - t_{m(i)}|$, *and is given by*:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) Let t_k, t_m be tuples consecutive at level i in T_A. Then d(1, i), the distance between t_k and t_m at level 1, is defined to be the absolute value of the difference between the level 1 elements of t_k and t_m, i.e., is defined to be $|t_{k(1)} - t_{m(1)}|$, and is given by:

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3})\ldots(2^{a_i})$$

Lemma 1.1 *Let T_A be a tuple-set defined by a sequence $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$. Then the distance d(j, i) between elements at level j, $1 \leq j \leq i$, of tuples t_k, t_m consecutive at level i is given by the following table:*

Table 1: Distances between elements of tuples t_k, t_m, consecutive at level i

Level	**Distances between elements of t_k, t_m at level**
i	$2 \cdot 3^{i-1}$
i - 1	$2 \cdot 3^{i-2} \cdot 2^{a_i}$
i - 2	$2 \cdot 3^{i-3} \cdot 2^{a_{i-1}}2^{a_i}$
i - 3	$2 \cdot 3^{i-4} \cdot 2^{a_{i-2}}2^{a_{i-1}}2^{a_i}$
...	...
2	$2 \cdot 3 \cdot 2^{a_3}\ldots2^{a_{i-1}}2^{a_i}$
1	$2 \cdot 2^{a_2}2^{a_3}...2^{a_{i-1}}\ 2^{a_i}$

Lemma 1.2 (Lemma 4.5 in [so]) *The number of i-level tuple-sets is countably infinite.*

Lemma 0.0 in [so] *Each i-level tuple-set, where $i \geq 2$, contains an infinity of tuples of each length j, where $1 \leq j \leq i$.*

Lemma 4.75 in [so] *For each $i \geq 2$, the set of all i-level elements of all i-level tuples in all i-level tuple-sets is the set of all range elements of the 3x + 1 function.*

Lemma 2.0 in [so] *For each exponent a_2, a tuple-set T_A, where $A = \{a_2\}$, exists.*

Lemma 4.0 in [so] *For each exponent sequence $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, there exists a tuple-set T_A determined by A.*

Lemma 1.3. *Let x, x′ be two odd, positive integers. Then the infinite upward exponent sequence $\overline{X}$ defined by the infinite upward tuple of which x is the first element, and the infinite upward exponent sequence $\overline{X}'$ defined by the infinite upward tuple of which x′ is the first element, cannot be the same.*

Lemma 1.4. *Let y, z be two range elements. Let $\{\overline{Y}\}$ be the* ***set*** *of infinite downward exponent sequences defined by all downward extensions of <y>, and let $\{\overline{Z}\}$ be the* ***set*** *of infinite downward exponent sequences defined by all downward extensions of <z>. Then $\{\overline{Y}\} \neq \{\overline{Z}\}$.*

Lemma 1.5. *Each cycle in the odd, negative integers defines an infinite exponent sequence $\underline{A}$ such that no odd, positive integer x generates $\underline{A}$. Examples of such sequences $\underline{A}$ are: A*{1, 1, 1, ... }, A*{1, 2, 1, 2, ..., } and A*{1, 1, 1, 2, 1, 1, 4}, where A is a finite (possibly empty) exponent sequence.* (Proof is under "Some Infinite Exponent Sequences That are Not Generated by Any Odd, Positive Integer, x" on page 25.)

Lemma 2.0. *Each i-level tuple-set can be extended by an arbitrary exponent a_{i+1}. Or, in other words, for each i-level tuple-set and for each a_{i+1}, the i-level row — though not every element in the i-level row — maps to a non-empty row in some (i+1)-level tuple-set.*

Lemma 3.0. *For each range element y there exists an i-level tuple-set in which y is an element of the first i-level tuple.*

Lemma 3.055. *The top row of an i-level tuple-set is a reduced residue class modulo $2 \cdot 3^{(i-1)}$.*

Lemma 3.057. *The set of minimum elements of all top rows in all i-level tuple-sets is the set of minimum residues of the set of reduced residue classes mod $2 \cdot 3^{i-1}$.* (Proof is under "Possible Strategies for Proving the 3x + 1 Conjecture Using Tuple-sets" on page 19.)

Lemma 3.0574. *Let M_i denote the set of all minimum residues of reduced residue classes mod $2 \cdot 3^{i-1}$. For each $i \geq 2$, the number of elements of M_i, which we will denote $|M_i|$, is $\varphi(2 \cdot 3^{(i-1)}) = 2 \cdot 3^{(i-2)}$, where φ is Euler's totient function, i.e., the function that returns the number of numbers less than its argument and relatively prime to its argument.* (Proof is under "Possible Strategies for Proving the 3x + 1 Conjecture Using Tuple-sets" on page 19.)

Lemma 3.059: *Let* **R** *denote the set of all range elements of the 3x + 1 function. Then for all $i \geq 2$, the set of all last elements of all i-level tuples in all i-level tuple-sets =* **R**.

Lemma (Conway or Thompson): *There are at most a finite number of cycles*.

Lemma 3.06. *If a cycle exists, it must be of length at least 10,700,000 iterations.* (This result is derived from (Eliahou 1993)[1].)

Lemma 3.07. *At most one cycle exists having a given sequence of exponents.*

Lemma 3.08. *No cycle exists in the sequence of tuple-sets defined by unlimited, successive concatenations of the exponent 1.*

Lemma 3.1. *Let A consist of an infinitely repeating cycle of exponents, i.e., let* $A = \{a_2, a_3, ..., a_m = a_2, a_{m+1} = a_3, ... \}$, $m \geq 3$. *Then, informally, no tuple such that elements don't repeat when elements of A repeat, can be an infinite-tuple. Formally, if* $a_{i+1} = a_{m+i+1}$ *for all* $i \geq 1$ *and* $t_{j(i)} \neq t_{j(m+i)}$, *then* t_j *is not an infinite-tuple.*
(Thus, e.g., let *A* denote any finite exponent sequence whatsoever, and let *A′* denote any finite exponent sequence such that, if the sequence is defined by the tuple <*x*, ..., *y*>, then $x < y$. Then there does not exist an odd, positive integer that generates an infinite tuple that defines the exponent sequence *A* * *A′* * *A′* * *A′* * ... , where * denotes concatenation of sequences.

Lemma 5.5 (in [ar]). *Let a be a finite exponent sequence such that if x maps to y via a, then y > x. Then there does not exist a counterexample x such that the infinite tuple <x, ... > is associated with the exponent sequence {a, a, a, ... }.*

Lemma 3.24. *Let x be a range element that is a minimum residue of a reduced residue class mod* $2 \cdot 3^{(i-1)}$*, and let*

$$\frac{3x+1}{2^j} = h$$

Then if

$$j \equiv k \mod (2 \cdot 3^{i-1})$$

there exists an x′ such that

$$\frac{3x'+1}{2^k} = h$$

and furthermore

$$x \equiv x' \mod (2 \cdot 3^{i-1})$$

1. Eliahou's result is actually as follows: if the 3*x* + 1 function is defined as: if *x* is odd, then (3*x* + 1)/2, else *x*/2, and if each possibility is considered to be an iteration of the function, then the maximum number of iterations in a cycle is 17,087,915. We convert this figure to the one given in Lemma 3.06 above by recognizing that a cycle in Eliahou's definition can be represented by 3/2 • 3/2 • ... • 3/2 • 1/2 • 1/2 •... • 1/2 = 1; where 3/2 is a close approximation (since the numbers involved are large) to the effect of an iteration if *x* is odd, and 1/2 is the effect if *x* is even. Since 100% of the denominators = 2, we then ask what percentage of the numerators must be 3 in order for the equation to hold. In other words, we need to solve $3^y/2 = 1$. The answer is that *y* is about 63%. But this corresponds to the number of iterations of our own definition, *C*, of the 3*x* + 1 function. Thus the equivalent of Eliahou's 17,087,915 is 0.63 • 17,087,915 or about 10,700,000.

Lemma 3.25. *The first elements of tuples consecutive at level 2 in all 2-level tuple-sets are as described in the following tables.*

Table 2: First elements of tuples consecutive at level 2: odd powers

Exponent a_2	**First elements of tuples consecutive at level 2**
1	$\{x \mid (x \equiv 3 \ \ mod \ (2 \cdot 2^1))\}$
3	$\{x \mid (x \equiv 3 + 5(2^1) mod \ (2 \cdot 2^3))\}$
5	$\{x \mid (x \equiv 3 + 5(2^1 + 2^3) mod \ (2 \cdot 2^5))\}$
7	$\{x \mid (x \equiv 3 + 5(2^1 + 2^3 + 2^5) mod \ (2 \cdot 2^7))\}$
...	...
$2k + 1$	$\{x \mid (x \equiv 3 + 5(2^1 + 2^3 + 2^5 + \ldots + 2^{2(k-1)+1}) mod \ (2 \cdot 2^{2k+1}))\}$

Table 3: First elements of tuples consecutive at level 2: even powers

Exponent a_2	**First elements of tuples consecutive at level 2**
2	$\{x \mid (x \equiv 1 mod \ (2 \cdot 2^2))\}$
4	$\{x \mid (x \equiv 1 + 1(2^2) mod \ (2 \cdot 2^4))\}$
6	$\{x \mid (x \equiv 1 + 1(2^2 + 2^4) mod \ (2 \cdot 2^6))\}$
8	$\{x \mid (x \equiv 1 + 1(2^2 + 2^4 + 2^6) mod \ (2 \cdot 2^8))\}$
...	...
$2k$	$\{x \mid (x \equiv 1 + 1(2^2 + 2^4 + 2^6 + \ldots + 2^{2(k-1)}) mod \ (2 \cdot 2^{2k}))\}$

Lemma 3.28. *For each $i \geq 2$, let $\{T_A\}_i$ denote the set of all i-level tuple-sets, i.e., the set of all tuple-sets defined by exponent sequences $A = \{a_2, a_3, ..., a_i\}$ where a_j is a positive integer. Then (a) for level 1, the set of all elements in all 1-level rows is the set of domain elements; (b) for level $1 < j \leq i$, the set of all elements in all j-level rows in $\{T_A\}_i$ is the set of all range elements.*

Lemma 4.0. *If t_1 is the first i-level tuple in an i-level tuple-set (i.e., if t_1 is an anchor tuple), then the extension of t_1 is the first (i + 1)-level tuple in the tuple-set its extension defines (i.e., the extension of t_1 is also an anchor tuple). And so on, recursively. "Once an anchor tuple, always an anchor tuple.")*

Lemma 3.0 (in [so]) *Each i-level tuple-set T_A, where $A = \{a_2, a_3, ..., a_i\}$ and $i \geq 2$, has an extension via each exponent a_{i+1}.*

Lemma 5.0 (Lemma 13.0 in [ar], Lemma 5.0 in [st]) *Each range element y is mapped to, in one iteration of the 3x + 1 function, by all exponents of one parity only.*

Lemma 5.0 (in [so]) *Assume a counterexample exists. Then for all $i \geq 2$, each i-level tuple-set contains an infinity of i-level counterexample tuples and an infinity of i-level non-counterexample tuples.*

Lemma 9.7 [so] *(a) If counterexamples do not exist, then for all i-level tuple-sets $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, if x is the first element of an i-level (necessarily non-counterexample) tuple in T_A, then the first element of the next i-level (necessarily non-counterexample) tuple is*

(1)

$$(x + (2 \cdot (2^{a_2})(2^{a_3})\ldots(2^{a_i})))$$

(b) If counterexamples exist, then in each i-level tuple-set $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, there exists an x which is the first element of an i-level non-counterexample tuple in T_A such that the first element of the next i-level non-counterexample tuple in T_A is greater than the value in (1).

Lemmas 5.5, 5.7. *If x maps to y in one iteration of the 3x + 1 function, then:*
If $x \equiv 1$ *mod* 4 *then the exponent of* 2 *is* ≥ 2*;*
If $x \equiv 3$ *mod* 4 *then the exponent of* 2 = 1*;*
If $y \equiv 1$ *mod* 3 *then the exponent of* 2 *is even*;
if $y \equiv 2$ *mod* 3 th*en the exponent of 2 is odd.*

Remark: these congruences are represented by elements of 2-tuples in 2-level tuple-sets.

We observe that $\{x \mid x \equiv 1 \text{ mod } 4\} = \{1, 5, 9, 13, 17, 21, 25, ...\}$.
The set of first elements of 2-tuples in the tuple-set $T_{\{2\}}$ is {1, 9, 17, 25, ... }. (Lemma 1.0, part (b))

We observe that $\{x \mid x \equiv 3 \text{ mod } 4\} = \{3, 7, 11, 15, 19, 23, 27, ...\}$.
The set of first elements of 2-tuples in the tuple-set $T_{\{1\}}$ is {3, 7, 11, 15, 19, 23, 27, ...}. (Lemma 1.0, part (b))

We observe that $\{y \mid y \equiv 1 \text{ mod } 3\} = \{1, 4, 7, 10, 13, 16, 19, ... \}$.
The set of second elements of 2-tuples in the tuple-set T_A, where $A = \{a_2\}$ and a_2 is even, is {1, 7, 13, 19, ...} (Lemma 1.0, part (a)}

We observe that $\{y \mid y \equiv 2 \text{ mod } 3\} = \{2, 5, 8, 11, 14, 17, 20, 23, ... \}$.
The set of second elements of 2-tuples in any 2-level tuple-set T_A, where $A = \{a_2\}$ and a_2 is odd, is {5, 11, 17, 23, ...}. (Lemma 1.0, part (a)}

Lemma 6.0 (Lemma 13.0 in [so], Lemma 14.0 in [ar])**.** *There exists an explicit recursive construction of the tuple-set produced by a given tuple.*

Lemma 7.0.(Lemma 14.0 and Lemma 18.0 in [so] and [ar]) . *For each range element y, and for each exponent sequence A, there exists an x that maps to y via A, possibly followed by an additional "buffer" exponent*

Lemma 7.1. *Let* $A = \{a_2, a_3, ..., a_i\}$ *be a sequence of exponents, and let y be an arbitrary range element. Then the maximum buffer exponent j (see proof of Lemma 7.0) required to ensure that an x exists that maps to y via A, is* $2 \cdot 3^{i-1}$.

Lemma 7.25. *Let* R_i *be the top-level row of an i-level tuple-set, where* $i \geq 2$. *Let* $f(R_i, a_{i+1})$ *denote the row produced by applying the 3x + 1 function to all elements x of* R_i*, and then selecting only those y yielded by* $ord_2(3x + 1) = a_{i+1}$. *Then the set* $\{ f(R_i, a_{i+1}) | a_{i+1} \geq 1\}$ *is the set of top rows of all (i + 1)-level tuple-sets, i.e., the set of reduced residue classes mod* $2 \cdot 3^{(i+1)-1}$.

Lemma 7.27. *For all* $i \geq 2$*, and for all (i + 1)-level top rows* R_{i+1}*, the minimum set of i-level top rows required to generate* R_{i+1} *via all possible exponents that can generate* R_{i+1}*, is* $\{R_i\}$*, the set of all i-level top rows. In other words, for all* $i \geq 2$*, and for all (i + 1)-level top rows* R_{i+1}*, if we generate* R_{i+1} *by a proper subset of* $\{R_i\}$*, then some elements of* R_{i+1} *will be generated by a proper subset of the set of exponents that can generate these elements.*

Lemma 7.3. *Let* R_i *be a top-level row of an i-level tuple-set,* $i \geq 2$. *Then all exponents* $a_{i+1} \geq 1$ *can be partitioned into* $(2 \cdot 3^{i-2})$ *equivalence classes such that all* a_{i+1} *that are in a given class, generate the same (i + 1)-level row* $R_{i+1} = f(R_i, a_{i+1})$*, where f is as defined in Lemma 7.25.*

Lemma 7.31. *Let* a_{i+1}*,* $i \geq 2$*, be an exponent that is "missing" from the set of exponents that generate a top row* R_{i+1} *because* a_{i+1} *is the exponent for a multiple-of-3. Then all exponents congruent to* a_{i+1} *mod* $2 \cdot 3^{i-2}$ *are likewise missing from the set of exponents that generate* R_{i+1}.

Lemma 7.32. *Let* R_{i+1} *be the top row of an (i + 1)-level tuple-set. Then* R_{i+1} *is generated by exponents of one parity only.*

Lemma 7.35. *Let* R_{i+1} *be the top row of an (i + 1)-level tuple-set. Let*

$$\{min\{a_{i+1}\}_{R_{i+1}}\}$$

be the set of minimum residues of all exponent congruence classes (i.e., equivalence classes) whose exponents map to R_{i+1}. *(By Lemmas 7.1 and 7.3 we know that each such class is a residue class mod* $2 \cdot 3^{(i+1)-1}$ *.) Then at least one element of*

$$\{min\{a_{i+1}\}_{R_{i+1}}\}$$

is ≤ 4.

Lemma 7.36. *Let R_i be the top row of an i-level tuple-set and let r_{0i} be its first element. Then the first element, $r_{0(i+1)}$, of the (i + 1)-level top row $R_{(i+1)}$ mapped to by R_i via the exponent a_{i+1} is given by*:

$$r_{0(i+1)} \equiv (2^{(j(2 \cdot 3^{i-1}) - a_{i+1})}(3r_{0i} + 1) - 3^i) mod(2 \cdot 3^{i-1})$$

where j is chosen *to make the exponent positive*.

Lemma 7.38. *Let $A = \{a_2, a_3, ..., a_i)$ be an exponent sequence, and let $a = a_2 + a_3 + ... + a_i$. Let r be as defined in the proof of Lemma 6.0. Then the smallest element r_{0i} of the top row of the tuple-set T_A is given by*

$$r_{0i} \equiv (2^{j(2 \cdot 3^{i-1}) - a}r - 3^i) \;\; mod(2 \cdot 3^i)$$

Lemma 7.4. *Let:*

$$y, y + lcm(2 \bullet 3^{i-1}, 2 \bullet 2^{a_i})$$

where lcm denotes the least common multiple and $i \geq 2$, be successive elements of the sub-row R'_i of the top row R_i that maps to the top row R_{i+1} via the exponent a_i. Then these successive elements map to successive elements of R_{i+1}. In other words, each such sub-row of a top row R_i maps to an entire top row R_{i+1}.

Lemma 8.0. *Let T be a 2-level tuple-set. Then the first 2-level tuple of T is an n-t-v-1.*

Lemma 10.0. (Lemma 5.0 in [so]) *Assume a counterexample exists. Then for all $i \geq 2$, every i-level tuple-set contains an infinity of i-level non-counterexample tuples and an infinity of i-level counterexample tuples*.

(This lemma establishes that there is no way to distinguish counterexamples from non-counterexamples on the basis of the ***finite*** exponent sequences generated by each.)

Lemma 10.5. *Let T_A be an i-level tuple-set defined by an exponent sequence $A * a_i$, where $i \geq 4$, "*" denotes concatenation of exponents, A is an arbitrary exponent sequence of length i - 2, and a_i is even if the last exponent of A is odd, and a_i is odd if the last exponent of A is even. Then the first i- level tuple of T_A is an n-t-v-1.*

Lemma 10.8. *The topology TT* [defined on tuples in tuple-sets] *is Hausdorff.*

Lemma 10.83. *A metric exists on the topological space TT.*

Lemma 7.0 (in {so]) *(a) For each i-level tuple-set T_A, where $A = \{a_2, a_3, ..., a_i\}$, the set of all i-level elements of all i-level tuples is a reduced residue class mod $2 \bullet 3^{(i-1)}$.*

(b) The set of all such reduced residue classes, over all i-level tuple-sets T_A, is a complete set of reduced residue classes mod $2 \cdot 3^{(i-1)}$.

Lemma 6.0 in [so] *Let t be the anchor tuple (by definition an i-level tuple) in an i-level tuple-set, where $i \geq 2$. Then the last element y of t, that is, the i-level element of t (this element being the anchor), is a number less than* $2 \cdot 3^{(i-1)}$.

Lemma 10.90. (Lemma 8.0 in [so] and [ar]) *For each odd, positive integer x there exists a minimum $i = i_0$ such that for all $i \geq i_0$, x is the first element of the first i-level tuple in some i-level tuple-set, i.e., x is the first element of an anchor tuple at i in some i-level tuple-set. In terms of infinite tuples, this lemma states: if x is an odd, positive integer, then in the infinite tuple $\bar{t}$ = <x, $\underline{y}$, y′, ... >, there exists a minimum level i_0 such that:*

$\bar{t}(i_0)$ is the i_0-level anchor tuple in an i_0-level tuple-set;
$\bar{t}(i_0 + 1)$ is the $(i_0 + 1)$-level anchor tuple in an $(i_0 + 1)$-level tuple-set;
$\bar{t}(i_0 + 2)$ is the $(i_0 + 2)$-level anchor tuple in an $(i_0 + 2)$-level tuple-set;
etc.

Lemma 10.905. *Let $M_{nc,\, i}$ denote the set of minimum non-counterexamples at level i. Then $M_{nc,\, i}$ constitutes the set of last elements of a set of i-level tuples that are complete at i. And similarly if $M_{c,\, i}$ denotes the set of minimum counterexamples at level i.* (Proof is in Appendix A1.)

Lemma 10.906. (a) *Let S be a set of elements that are i-level downward- or upward-complete, $i \geq 2$. Then for all S such that $S \subset S'$, S′ is also downward or upward i-level complete.*

(b) *Let S′ be a set of elements that are i-level downward- or upward-incomplete. Then for all S such that $S \subset S'$, S is also i-level downward- or upward incomplete.*

(c) *Let S be a set of elements that are i-level downward- or upward-incomplete. Then if $S \subset S'$, S′ may be i-level downward- or upward-incomplete or -complete.*

Lemma 10.907. *Let M_i denote the set of anchors at i, i.e., the set of minimum residues of the set of reduced residue classes mod $2 \cdot 3^{i-1}$. Then every element y of M_i must be the first element of some i-level anchor tuple. Furthermore, if an element y of M_i is the first element of the kth i-level tuple in an i-level tuple-set, then so must the first elements of the 1st, 2nd, ..., (k - 1)th i-level tuples be elements of M_i.*
(Proof is in Appendix A1.)

Lemma 10.91. *Let t be an i-level anchor tuple. Then t is the first i-level tuple in an i-level tuple-set.*
(Proof is in Appendix A1.)

Lemma 10.92. *Let t = <$x_1, x_2, ..., x_i$> be an i-level anchor tuple, and let t′ = <$x_1, x_2, ..., x_i, x_{i+1}$> be the extension of t. Then t′ is an (i+1)-level anchor tuple.*
(Proof is in Appendix A1.)

Lemma 10.93. *Let t = <$x_1, x_2, ..., x_i$> be an i-level anchor tuple that generates the exponent sequence A = {a2, a3, ..., ai}. By definition, x_i is an anchor. At level i+1, there exists an (i+1)-*

level anchor tuple having x_i (now an (i+1)-level anchor), as its last element, that generates one of two exponent sequences: either $\{a_1, a_2, a_3, ..., a_i\}$, or $\{a_1', a_2', a_3, ..., a_i\}$.
(Proof is in Appendix A1.)

Lemma 10.95.
Let y be a range element. Then y is mapped to by all (i -1)-level exponent sequences A, $i \geq 3$, followed by some exponent a_i.
(Proof is in Appendix A1.)

Lemma 10.96.
(a) If a counterexample exists, then for all $i \geq i_0$, where i_0 is the smallest i such that a counterexample is an anchor at i, the set of anchor tuples at i is partitioned into two disjoint sets: the set $\{t_c\}$ of counterexample anchor tuples and the set $\{t_{nc}\}$ of non-counterexample anchor tuples. Otherwise, if there are no counterexamples, the set of anchor tuples at i, $i \geq 2$, consists exclusively of non-counterexample anchor tuples.

(b) For each $i \geq i_0$, let $\{A_{nc}\}$ denote the set of all exponent sequences defined by $\{t_{nc}\}$ in part (a), and let $\{A_c\}$ denote the set of all exponent sequences defined by $\{t_c\}$ in part (a). Then $\{A_{nc}\} \cap$ and $\{A_c\} = \varnothing$. (Proof is in Appendix A1.)

Lemma 10.965. *In the set of all anchor tuples at i, it must always be the case that if $A * \{a\}$ maps to the anchor y, and $A * \{a'\}$ maps to the anchor y', where $y \neq y'$, then necessarily $a \neq a'$.* (Proof is in Appendix A1.)

Lemma 10.97.
(**a**) *If no counterexamples exist, then for all $i \geq 2$ the set of non-counterexample anchor tuples at i is complete.*

(**b**) *If counterexamples exist, then for all $i \geq i_0$, where i_0 is the smallest i such that a counterexample is an anchor at i:*

the set $\{t_{nc}\}$ of non-counterexample anchor tuples at i is not complete at i, and
the set $\{t_c\}$ of counterexample anchor tuples at i is not complete at i, and

$\{t_{nc}\} \cap \{t_c\} = \varnothing$ *and*

$\{t_{nc}\} \cup \{t_c\} = \{A\}$, *where {A} is the set of all i-level exponent sequences.*
(Proof is in Appendix A1.)

Lemma 10.981. *For every suffix s, there exists an infinity of* $\mathbf{R}'_{nc}$ *that are mapped to by s, where* .$\mathbf{R}'_{nc}$ *denotes the set of non-counterexample range elements.* (Proof is in Appendix A1.)

Lemma 10.982. *In order for a suffix s of length (i - 1) to map to an anchor, it is necessary that s map to a y_{nc} that is less than $2 \cdot 3^{(i-1)}$. s then maps to the anchor y_{nc} at i + 1, i + 2, ...* (Proof is in Appendix A1.)

Lemma 10.983. *Every suffix s_1 of length 1 maps to an anchor y_{nc} at i for i = 2, 3, 4, ...*
Every suffix s_2 of length 2 maps to an anchor y_{nc} at i for i = 3, 4, 5, ...
...
Every suffix s_{34} of length 34 maps to an anchor y_{nc} at i for i = 35, 36, 37,...
(Proof is in Appendix A1.)

Lemma 10.984. *If a counterexample exists, then for every suffix s there exists a least i such that s maps to a y_{nc} anchor at i and s maps to a y_c anchor at i. This fact generalizes to any finite number of anchors y_{nc} and y_c.* (Proof is in Appendix A1.)

Lemma 10.985. *Let s be a suffix of length (i - 1). If s is an anchor sequence at all, it must be an anchor sequence at i and i alone.* (Proof is in Appendix A1.)

Lemma 10.986. *Every suffix s is a suffix of an infinity of anchor sequences.* (Proof is in Appendix A1.)

Lemma 11.0 in [so] (Lemma 12.0 in [ar]) *For each range element y there exists an infinity of x that map directly to y. Specifically,*
If

$$\frac{3x+1}{2^a} = y$$

Then, for each n ≥ 1,

$$\frac{3(x + (2^{a+2(0)} + 2^{a+2(1)} + \ldots + 2^{a+2(n-1)})y) + 1}{2^{a+2(n)}} = y$$

Lemma 8.5 (in [ar]) *Assume counterexamples exist. Let $\bar{t}_{nc}$, $\bar{t}_c$ be non-counterexample and counterexample infinite tuples, respectively, with marks m_{nc}, m_c respectively.*
Then for all levels $i \geq \max(m_{nc}, m_c) = i_0$, $A(\bar{t}_{nc}(i)) \neq A(\bar{t}_c(i))$, where max(u, v) denotes the maximum of u, v, and A(t) denotes the exponent sequence associated with the tuple t.

Lemma 8.7 (in [ar]) *If counterexamples do not exist, then*
(a) *For each $i \geq 2$, the set of i-level non-counterexample anchor tuples is complete.*

(b) *Each non-counterexample infinite tuple has a prefix, namely, that determined by its mark, such that that prefix, and all larger prefixes, are elements of* ***complete*** *sets of non-counterexample anchor tuples.*

If counterexamples exist, then

(c) F*or each* $i \geq$ *some* i_0*, the set of i-level non-counterexample anchor tuples is incomplete, so that a complete set of i-level non-counterexample tuples must include tuples other than anchor tuples.*
(d) *Each non-counterexample infinite tuple has a prefix, namely, that determined by its mark, such that that prefix, and all larger prefixes, are elements of* ***incomplete*** *sets of non-counterexample anchor tuples.*

Recursive "Spiral"s

Lemma 8.8. [so] *Exactly one set J of odd, positive integers maps to 1, regardless whether counterexamples exist or not. In other words:*

If counterexamples exist, then the set of odd, positive integers that map to 1 is J.
If counterexamples do not exist, then the set of odd, positive integers that map to 1 is J.

Lemma 11.0. *(a) The distance between the jth element,* $j \geq 1$*, of a "spiral", and the base element y of the "spiral", is given by* $|(2^k y - 1)/3 - y|$*, where k is the jth element in the sequence <1, 3, 5, ..., > or the sequence <2, 4, 6, ..., > as established by y.*
(b) The distance between successive elements x, x′ of a "spiral" is given by 3x + 1. In other words, if x is an element of a "spiral", then 4x + 1 is the next element.
(*c*) *If x, x′ are elements of a* "*spiral*" *then* $x, x' \equiv 5$ *mod* 8.
(d) The distance between successive elements x, x′ of a "spiral" is given by $y \cdot 2^{e(x)}$ *, where y is the base element and e(x) is the exponent of 2 by which x maps to y.*
(e) *The distance between the jth element,* $j \geq 1$*, of a "spiral", and the (j + 1)th element is given by* $y \cdot 2^k$*, where k is the jth element in the sequence <1, 3, 5, ..., > or the sequence <2, 4, 6, ..., > as established by y.* (Proof is given under "Distance Functions on "Spiral"s" on page 32.)

Lemma 12.1: *Let s(j) denote the jth element of the base sequence relative to 1, that is, of the sequence {1, 5, 21, 85, 341, ...}. Then*

$$2^2 + 2^4 + 2^6 + \ldots + 2^{2j} = s(j+1) - 1 = 2 \cdot (2 \cdot s(j))$$

Lemma 12.2: *Let s(j) denote the jth element of the base sequence relative to 1, that is, of the sequence {1, 5, 21, 85, 341, ...}. Then*

$$2^1 + 2^3 + 2^5 + \ldots + 2^{2j-1} = (2 \cdot s(j))$$

Lemma 12.5. *Let* $\underline{A}$ = *... A * A * A * A′ be an infinite sequence of positive integers, where A, A′ are finite sequences, A is not empty, A′ is possibly empty, A is repeated infinitely and successively, and is such that in every path <x, ..., y> defined by A, x is less than y. Then no range element y′ defines* $\underline{A}$ *.*

An example of such an $\underline{A}$ *is {...1, 1, 1}*. (Proof is under "Some Infinite Inverse Exponent Sequences That are Not Generated by Any Range Element, y" on page 35.)

Lemma 13.0. *A bijection exists between tuples and "spiral" elements that eventually map to 1.*

Lemma 13.5. *For all base sequences, and for all levels i ≥ 2 relative to a base sequence, no element of a level i sequence is an element of the base sequence. Thus, in particular, no level i sequence, i ≥ 2, is an element of the base sequence relative to 1, i.e., of the base sequence {1, 5, 21, 85, 341, ... }*. (Proof is given in second file of the paper, "The Structure of the 3*x* + 1 Function" on occampress.com..)

Lemma 14.0. *For each "spiral" at each level i ≥ 1, the sequence of elements of the "spiral" map to successive intervals of the base sequence relative to 1.* (Proof is given in second file of the paper, "The Structure of the 3*x* + 1 Function" on occampress.com..)

Lemma 15.0. *The parity of exponents mapping to successive range elements of a "spiral" alternates. The alternation sequence is not affected by the presence of multiples-of-3 in the "spiral".*

Lemma 15.5. *For each element y of the two top rows {5, 11, 17, 23, ... } and {1, 7, 13, 19, ...} of all 2-level tuple-sets, and hence for each y in the range of the 3x + 1 function, the smallest exponent mapping to y by a multiple-of-3 is given by the following table:*

Table 4:

y	**Smallest exponent a_2 such that** $\frac{3(3x)+1}{2^{a_2}} = y$ **x odd**
$y \equiv 5 \bmod 3 \cdot 2 \cdot 3^1$	1
$y \equiv 11 \bmod 3 \cdot 2 \cdot 3^1$	5
$y \equiv 17 \bmod 3 \cdot 2 \cdot 3^1$	3
$y \equiv 1 \bmod 3 \cdot 2 \cdot 3^1$	6
$y \equiv 7 \bmod 3 \cdot 2 \cdot 3^1$	2
$y \equiv 13 \bmod 3 \cdot 2 \cdot 3^1$	4

Lemma 15.75. *In the following table, let the values of y (left-most column) be the minimum residues of the reduced residue classes mod 18 to which a base element of a recursive "spiral" must belong (see Lemma 15.5). Let 1, 2, 3, ..., 20 in the top horizontal column denote exponents. Then:*

(a) the value x in the cell defined by any y and any exponent a is the minimum residue of the reduced residue class mod 18 of the "spiral" element that maps to y in one iteration of the 3x + 1 function via the exponent a. Thus, e.g., 85 maps to y =1 via the exponent a = 8, and 85 ≡ 13 mod 18. *x = 13. Furthermore,*

(b) the table holds for all exponents congruent mod 18. Thus, e.g., if y is a base element that is congruent to 7 mod 18, then all "spiral" elements x mapping to y via any exponent congruent to 10 mod 18, are congruent to 13 mod 18.

Table 5: Description of all "spiral" elements mapping to any base element *y* in one iteration of 3*x* + 1 function ("*" denotes a multiple of 3. No multiple of 3 is a minimum residue of the reduced residue class mod 18.)

y	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1		1		5		*		13		17		*		7		11		*		1
5	*		13		17		*		7		11		*		1		5		*	
7		*		1		5		*		13		17		*		7		11		*
11	7		11		*		1		5		*		13		17		*		7	
13		17		*		7		11		*		1		5		*		13		17
17	11		*		1		5		*		13		17		*		7		11	

Lemma 15.85. *For each i ≥ 2, and for each base element y, the successive elements (ignoring multiples of 3) in any "spiral" whose base element is y consist of elements from a fixed sequence of all reduced residue classes mod* $2 \cdot 3^{i-1}$*, the same sequence of classes being repeated endlessly in any given "spiral".*

Lemma 15.87. *For each base element y, and for each exponent sequence A, A maps to an infinite number of elements of the base sequence relative to y.*

Lemma 16.0. *Every finite sequence of* parities *of exponents occurs in the structure of all "spirals" mapping directly or indirectly to a given base element*.

Lemma 17.0. *For all m* ≥ 2, *and for all k* ≥ 2, *there exists an infinity of consecutive intervals in the base sequence relative to 1, i.e., in the sequence {1, 5, 21, 85, 341, ... }, such that each such interval contains* $((m^k - 1)/(m - 1) - 1)$ *numbers that map to 1.* (Proof is given under "Three Important Lemmas" in first file of this paper on occampress.com. .

Lemma 18.0 (in first file of this paper). *Let S denote the level 1 "spiral" (i.e., the base sequence) relative to 1, {1, 5, 21, 85, 341, ... }. Let x, x′ be successive elements of the sequence S. Then the set of odd, positive integers lying between x and x′, i.e., the set {x + 2, x + 4, ..., x′ – 4, x′ – 2} we* *call an interval of the sequence S. The intervals of S are numbered 1, 2, 3, ... Thus the odd positive integer 3 is the sole element of interval 1; the odd, positive integers 7, 9, 11, 13, 15, 17, 19, are all the elements of interval 2, etc.*

Let y be a range element in the interval $k \geq 2$. Then y is mapped to, in one iteration of the 3x + 1 function, by a range element that lies either in interval $k - 1$, k, or $k + 2$.

Lemma 20.0. *Assume a counterexample exists. Then for all counterexamples y_c and for all exponent sequences A, there exists an infinity of tuples t, t′, t″, ... having the following properties:*

- $t = t_s(A_s)*\{y_{\text{PAE}}\}*t_f(A)$,

where: $t_f(A) = \langle y, ..., y_c \rangle$ *for some y*;

A_s *is any exponent sequence*;

y_{PAE} *may vary with* A_s*, and may, of course, not be present*;

- *t′, t″*, etc., *are as t except that the length of* A_s *is increased by 1, 2, etc., and, of course,* y_{PAE} *may vary with each such* A_s.

Lemma 45.0. *Let* $\bar{t}_{nc}$, $\bar{t}_c$ *be non-counterexample and counterexample infinite tuples respectively. Then if* $A(\bar{t}_{nc}(k)) = A(\bar{t}_c(k))$, $k \geq 2$, *then the mark of at least one of the tuples must be > k.*

(For proof, see Appendix B.)

Lemma 50.0. *(See also Lemma 8.8 above) Let S denote the set {1, 5, 21, 85, 341, ... } of odd, positive integers that map to 1 in one iteration of the 3x + 1 function. Let S′ denote the set of odd, positive integers that map to1 in two iterations of the 3x + 1 function. (S′ is the set of inverses of all elements of S that are non-multiples-of-3, since no odd, positive integer maps to a multiple-of-3 via the 3x + 1 function.)* And similarly for the set *S″ of odd, positive integers mapping to 1 in three iterations of the 3x + 1 function. Etc.*

Define the set $\underline{S}$ *to be* $\underline{S} = S \cup S' \cup S'' \cup ...$

Then whether or not counterexamples exist, $\underline{S}$ *is the set of odd, positive integers that map to 1.*

3*x* + C Functions

Lemma 14.8 (in [ar]). If $\langle Cx, Cy, Cy', ..., Cz \rangle$ is a 3*x* + C tuple, then $\langle x, y, y', ..., z \rangle$ is a 3*x* + 1 tuple.

Lemma 15.0. (in [so] and [ar]) *Let C define a 3x + C function* F_C. *Then* F_C *gives rise to a 3x + 1-like Problem iff* $C = -1$ *or* $C = -1 + 2^1 + 2^2 + 2^3 + ... + 2^k$.

Corollary to Lemma 15.0. [so]

(a) For each C such that 3x + C is a 3x + 1-like function, the exponent sequence associated with the infinite tuple <1, 1, 1, ... > is as follows:

$3x - 1$: exponent sequence {1, 1, ...}
$3x + 1$: exponent sequence {2, 2, ...}
$3x + 5$: exponent sequence {3, 3, ...}
$3x + 13$: exponent sequence {4, 4, ...}

etc.

(*b*) For each *C* (except *C* = 1 and *C* = –1) such that $3x + C$ is a $3x + 1$-like function, *C* is a counterexample to the 3x + C Conjecture because it yields the infinite cycle, <*C*, *C*, *C*, ... >. Thus:
$3x + 5$: $(3(5) + 5)/2^2 = 5$, giving rise to the infinite tuple <5, 5, 5, ...>
$3x + 13$: $(3(13) + 13)/2^2 = 13$, giving rise to the infinite tuple <13, 13, 13, ...>

etc.

Part (b) was brought to our attention by a computer scientist.

Lemma 9.0

For no odd, negative integer –u is it the case that C′(–u) is positive, where C′ is the 3x – 1 function.

Lemma 9.05.

(*a*) *The negative of the 3x – 1 function over the odd, positive integers = the 3x – 1 function over the odd, negative integers. That is, for all odd, non-zero integers u,*

$$-\left(\frac{3(u) - 1}{2^2}\right) = -w = \left(\frac{3(-u) + 1}{2^2}\right)$$

(*b*) *The negative of the 3x + 1 function is embedded in the 3x – 1 function. That is, <x, y> is a tuple in the 3x + 1 function iff <–x, –y> is a tuple in the 3x – 1 function.*

Lemma 9.0. [so] Let *C′*(u) denote the $3x - 1$ function. *Then for no odd, negative integer –u is it the case that C′(–u) is positive.*

Lemma 9.1 [so] *If y is an anchor for the 3x + 1 function at level i , then* $y - 2 \cdot 3^{(i-1)}$ *is an i-level anchor for the 3x – 1 function.*

Lemma 9.2. [so] *For each* $i \geq 2$*, the set of all i-level anchor tuples for the 3x – 1 function is complete.*

Lemma 9.3. [so] *Lemma 1.0 and Lemma 5.0 (in [so]) apply to the 3x – 1 function.*

Lemma 9.4 [so] *Let u be an odd, negative integer, and let* $\bar{t}_u$ = *<u, u′, ... > be the infinite tuple it generates. Let* $A(\bar{t}_u)$ *be the infinite exponent sequence associated with* $\bar{t}_u$*. Let x be an odd, pos-*

itive integer, and let $\bar{t}_x$ = <x, x′, ... > be the infinite tuple it generates. Let $A(\bar{t}_x)$ be the infinite exponent sequence associated with $\bar{t}_x$.

Then $A(\bar{t}_u) \neq A(\bar{t}_x)$.

Lemma 9.5. [so] *Let u be a counterexample to the 3x – 1 Conjecture, and let $\bar{t}_u$ = <u, u′, ... > be the infinite tuple it generates. Let $A(\bar{t}_u(j))$ be the exponent sequence associated with the prefix $\bar{t}_u(j)$. And similarly for counterxamples x to the 3x + 1 Conjecture. Then for all counterexamples x to the 3x + 1 Conjecture:*

If $A(\bar{t}_x(2)) = A(\bar{t}_u(2))$ then x - u must be $\geq 2 \bullet 2^{a_2}$; and

If $A(\bar{t}_x(3)) = A(\bar{t}_u(3))$ then x - u must be $\geq 2 \bullet 2^{a_2}2^{a_3}$; and

If $A(\bar{t}_x(4)) = A(\bar{t}_u(4))$ then x - u must be $\geq 2 \bullet 2^{a_2}2^{a_3}2^{a_4}$; and

...

Appendix A1 — Lemmas and Definitions Used in Implementations of the "Pushing Away" and "Missing Sequences" Strategies

The following definitions and lemmas are used in the implementations of the "Pushing Away" and "Missing Sequences" strategies that yield possible proofs of the $3x + 1$ Conjecture. These possible proofs are given in Appendices B and C.

Understanding the full contents of this Appendix is not necessary in order to understand the possible proofs, but it will definitely deepen the reader's understanding of the effect of counterexamples on tuple-sets.

On the Set of All Last Elements of All *i*-Level Tuples in All Tuple-sets

Lemma 3.059: *Let R denote the set of all range elements of the 3x + 1 function. Then for all* $i \geq 2$, *the set of all last elements of all i-level tuples in all i-level tuple-sets = R.*

Proof: Follows directly from Lemma 0.0 by induction..□

On Infinite Tuples

Definition: If the tuple *t* is defined in the standard way, i.e., as given under "Trajectory" on page 4 and "Tuple" on page 5, then we say that the tuple is defined in the *upward direction*. We also say that such tuples define exponent sequences in the upward direction.

Definition: Let $t = <y_1, y_2, ..., y_i>$ be a tuple, $i \geq 1$. Then if y_1 is not a multiple-of-3, there exists an infinity of tuples, $<y_0, y_1, y_2, ..., y_i>$ (by Lemma 5.0). We call each of these a *downward extension* of *t*.

Definition: If the tuple *t* is defined by a sequence of downward extensions, then we say that *t* is defined in the *downward* or *inverse direction*. We also say that such tuples define exponent sequences in the downward direction.

Lemma 1.3. *Let x, x′ be two odd, positive integers. Then the infinite upward exponent sequence* $\overline{X}$ *defined by the infinite upward tuple of which x is the first element, and the infinite upward exponent sequence* $\overline{X}'$ *defined by the infinite upward tuple of which x′ is the first element, cannot be the same.*

Proof: Very easy using the distance functions. See proof in our "The Structure of the $3x + 1$ Function", accessible on the web site www.occampress.com. □

Lemma 1.4. *Let y, z be two range elements. Let* $\{\underline{Y}\}$ *be the **set** of infinite downward exponent sequences defined by all downward extensions of* $<y>$, *and let* $\{\underline{Z}\}$ *be the **set** of infinite downward exponent sequences defined by all downward extensions of* $<z>$. *Then* $\{\underline{Y}\} \cap \{\underline{Z}\} = \varnothing$.

Proof: Very easy using the distance functions. See proof in our "The Structure of the $3x + 1$ Function", accessible on the web site www.occampress.com. □

On Minimum Sets of Range Elements That Are Downward-Complete at *i*

Definition: $M'_{nc, i}$ denotes the set of smallest non-counterexamples in each of the reduced residue classes mod $2 \cdot 3^{i-1}$. That is,

$M'_{nc, i}$ = {the smallest non-counterexample in the first such residue class} ∪
{the smallest non-counterexample in the second such residue class} ∪ ... ∪
{the smallest non-counterexample in the last such residue class}.

$M'_{c, i}$ is defined similarly, but for counterexamples.

(We would have a proof of the $3x + 1$ Conjecture if we could prove that, for at least one $i \geq i_0$, $M'_{nc, i} = M_i$ or $M'_{c, i} = M_i$, because then we would have a contradiction to Lemma 10.97, which is stated and proved below.)

Lemma 10.905. *$M'_{nc, i}$ constitutes a set of last elements of a set of i-level tuples that are downward-complete at i, and similarly for $M'_{c, i}$, where "downward-complete at i" means that the set of last elements is mapped to by every i-level exponent sequence.* ("*Upward-complete at i*" means that a set of elements is the set of first elements of tuples that together define every *i*-level exponent sequence.)

Proof: The set of last elements of all *i*-level tuples in any *i*-level tuple-set is the set of elements in a reduced residue class mod $2 \bullet 3^{i-1}$ (by Lemma 3.055).

Each *i*-level tuple in the tuple-set T_A defines the exponent sequence A (by definition of tuple-set).

If a counterexample exists, then in each *i*-level tuple-set, $i \geq 2$, there exist an infinity of *i*-level counterexample tuples, and an infinity of i-level non-counterexample tuples, in T_A (by Lemma 10.0).

Therefore, in particular, the set of smallest non-counterexample elements of all *i*-level tuples in all *i*-level tuple-sets, i.e., the set $M'_{nc, i}$, defines a complete set of non-counterexample tuples at *i*. And similarly for $M'_{c, i}$. □

Basic Facts About Downward- and Upward- Complete and Incomplete Sets

Lemma 10.906. (a) *Let S be a set of elements that are i-level downward- or upward-complete. Then for all S such that $S \subset S'$, S' is also downward or upward complete at i, $i \geq 2$.*

(b) *Let S' be a set of elements that are i-level downward- or upward-incomplete. Then for all S such that $S \subset S'$, S is also i-level downward- or upward incomplete.*

(c) *Let S be a set of elements that are i-level downward- or upward-incomplete. Then if $S \subset S'$, S' may be i-level downward- or upward-incomplete or -complete.*

Proof of (a) Adding tuples to a set of complete tuples does not remove any exponent sequences from the set defined by the complete tuples. □

Proof of (b) This is the contrapositive of (a). □

Proof of (c) *S'* may or may not contain tuples to define the exponent sequences missing from the sequences defined by tuples defined by elements of *S*. □

Anchors and Anchor Tuples

Definition of "Anchor"

For each range element *y*, there exists a minimum $i = i_0$ such that $y < 2 \cdot 3^{(i-1)}$. We say that *y* is an *anchor at level i*, or an *i-level anchor*. Clearly, if *y* is an anchor at $i = i_0$, then *y* is also $< 2 \cdot 3^{(i+j-1)}$, $j \geq 0$. We say that *y* is also an anchor at each such level $i + j$, and we summarize the fact via an informal saying, "Once an anchor, always an anchor". In general, we will use the terms *anchor at level i* and *i-level anchor* to refer to any such $i + j$, not merely that at which $j = 0$.

Thus, for example, the set of 2-level anchors is {1, 5}, and the set of 3-level anchors is {1, 5, 7, 11, 13, 17}.

Definition of "Anchor Tuple"

Let *y* be an *i*-level anchor, $i \geq 2$. Then any *i*-level tuple of which *y* is the last element is called an *anchor tuple at level i* or *an i-level anchor tuple*.

Definition of "*i*-sequence"

Let *y* be a range element. Then the sequence of *i*'s $\geq i_0$ in the above definition of "anchor" we call the *i-sequence of y*, or *y's i-sequence*.

Thus, for example:

the *i*-sequence of 1 is {2, 3, 4, 5, ... } because 1 is an anchor at *i* for all levels $i \geq 2$.

the *i*-sequence of 17 is {3, 4, 5, 6, ... } because 17 is an anchor at *i* for all levels $i \geq 3$. But 17 is 17 is not an anchor at *i* for $i = 2$.

Properties of Anchors and Anchor Tuples

Every *i*-Level Anchor Tuple is the First *i*-Level Tuple in its Tuple-set

Lemma 10.91. *Let t be an i-level anchor tuple. Then t is the first i-level tuple in an i-level tuple-set.*

Proof: By the distance function defined in Lemma 1.0 (a), the distance between the last elements of *i*-level tuples in an *i*-level tuple-set is $2 \cdot 3^{(i-1)}$. But since by definition the last element *y* of *t* is an anchor, and since an anchor is $< 2 \cdot 3^{(i-1)}$, there is no *i*-level tuple with a last element $< y$, hence *t* must be the first *i*-level tuple in the tuple-set. □

Not All Elements of Anchor Tuples are Anchors

The last element of an anchor tuple is an anchor (follows directly from the definition of *anchor tuple*). But it is by no means the case in general that all elements of an anchor tuple are anchors at their respective levels. For example, 7 is an anchor at level 4, and the following are two of the infinite set of level-4 anchor tuples having 7 as their last element: <99, 149, 7> and <3185, 2389, 7>. But neither 149 nor 2389 are level-4 (or level-3, or level 2) anchors.

There Is Exactly One *i*-Level Anchor Tuple in Each *i*-Level Tuple-set

Since all *i*-level tuples in a tuple-set are ordered by the first element of each tuple, and since the first element of each tuple is an odd, positive integer, there can be only one first *i*-level tuple .

Definition of "Upward Extension" of Anchor Tuple

Let $t = <x_1, x_2, ..., x_i>$ be an *i*-level anchor tuple. Then the tuple $<x_1, x_2, ..., x_i, x_{i+1}>$ is an *upward extension* of *t*, where $x_{i+1} = C(x_i)$, i.e., x_{i+1} is the result of one iteration of the $3x + 1$ function on x_i.

Every Upward Extension of an Anchor Tuple Is an Anchor Tuple

Lemma 10.92.

Let $t = <x_1, x_2, ..., x_i>$ *be an i-level anchor tuple, and let* $t' = <x_1, x_2, ..., x_i, x_{i+1}>$ *be the extension of t. Then t′ is an (i+1)-level anchor tuple.*

Proof: The only way that $C(x_i) = x_{i+1}$ can be greater than x_i is if $e(x_i) = 1$, i.e., if $(3x_i + 1)/2^1 = x_{i+1}$, since for all other exponents, x_{i+1} is $< x_i$. But for all range elements x_i, $x_i \geq 1$. Then $x_{i+1} = (3x_i + 1)/2^1 < 3x_i$. Therefore $x_{i+1} < 3x_i < 3 \cdot 2 \cdot 3^{(i-1)}$, since x_i must be less than $2 \cdot 3^{(i-1)}$. So x_{i+1} must be $< 2 \cdot 3^{((i+1)-1)}$, hence x_{i+1} must be an $(i+1)$-level anchor. □

We sometimes summarize Lemma 10.92 informally, and by abuse of language, as "once a anchor tuple, always an anchor tuple."

Definition of "Downward Extension" of Anchor Tuple

Let $t = <x_1, x_2, ..., x_i>$ be an *i*-level anchor tuple. Then any tuple $<x_0, x_1, x_2, ..., x_i, x_{i+1}>$ is a *downward extension* of *t*. Not all anchor tuples (or, in fact, tuples in general) have a downward extension. In particular, if the first element of *t* is a multiple-of-3, then *t* has no downward extensions. On the other hand, if *t* has one downward extension (i.e., if x_1 is a range element of the $3x + 1$ function), then *t* has an infinity of downward extensions (Lemma 5.0).

On Exponent Sequences Generated by Downward Extensions of Anchor Tuples

Lemma 10.93. *Let* $t = <x_1, x_2, ..., x_i>$ *be an i-level anchor tuple that generates the exponent sequence* $A = \{a_2, a_3, ..., a_i\}$*. By definition,* x_i *is an anchor. At level i+1, there exists an (i+1)-level anchor tuple having* x_i *(now an (i+1)-level anchor), as its last element, that generates one of two exponent sequences: either (a)* $\{a_1, a_2, a_3, ..., a_i\}$*, or (b)* $\{a_1', a_2', a_3, ..., a_i\}$*.*

Proof: if x_1 is a range element, then *(a)* applies, yielding an $(i + 1)$-level exponent sequence that is generated by the $(i + 1)$-level anchor tuple t'; if x_1 is not a range element, then *(b)* applies, because in that case a range element x_1' must map to x_2 and that will necessarily be by a different exponent than a_2, namely, by a_2'. (Lemma 5.0 guarantees that an infinity of elements map to each range element; in this case, to the range element x_2. Lemmas 15.0 and 15.85 guarantee that an infinity of these elements will not be multiples-of-3, i.e., will be range elements.) So some x_0 must map to x_1' via the exponent a_1', again yielding an $(i + 1)$-level exponent sequence that is generated by an $(i + 1)$-level anchor tuple t''. □

On First Elements of Anchor Tuples

Lemma 10.90. *For every odd, positive integer x there exists a minimum* $i = i_0$ *such that for all* $i \geq i_0$*, x is the first element of the first i-level tuple in some i-level tuple-set, i.e., x is the first element of an anchor tuple at i in some i-level tuple-set.*

Proof: x gives rise to a sequence of tuples <*x*>, <*x*, *C*(*x*)>, <*x*, *C*(*x*), $C^2(x)$>, ... t = <*x*, *C*(*x*), $C^2(x)$, ..., $C^{i-1}(x)$> will be an anchor tuple when, by Lemma 1.0 (b), x is less than

$$2 \bullet 2^{a_2} \bullet 2^{a_3} \bullet \ldots \bullet 2^{a_i}$$

i.e., when there are no *i*-level tuples to the left of *t*. Clearly, *x* will continue to be less than this product of powers of 2 for all larger *i*, hence every extension of *t* will likewise be the first *i*-level tuple in an *i*-level tuple-set, i.e., an anchor tuple at *i*. □

Lemma 10.907. *Let* M_i *denote the set of anchors at i, i.e., the set of minimum residues of the set of reduced residue classes mod* $2 \cdot 3^{i-1}$*. Then every element y of* M_i *must be the first element of some i-level anchor tuple. Furthermore, if an element y of* M_i *is the first element of the kth i-level tuple in an i-level tuple-set, then so must the first elements of the 1st, 2nd, ..., (k - 1)th i-level tuples be elements of* M_i*.*

Proof: First of all, we know that *y* must be the first element of a tuple in at least one tuple-set, since *y*, being an odd, positive integer, generates an infinite tuple, i.e., an infinite sequence of finite tuples. Now if *y* were not the first element of some *i*-level anchor tuple, i.e., if *y* were only the first element of the second or third or... or *m*th tuple in some *i*-level tuple-set T_A, then, by Lemma 1.0 (b) the first element *x* (not an element of M_i) of the *i*-level anchor tuple in T_A would be

$$x = y - m \cdot 2 \cdot 2^{a_2} \cdot 2^{a_3} \cdot \ldots \cdot 2^{a_i}$$

where $m \geq 1$. But this would imply that a number (namely, *x*) less than an element of M_i were not in M_i, which is false. The proof of the second part follows by a similar argument. □

A "Coincidence" Between First and Last Elements of Anchor Tuples

The previous two sub-sections bring to light a remarkable coincidence, namely, that when an odd, positive integer *x* becomes the first element of an anchor tuple, say, an anchor tuple *t* at *i*, then *simultaneously* it must be the case that the last element of the anchor tuple (which, by definition, is an anchor at *i*) is less than $2 \cdot 3^{(i-1)}$. Naively, we would suppose that, if

$$x < 2 \bullet 2^{a_2} \bullet 2^{a_3} \bullet \ldots \bullet 2^{a_i}$$

(which must be the case if *x* is the first element of an anchor tuple at *i*), then the last element of the anchor tuple might well be greater than $2 \cdot 3^{(i-1)}$. But this can never be the case. Why? The answer is that the last tuple elements in a succession of tuple extensions can not increase more rapidly than that allowed by the exponent sequence {1, 1, 1, ..., 1}. Specifically, if *y* is the last element of a tuple extension t, then $C(y) = (3y + 1)/2^1 \approx (3/2)y$. So the size of the last element in any succession of tuple extensions is constrained by this fact.

Definition of "Prefix" and "Suffix" of an Exponent Sequence

:Let $A * A'$ denote the concatenation of the exponent sequence A' onto the right-hand end of the exponent sequence A, where at least one of A, A' is not empty. We say that A is the *prefix*, and A' is the *suffix*, of $A * A'$. The *length* of a prefix or a suffix is the number of exponents it contains.

Definition of "Suffix Maps to y", "Prefix Maps to y", "Maps to"

Let y be a range element. Sometimes we say that a *suffix* (of an exponent sequence) *maps to y*, meaning that we are not concerned with the prefix in that case. Similarly, we will sometimes say that a *prefix* (of an exponent sequence) *maps to y*, meaning that we are not concerned with the suffix in that case. Finally, we will sometimes say that the exponent sequence generated by a tuple t in a tuple-set *maps to* the last element y of t.

Definition of A "Touches Down" at *i* via *y*

Let A be a suffix of an exponent sequence that maps to a range element y. Then if y is an anchor at $i \geq 2$, and A is generated by an anchor tuple at i of which y is the last element, we say that *A touches down* at *i via y*.

Every Range Element Is Mapped to By Every (*i* - 1)-Level Exponent Sequence

Lemma 10.95. *Let y be any range element. Then y is mapped to by all (i -1)-level exponent sequences A, $i \geq 3$, followed by some exponent a_i.*

Proof: In the sub-section, "Definitions" on page 31, a range element of the $3x + 1$ function is called a "base element". It is clear from that sub-section that each base element establishes an infinitary tree consisting of an infinite set of recursive "spiral"s. Fig. 4 in that sub-section gives a graphic view of one particular base element (the range element 1) and some of the numbers mapping to it. (All numbers mapping to a base element in one iteration of the $3x + 1$ function constitute a "spiral".)

For any range element (i.e., base element) we orient the infinite tree it gives rise to, opposite to that in Fig. 4, i.e., we orient the tree so that the base element is on top, and the "spiral"s are below it. Each branch in the infinite tree is labelled with an exponent as described in the sub-section.

We adopt a left-to-right convention for writing exponent sequences that are derived from the tree. Given an i-length path downward from the base element y, $i \geq 3$, we write the exponent sequence it defines as follows: the exponent on the lowest branch of the path we call a_2 and write it on the left, then the exponent on the next-lowest branch of the path we call a_3 and write it immediately to the right of the first exponent, etc., yielding an exponent sequence $\{a_2, a_3, ..., a_i\}$.

Lemma 7.0 then implies that for any range element y and for all $i \geq 3$, the corresponding set of all downward paths of length i will define the set of all $(i - 1)$-level exponent sequences, with the i-level exponent a_i either being indeterminate or else simply ignored by us.. □

Readers sometimes wonder why it is that Lemma 7.0 does not leave the way open for a contradiction among non-counterexample anchor tuples themselves — a contradiction arising from the possibility that a given $(i - 1)$ exponent sequence A mapping to both an i-level non-counterexample anchor y_{nc} and another *i*-level non-counterexample anchor y_{nc}', both have the same "buffer" exponent.

The briefest answer is that, if there are no *i*-level counterexample anchor tuples, as is the case by computer test (see second paragraph of "First Plausibility Argument" on page 85) for the set of *i*-level anchor tuples, $2 \le i \le 35$, then the set of *i*-level exponent sequences generated by *i*-level non-counterexample anchor tuples is precisely the set of all *i*-level exponent sequences. Thus, the existence of counterexample *i*-level anchor tuples merely reduces the set of *i*-level exponent sequences generated by *i*-level non-counterexample tuples. This cannot introduce a contradiction where none existed before.

Lemmas 15.0 through 16.0 give further insight into the nature of mappings to anchors.

On Exponent Sequences That Map to Anchors

Lemma 10.965. *In the set of all anchor tuples at* $i \ge 2$, *it must always be the case that if* $A * \{a\}$ *maps to the anchor* y, *and* $A * \{a'\}$ *maps to the anchor* y', *where* $y \ne y'$, *then necessarily* $a \ne a'$.

Proof: Follows directly from the fact that each *i*-level tuple-set, hence the first *i*-level tuple (i.e., anchor tuple) of each *i*-level tuple-set, is defined by a unique *i*-level exponent sequence. □

Definition of "Complete" Exponent Sequences and "Missing" Exponent Sequences

Let $\{t\}$ be a set of *i*-level tuples. Then if $\{t\}$ generates all *i*-level exponent sequences, we say that $\{t\}$ is *i-level complete*. We sometimes express this as $A(\{t\})$ is complete, where the i is understood. For example, the set of all anchor tuples at i, for each $i \ge 2$, is complete.

If a set $\{t\}$ is *not complete*, or *incomplete*, we say that one or more sequences is *missing* from $A(\{t\})$.

By abuse of language, we will sometimes say that a set M of range elements is complete, or incomplete, if the set of tuples having M as the set of last elements of the tuples, is complete, or incomplete. Thus, e.g., the set M_i of anchors at level i is *i*-level complete, because this set is the set of last elements of all anchor tuples at i, and these tuples define all *i*-level exponent sequences.

On "Missing" Exponent Sequences

Lemma 10.97.

(a) *If no counterexamples exist, then for all* $i \ge 2$ *the set of non-counterexample anchor tuples at i is complete.*

(b) *If counterexamples exist, then for all* $i \ge i_0$, *where* i_0 *is the smallest i such that a counterexample is an anchor at i:*

the set $\{t_{nc}\}$ *of non-counterexample anchor tuples at i is not complete at i, and*
the set $\{t_c\}$ *of counterexample anchor tuples at i is not complete at i, and*

$\{t_{nc}\} \cap \{t_c\} = \varnothing$ *and*

$\{t_{nc}\} \cup \{t_c\} = \{A\}$, *where* $\{A\}$ *is the set of all i-level exponent sequences.*

In other words, some i-level exponent sequences are missing from those defined by $\{t_{nc}\}$ (because they are defined instead by tuples in $\{t_c\}$), and some i-level exponent sequences are missing from those defined by $\{t_c\}$ (because they are defined by tuples in $\{t_{nc}\}$).

Proof: (a) and (b) follow from the definition of anchor tuple and the fact that each i-level tuple-set is defined by a unique i-level exponent sequence, and the fact that the set of all i-level tuple-sets, hence the set of anchor tuples of all i-level tuple-sets, define the set of all i-level exponent sequences, and the fact that the set of non-counterexample tuples and the set of counterexample tuples are disjoint. □

Definition of "Prefix Complete Up to *j*"
If a set of i-level tuples defines *all* exponent sequences up to and including some level j but not beyond, $2 \le j \le i$, then we say that the tuples are *prefix-complete up to j*. (Note that the term *prefix* refers to the set of exponent sequences defined by the set of tuples, not to the set of tuples themselves.)

Definition of "Suffix Complete Down to *j*"
If a set of i-level tuples defines *all* suffixes down to some level j but no lower, $2 \le j \le i$, then we say that the tuples are *suffix-complete down to j*. (Note that the term *suffix* refers to the set of exponent sequences defined by the set of tuples, not to the set of tuples themselves.)

On Suffixes of Exponent Sequences That Map to Non-Counterexamples
Lemma 10.981.
Let $\mathbf{R}'_{\text{nc}}$ denote the set of all non-counterexample range elements. For every exponent sequence suffix s, there exists an infinity of $\mathbf{R}'_{\text{nc}}$ that are mapped to by s.

Proof: follows directly from Lemma 10.0. □

Lemma 10.982. *In order for a suffix s of length (i - 1) to map to an anchor, it is necessary that s map to a y_{nc} that is less than $2 \bullet 3^{(i-1)}$. s then maps to the anchor y_{nc} at i + 1, i + 2, ...*

Proof: by Lemma 10.91. .□

Lemma 10.983. *Every suffix s_1 of length 1 maps to an anchor y_{nc} at i for i = 2, 3, 4, ...*
Every suffix s_2 of length 2 maps to an anchor y_{nc} at i for i = 3, 4, 5, ...
...
Every suffix s_{34} of length 34 maps to an anchor y_{nc} at i for i = 35, 36, 37,...

Proof: follows directly from the fact that all anchors through at least level 35 are known, by computer test, to map to 1 (see second paragraph of "First Plausibility Argument" on page 85). □

Lemma 10.984. *If a counterexample exists, then for every suffix s there exists a least i such that s maps to a y_{nc} anchor at i and s maps to a y_c anchor at i. This fact generalizes to any finite number of anchors y_{nc} and y_c.*

Proof: Let i' be the smallest i such that s maps to an anchor y_{nc} at i, and let i'' be the smallest i such that s maps to an anchor y_c at i. Then the desired i is simply the larger of i', i''. □

Definition of "Anchor Sequence"
In the tuple-set T_A, we sometimes say that A is an *anchor sequence*.

On Anchor Sequences
Lemma 10.985. *Let s be a suffix of length (i - 1). If s is an anchor sequence at all, it must be an anchor sequence at i and i alone.*

Proof: By definition, an anchor sequence at i is an i-level sequence, i.e., a sequence of length (i - 1). For all other i, s is either too long or too short to be an anchor sequence. □

Lemma 10.986. *Every suffix s is a suffix of an infinity of anchor sequences.*

Proof: Follows directly from Lemma 10.0. .□

Lemma 45.0. *Let $\bar{t}$, $\bar{t}'$, be two infinite tuples. They may both be non-counterexample tuples, both counterexample tuples, or one may be a non-counterexample and the other a counterexample tuple. Then if $A(\bar{t}(k)) = A(\bar{t}'(k))$, $k \geq 2$, the mark of at least one of the tuples must be > k.*

In words: if two infinite tuples both generate the same k-level exponent sequence, then the mark of at least one of the tuples must be > k.

(Proof is given in Appendix B.)

A Convenient Representation of Anchors, Anchor Tuples, and their Exponent Sequences

Definition: Let $\mathbf{R}'_{nc}$ = the set of non-counterexample range elements. We denote elements of $\mathbf{R}'_{nc}$ by y_{nc}. We now define, for $\mathbf{R}'_{nc}$, the *anchor rectangle at i*, $i \geq 2$. The "limit" of the set of all anchor rectangles for $i \geq 2$ constitutes the *Infinite Anchor Rectangle for $\mathbf{R}'_{nc}$*. The reader will find it helpful to refer to the table below while reading the definition.

For i = 2, the *anchor rectangle at i* consists of all anchor tuples at i = 2, plus the marker $2 \cdot 3^{(2 - 1)} = 6$. By definition of anchor tuple, this means that the anchor rectangle at 2 consists of all (necessarily 2-level) anchor tuples whose last element is the anchor 1, and all (necessarily 2-level) anchor tuples whose last element is the anchor 5. In the table below, anchors run across the top of the table, along with markers equal to $2 \cdot 3^{(i - 1)}$. The asterisk immediately below the anchor 1 denotes the anchor 1, and the asterisk below that asterisk denotes the set of all first elements of all anchor tuples whose last element is 1. Similarly for the anchor 5. In the table, the anchor rectangle at 2 is bordered by a double line.

For i = 3, the anchor rectangle at i consists of all anchor tuples at i = 3. As for the case i = 2, asterisks below anchors denote elements of anchor tuples.

And similarly for i = 4, 5, 6, ...

It is clear that, for all $i \geq 2$, the anchor rectangle at $i + 1$ contains, in its upper left-hand corner, the anchor rectangle at i. In other words, the anchor tuples at i include all downward extensions (of length i) of all anchor tuples at levels 2, 3, 4, ..., i - 1. **This is an important fact**.

Observe that a given asterisk does not always represent the same set. This is due to the fact that multiples-of-3 are not mapped to by any integers. Thus in the anchor rectangle at 2, the second asterisk from the top for 1 and 5 represents, among other integers, multiples-of-3. However, for the anchor rectangle at 3, these same asterisks cannot represent any multiples-of-3.

If we allow i to increase without limit, we have the *Infinite Anchor Rectangle for* $\boldsymbol{R}'_{nc}$. In this case, each anchor tuple is infinitely long, and defines an infinite downward or inverse exponent sequence.

Table 6: Initial part of the Infinite Anchor Rectangle for $\mathbf{R}'_{nc}$

	1	**5**	**2 • $3^{(2-1)}$ = 6**	**7**	**11**	**13**	**17**	**2 • $3^{(3-1)}$ = 18**	**19**	**...**
1	*	*		*	*	*	*		*	...
2	*	*		*	*	*	*		*	...
3	*	*		*	*	*	*		*	...
4	*	*		*	*	*	*		*	...
...	...	...		...		...				...

Let $\mathbf{R}'_c$ = the set of counterexamples. Then, similarly, for each $i \geq 2$, there exists an anchor rectangle at i for $\mathbf{R}'_c$, and an Infinite Anchor Rectangle for $\mathbf{R}'_c$. We denote elements of $\mathbf{R}'_c$ by y_c.

We now establish certain facts about the Infinite Anchor Rectangle for $\mathbf{R}'_{nc}$. Most of these facts also apply to the Infinite Rectangle at $\mathbf{R}'_c$.

Appendix A2 — Challenge to Readers Regarding the "Pushing Away" Strategy

This Appendix reviews the "Pushing Away" Strategy, then describes an approach to possibly implementing it, then challenges the reader to prove a related conjecture.

Brief Review of the "Pushing Away" Strategy

The "Pushing Away" Strategy is motivated by the following facts:

(1) Whether or not counterexamples exist, each finite exponent sequence is generated by exactly one anchor tuple.

(2) Every tuple, counterexample or non-counterexample, is the prefix of an infinite tuple, and each infinite tuple contains a mark (for definition of "mark", see under "First Possible Proof of the 3x + 1 Conjecture Using the "Pushing Away" Strategy" on page 80). Thus every infinite tuple has a prefix that is eventually an anchor tuple. All longer prefixes are likewise anchor tuples ("once an anchor tuple always an anchor tuple").

(3) If no counterexamples exist, then, trivially, every tuple-set contains a countable infinity of non-counterexample tuples, and, trivially, every anchor tuple is a non-counterexample anchor tuple.

(3) If counterexamples exist, then, by Lemma 10.0, every tuple-set contains an infinity of counterexamples and (again) an infinity of non-counterexample tuples.

We ask (informally): is it really possible that all those counterexample anchor tuples and non-counterexample anchor tuples, can somehow "fit" into the set of all anchor tuples without a counterexample tuple and a non-counterexample tuple generating the same exponent sequence (which would imply that two different anchor tuples generate the same exponent sequence, an impossibility)?

The "Pushing Away" Strategy attempts to answer that question with a No by showing that, to avoid that impossibility, either non-counterexample tuples, or counterexample tuples, are forever "pushed away" from anchor tuple status, and thus, do not exist. (An odd, positive integer that is never an anchor does not exist, and an infinite tuple that has no prefix that is eventually an anchor tuple, likewise does not exist.)

Separating Non-Counterexample Tuples From Counterexample Tuples In A Tuple-set

A natural approach to implementing the Strategy is to attempt to "separate" the counterexample tuples from the non-counterexample tuples in each tuple-set — to see if we can, in fact, show how the two different sets of tuples can co-exist — or cannot co-exist — in every tuple-set.

Before continuing, the reader should read "Why Are There An Infinite Number of Tuples in Each Tuple-set?" on page 21.

The task of separating the two sets seems very difficult. Consider the tuple-set T_A, where $A = \{a_2, a_3, a_4, ..., a_i\}$, $i \geq 2$. In accordance with Lemmas 10.0 and 2.0, we know that in T_A there must be:

an infinity of non-counterexample tuples that extend via the exponent 1, and

an infinity of non-counterexample tuples that extend via the exponent 2, and an infinity of non-counterexample tuples that extend via the exponent 3, and ...

And similarly for counterexample tuples. Furthermore among those non-counterexample tuples that extend via the exponent 1, there must be:

an infinity that extend one level further via the exponent 1, and
an infinity that extend one level further via the exponent 2, and
an infinity that extend one level further via the exponent 3, and ...

And similarly for counterexample tuples.

How can we separate the counterexample from the non-counterexample tuples in a way that guarantees that Lemma 10.0 will hold for all arbitrarily long extensions of T_A? The problem is further compounded by the following fact:

Lemma 45.0. *Let $\bar{t}$, $\bar{t}'$, be two infinite tuples. They may both be non-counterexample tuples, both counterexample tuples, or one may be a non-counterexample and the other a counterexample tuple. Then if $A(\bar{t}(k)) = A(\bar{t}'(k))$, $k \geq 2$, the mark of at least one of the tuples must be $> k$.*
(See definition of "mark" under "First Possible Proof of the 3x + 1 Conjecture Using the "Pushing Away" Strategy" on page 80).

In words: if two infinite tuples both generate the same *k*-level exponent sequence, then the mark of at least one of the tuples must be $> k$.

Proof: If, to the contrary, the mark of each tuple is $< k$, then that means that two anchor tuples generate the same exponent sequence, contrary to the definition of anchor tuple and tuple-set. □

Suppose, in a tuple-set T_A, where $A = \{a_2, a_3, a_4, ..., a_i\}$, $i \geq 2$, we attempt to match, one-for-one, each extension, regardless how long, of a non-counterexample tuple, with an extension of a counterexample tuple that generates the same exponent sequence. Surely this will accomplish our goal of separating the two sets of tuples!
Unfortunately, it won't, because there are an infinity of extensions of non-counterexample tuples that generate a given exponent sequence. And an infinity of extensions of counterexample tuples that generate the same exponent sequence.

We challenge the reader to prove the following Conjecture, which is motivated by the attempt to separate the two sub-sets of tuples in each tuple-set. If true, the Conjecture implies the truth of the $3x + 1$ Conjecture.

Conjecture 8. *Let $\{\bar{t}_{nc}\}$, $\{\bar{t}_c\}$ denote the set of all non-counterexample infinite tuples and the set of all counterexample infinite tuples, respectively. Then there exists a non-counterexample infinite tuple $\bar{t}_{nc}$ and a counterexample infinite tuple $\bar{t}_c$ and a $k \geq 2$ such that $\bar{t}_{nc}(k)$ and $\bar{t}_c(k)$ violate Lemma 45.0.*

Let us now consider several approaches that might prove Conjecture 8.

Separating the Tuples by the "No-Redundancy" Argument

Suppose no counterexamples exist. Then each tuple-set contains an infinity of non-counterexample tuples (which is also true if counterexamples exist, by Lemma 10.0) and each anchor tuple is a non-counterexample tuple. Whether or not counterexamples exist, we know (by definition) that each i-level tuple-set contains exactly one i-level anchor tuple (first i-level tuple). If each i-level tuple-set contained *two* i-level anchor tuples, then we could say, informally, that there is "room for redundancies", because one of the i-level anchor tuples could be a non-counterexample anchor tuple and the other could be a counterexample tuple. But there is only one i-level anchor tuple in each i-level tuple-set, and when no counterexamples exist, we see that each i-level anchor tuple 'space" is "needed" by the non-counterexample anchor tuples. We say that there are "no redundancies" in the no-counterexamples case.

Suppose we take the no-counterexamples case as a model of what is required if every tuple-set is to contain an infinity of non-counterexample tuples. Then, clearly, there is "no room" for an infinity of counterexample tuples in each tuple-set, and for the corresponding counterexample anchor tuples, in addition to the infinity of non-counterexample tuples in each tuple-set required by Lemma 10.0, and for the corresponding non-corresponding non-counterexample anchor tuples.

Can this argument be made valid?

Separating the Tuples by "Pushing Up" the Counterexample Tuples

We begin by stating a fact:

Every exponent sequence that is generated by a sufficiently long extension of a non-counterexample tuple prefix $\bar{t}_{nc}(i)$, cannot be generated by a counterexample anchor tuple. "Sufficiently long" is, of course, $\geq \bar{t}_{nc}(m(\bar{t}_{nc}))$, where $m(\bar{t}_{nc})$ is the mark of $\bar{t}_{nc}$, because for all these extensions the prefix of $\bar{t}_{nc}$ is an anchor tuple.

Suppose we try to find a counterexample anchor tuple that generates an exponent sequence that is not generated by any sufficiently long non-counterexample tuple. But by Lemma 10.0 we know that, for each exponent sequence $A(\bar{t}_{nc}(m(\bar{t}_{nc})+j)))$, $j \geq 0$, there are an infinity of counterexample tuples that generate that exponent sequence. So we must conclude that none of these $(m(\bar{t}_{nc})+j)$-level counterexample tuples can be an $(m(\bar{t}_{nc})+j)$-level anchor tuple generating the sequence.

We also know that, for each finite exponent sequence there is an infinity of counterexample tuples generating that sequence (by Lemma 10.).

The reader is encouraged to try to see if he or she can finish a proof, given these facts.

Separating the Tuples by "Turning out" the Counterexample Tuples

Another approach is the following.

Let T_A be any tuple-set T_A, where $A = \{a_2, a_3, a_4, ..., a_i\}$, $i \geq 2$. We conceive of the tuple-set as an infinite picket fence, as described under "Graphical View of a Tuple-set" on page 8. T_A contains all i-level prefixes of infinite non-counterexample tuples that generate A, and all i-level prefixes of counterexample tuples that generate A.

Let $\bar{t}_{nc}$ be any non-counterexample infinite tuple having a prefix in $T_{A.}$. $\bar{t}_{nc}$ has a mark, $m(\bar{t}_{nc})$, at the lowest level at which a prefix of $\bar{t}_{nc}$ first becomes an anchor tuple. (See definition of "mark" under "First Possible Proof of the 3x + 1 Conjecture Using the "Pushing Away" Strategy" on page 80).

There exists an infinite sequence of tuple-set extensions defined by the anchor tuples $\bar{t}_{nc}(m(\bar{t}_{nc}))$, $\bar{t}_{nc}(m(\bar{t}_{nc}) + 1))$, $\bar{t}_{nc}(m(\bar{t}_{nc}) + 2))$, $\bar{t}_{nc}(m(\bar{t}_{nc}) + 3))$, ...

Orient this sequence of tuple-set extensions in a single plane *perpendicular* to the plane of the "picket fence" that is $T_{A.}$, at the tuple (prefix) $\bar{t}_{nc}(i)$.

Do this for every non-counterexample tuple in $T_{A.}$. We have thus, by a simple geometrical device, simultaneously made every non-counterexample infinite tuple having a prefix in T_A, an anchor of all the tuple-sets it generates.

Suppose now that we argue that we have "removed" all the non-counterexample tuples from occupying any position in these tuple-sets except the first (anchor) position.

Then all that is left in the tuple-sets are counterexample tuples. Is this a proof that no counter-example tuple can ever be an anchor tuple? Can the reader construct such a proof, beginning at this point?

Appendix B — Possible Proofs of the 3*x* + 1 Conjecture Using the "Pushing Away" Strategy

There are at least two implementations of the "Pushing Away" Strategy: in one, we attempt to show that every tuple containing an assumed counterexample is "pushed away" from tuples whose elements map to 1, i.e., every tuple containing a counterexample must always be the second, or third, or fourth, or ... tuple in any tuple-set, but never the first. Thus counterexample tuples never become anchor tuples, hence counterexample tuples do not exist (by the Corollaries to Lemmas 10.90 and 10.91 (see Appendix A1)).

In the other, we assume counterexamples exist, then use the same argument as that used in the above implementation and show that non-counterexamples do not exist. But we know that non-counterexamples exist, and so we have a contradiction, implying that counterexamples do not exist.

In this Appendix, we devote our efforts to the first implemenation, of which there are several versions.

First Possible Proof of the 3x + 1 Conjecture Using the "Pushing Away" Strategy

Definition of *Mark*

1. *Definition*: Since every odd, positive integer is the first element of an infinite tuple, and since, by Lemma 10.90, there exists a minimum level i_0 at which a prefix of an infinite tuple is first an anchor tuple, we place a *mark* at this level i_0 on each infinite tuple. We denote the mark of a given infinite tuple $\bar{t}$ by $m(\bar{t})$. (By "Lemma 10.92." on page 69, every extension of an anchor tuple is likewise an anchor tuple.)

2. We know that non-counterexamples exist. The marks of all non-counterexample infinite tuples can be placed in a monotonic increasing sequence $\{m_1, m_2, m_3, ...\}$. Of course, more than one non-counterexample infinite tuple can have a given mark m_i, $i \geq 1$. For example, an infinite number of non-counterexample infinite tuples have a mark at $m_1 = 2$, namely, all non-counterexample infinite tuples $<x, 1, 1, 1, ... >$ where x = 1, 5, 21, 85, 341, ... and all non-counterexample infinite tuples $<x, 5, 1, 1, ... >$ where x = 3, 13, 53, 213, 853, ..., because 1 and 5 are level-2 anchors.

Assume counterexamples exist.

3. Let A be any i-level exponent sequence, $i \geq 2$, such that for all j, $2 \leq j \leq i$, the anchor tuple in every j-level tuple-set is a non-counterexample anchor tuple. By known computer results, $2 \leq i \leq 35$ (see, e.g., step 1 under "First Plausibility Argument" on page 85). By Lemma 10.0, we know that the tuple-set T_A contains an infinity of non-counterexample tuples, as well as an infinity of

counterexample tuples. Each of these non-counterexample tuples is the prefix of an infinite tuple, and by step 1, each such infinite tuple has a mark. Similarly for the counterexample tuples in T_A .

4. Since every tuple-set $T_{A*A'}$, where A' is any exponent sequence, contains an infinity of non-counterexample tuples and an infinity of counterexample tuples (Lemma 10.0), it follows that the set of all finite extensions of all non-counterexample tuples in T_A generates the set of all exponent sequences that are finite extensions of A. And similarly for the set of all finite extensions of all counterexample tuples in T_A .

5. Select any non-counterexample tuple t_{nc} in T_A. t_{nc} is the prefix of an infinite non-counterexample tuple $\bar{t}_{nc}$. $\bar{t}_{nc}$ has a mark, $m(\bar{t}_{nc})$. Extend t_{nc} to the level $m(\bar{t}_{nc})$. The extended tuple, i.e., in prefix terms, $t_{nc}(m(\bar{t}_{nc})$ generates an exponent sequence $A'(\bar{t}_{nc}(m(\bar{t}_{nc})))$.

By step 4, there must be an infinity of counterexample infinite tuples $\bar{t}_c$ having i-level prefixes in T_A, each having an extension to (i.e., prefix at) level $m(\bar{t}_{nc})$ and each generating the exponent sequence $A'(\bar{t}_{nc}(m(\bar{t}_{nc})))$. But to avoid the contradiction of two i-level anchor tuples generating the same i-level exponent sequence (here $i = m(\bar{t}_{nc})$), the mark $m(\bar{t}_c)$ of each of these extended infinite tuples $\bar{t}_c$ must be $> m(\bar{t}_{nc})$.

6. Step 5 applies to *each* non-counterexample tuple in T_A, where A is any i-level exponent sequence, $2 \leq i \leq 35$. These tuples generate the set of all i-level exponent sequences, $2 \leq i \leq 35$. All greater-level exponent sequences, i.e., all $(i + k)$-level exponent sequences, $k \geq 1$, are extensions of these i-level sequences, and all greater-level exponent sequences are generated by infinite tuples whose i-level prefixes generate the set of all these exponent sequences A.

Therefore for all exponent sequences the mark of every counterexample tuple is greater than the mark of every non-counterexample tuple . But then there is no finite counterexample tuple mark, hence counterexample tuples do not exist, hence counterexamples do not exist, and the $3x$ + 1 Conjecture is true. (End of "First Possible Proof...")

Second Possible Proof of the 3x + 1 Conjecture Using the "Pushing Away" Strategy

1. Every finite exponent sequence is generated by an infinity of non-counterexample (finite) tuples (Lemma 10.0).

Since every finite tuple is the prefix of an infinite tuple, and since every infinite tuple has a mark, it follows (from Lemma 10.0) that every finite exponent sequence is generated by the prefixes of an infinity of non-counterexample *anchor* tuples. (Note that we are not saying here that every finite exponent sequence is generated by a non-counterexample anchor tuple.)

2. Similarly, every finite exponent sequence is generated by the suffixes of an infinity of non-counterexample finite tuples (by Lemma 10.0, and the fact that, since every tuple in every tuple-set has a last element, the tuple can be viewed as a suffix (e.g., of itself)).

Since the last element of every tuple in every tuple-set is eventually an anchor, it follows that every finite exponent sequence is generated by the suffixes of an infinity of non-counterexample *anchor* tuples.

3. But then:

there is no *longest* exponent sequence prefix such that all longer exponent sequence prefixes generated by non-counterexample anchor tuples are incomplete, i.e., are a proper subset of all exponent sequences of that length. (The missing sequences in each set of incomplete sequences could be generated by counterexample anchor tuples.)

Furthermore, there is no *longest* exponent sequence suffix such that all longer exponent sequence suffixes generated by non-counterexample anchor tuples are incomplete, i.e., are a proper subset of all exponent suffix sequences of that length. (The missing sequences in each set of incomplete suffix sequences could be generated by counterexample anchor tuples.)

The possibility remains, however, that the exponent sequences generated by non-counterexample anchor tuples are incomplete "in the middle", even though they are complete "at the ends" for arbitrarily large ends. The incompleteness in the middle could then allow for counterexample anchor tuples to exist. More precisely, the possibility remains that, for all $i \geq i_c$, where i_c is the minimum level at which there is a counterexample anchor tuple, it is possible that the maximum length $l(s_i)$ of the set of complete exponent sequences generated by suffixes of anchor tuples , + the maximum length $l(p_i)$ of the set of complete exponent sequences generated by prefixes of anchor tuples, is less than $i - 1$, the length of the exponent sequence generated by any i-level tuple.

However, Lemma 7.0 rules out this possibility, since it states that if a non-counterexample and a counterexample anchor tuple generate different exponent sequences, then those exponent sequences can differ in at most the last element.

Therefore all sequences generated by anchor tuples are generated by non-counterexample anchor tuples, and the 3*x* +1 Conjecture is true. (End of First Possible Proof)

Third Possible Proof of the 3x + 1 Conjecture Using the "Pushing Away" Strategy

The reader is urged to read all of the section "Anchors and Anchor Tuples" on page 68 through page 68, before reading this possible proof.

1. Assume counterexamples exist. Then there exists a smallest $i = i_0$ such that the set of i-level non-counterexample anchor tuples is incomplete ("Lemma 10.97." on page 72). By computer test, we know that $i_0 \geq 35$ (see, e.g., step 1 under "First Plausibility Argument" on page 85).

For all levels $i_0 + 1$, $i_0 + 2$, $i_0 + 3$, ..., $i_0 + k$, ... the set of $(i_0 + k)$-level non-counterexample anchor tuples is likewise incomplete ("Lemma 10.97." on page 72).

2. Let t_{nc} be an i-level non-counterexample anchor tuple, where $i \geq i_0$. t_{nc} generates an i-level exponent sequence $A^*\{a\}$. By "Lemma 10.95." on page 58 we know that there exists an i-level counterexample anchor tuple t_c that generates an i-level exponent sequence $A^*\{b\}$.

In order to avoid the impossibility of two different i-level anchor tuples generating the same i-level exponent sequence, it is necessary that $a \neq b$.

3. Now since by "Lemma 10.981." on page 73, every exponent sequence suffix s maps to the set of non-counterexample range elements, and similarly for counterexample range elements, we know that for some $j > i$, there must be a j-level non-counterexample anchor tuple t_{nc}' and a j-level counterexample tuple t_c' that are mapped to by the same exponent suffix, s. In order to avoid the impossibility of two different j-level anchor tuples generating the same j-level exponent sequence, it must be the case that the j-level exponent sequence B^*s generated by t_{nc}' and the j-level exponent sequence C^*s generated by t_c' must be such that $B \neq C$.

But this contradicts "Lemma 10.95." on page 58, hence the $3x + 1$ Conjecture is true. (End of "Third Possible Proof...")

Fourth Possible Proof of the 3x + 1 Conjecture Using the "Pushing Away" Strategy

1. Assume counterexamples exist. By Lemma 10.0 we know that there is an infinity of infinite non-counterexample tuples and that the set of all prefixes of these generate the set of all finite exponent sequences. Each infinite tuple has a mark m which defines when that tuple is first an anchor tuple (see step 1 under "First Possible Proof of the 3x + 1 Conjecture Using the "Pushing Away" Strategy" on page 80).

2. Let $\{T_A(\bar{t}_{nc})\}$ = {tuple-sets T_A | T_A is an m-level tuple-set whose anchor tuple is the prefix $\bar{t}_{nc}(m)$ of an infinite non-counterexample $\bar{t}_{nc}$ whose mark is m}.

3. Then all that remains in each tuple-set in $\{T_A(\bar{t}_{nc})\}$, besides the anchor tuple, is counterexample tuples. As each non-counterexample anchor tuple is extended through higher and higher levels, producing extensions of T_A, the counterexample tuples are "pushed farther and farther away" (by Lemma 1.0). (The non-counterexample anchor tuples in these extensions remain anchor tuples ("once an anchor tuple, always an anchor tuple").)

But these counterexample tuples can never become anchor tuples, because the non-counterexample anchor tuples generate the set of all 2-level exponent sequences, and the set of all 3-level exponent sequences, and the set of all 4-level exponent sequences, and...

If no extension of any counterexample tuple becomes an anchor tuple, then counterexample tuples do not exist (by "Definition of "Anchor"" on page 68), hence counterexamples do not exist. (End of Fourth Possible Proof)

Remark on Fourth Possible Proof

The identical argument, with "non-counterexample" and "counterexample" interchanged, can be used to prove that non-counterexamples do not exist! Which, of course, is false.

Appendix C — Plausibility Arguments for the Truth of the 3*x* + 1 Conjecture Using the "Missing Sequences" Strategy

First Plausibility Argument

1. As we state under "Preliminary Discussion of Strategies" on page 19, "whether or not there is a counterexample, the set of all tuple-sets will remain unchanged. Every odd, positive integer, whether counterexample or non-counterexample, will occupy exactly the same place in every tuple of which the integer is a member."

We can say more: according to a source we consider reliable[1], the $3x + 1$ Conjecture has been found to be valid, by computer testing, for all odd, positive integers $< 56 \cdot 10^{15}$ (56 quadrillion). Since $2 \cdot 3^{35 - 1} < 34 \cdot 10^{15} < 56 \cdot 10^{15}$ this means, by Lemma 3.057, that *all* tuples of length 35 that end in a $y < 2 \cdot 3^{35 - 1}$ — hence all 2-level, 3-level, 4-level, ..., 35-level anchor tuples — are tuples whose elements map to 1.

We cannot tell, by examining the set of all tuple-sets, whether or not a counterexample exists. We can only *state* (by Lemma 10.97) that, if a counterexample exists, then some exponent sequences will be missing from the set of exponent sequences generated by non-counterexample anchor tuples beyond some minimum i_0, and similarly for counterexample anchor tuples beyond that minimum i_0 (i_0 is the smallest i such that a counterexample is an anchor).

2. The statements in the sub-section, "Why Are There An Infinite Number of Tuples in Each Tuple-set?" on page 21, imply the following:

Let $A = \{a_2, a_3, ..., a_i\}$, $i \geq 2$, be any finite sequence of exponents. A defines a tuple-set, T_A. Now by Lemma 10.0, whether or not a counterexample exists, there must exist the following non-counterexample tuples in T_A:

For *each* possible exponent $a_{i+1} \geq 1$ an *infinity* U of tuples that each have an extension via a_{i+1}. (We know there must be such an infinity of tuples U because if there were only a finite number, there would only be a finite number of $(i + 1)$-level tuples in the tuple-set $T_{A'}$, where $A' = \{a_2, a_3, ..., a_i, a_{i+1}\}$, which would be contrary to Lemma 2.0.)

But furthermore, U must contain, for *each* possible exponent a_{i+2}, an infinity U' of tuples that each have an extension via the sequence a_{i+1}, a_{i+2}.

But furthermore, U' must contain, for *each* possible exponent a_{i+3}, an infinity U'' of tuples that each have an extension via the sequence a_{i+1}, a_{i+2}, a_{i+3}.

etc.

3. If no counterexamples exist, there is no "redundancy" in the set of all tuple-sets. By this we mean that:

(3.1) Every infinite non-counterexample tuple is unique (Lemma 1.3). (This is true whether or not counterexamples exist.)

1. At time of writing, we do not know if the paper has been published.

(3.2) Every i-level tuple-set has exactly one first, i-level tuple (i.e., anchor tuple). (If a tuple-set could have *two* anchor tuples, that redundancy would allow for counterexamples and non-counterexamples to co-exist).

4. The above statements, we argue, constitute a plausibility argument that there is not enough "room" in the set of all tuple-sets for non-counterexamples and counterexamples. Hence the $3x + 1$ Conjecture is true. (End of plausibility argument)

Second Plausibility Argument

1. We begin by observing that, if counterexamples exist, then no counterexample can generate an infinite tuple whose exponent sequence is the same as that of any non-counterexample tuple.

2. Lemma 7.0 asserts:

Lemma 7.0. For each range element y and for each exponent sequence A, there exists an x that maps to y via A, possibly followed by an additional "buffer" exponent.

Assume counterexamples exist and let i_0 denote the smallest level i such that a counterexample is an anchor at level i. Thus at level i_0 there are both counterexample and non-counterexample anchors. Now since all anchors are range elements, we can assert that:

(1) The set of all (i_0 - 1)-level exponent sequences A is defined by counterexample anchor tuples, and the set of all (i_0 - 1)-level exponent sequences A is defined by counterexample anchor tuples.

3. Now a naive reader might be inclined to say: "Well, that's it, then! The infinite exponent sequences generated by counterexamples and those generated by non-counterexamples, first differ at level i_0."

4. But we now advance to level $i_0 + 1$, and assert the equivalent of (1).

The naive reader might be inclined to say, "Well, OK, I was wrong. The infinite exponent sequences generated by counterexamples and those generated by non-counterexamples, first differ at level $i_0 + 1$, not i_0."

5. But we now advance to level $i_0 + 2$, etc. From which we conclude:

There does not exist a level i such that the infinite exponent sequences generated by counterexamples, and those generated by non-counterexamples, differ at level i.

And therefore infinite exponent sequences generated by counterexamples, and those generated by non-counterexamples, do not differ, which contradicts what we said at the start. (End of Second Plausibility Argument)

Probable Error in Second Plausibility Argument

Step 5 in the the Second Plausibility Argument makes an assumption which does not appear to be justified. We can describe this assumption via an example.

Let $\bar{t}_c$ be an infinite counterexample tuple. Thus $\bar{t}_c$ generates an infinite exponent sequence $\overline{A}_c$ containing an infinite sub-sequence *not* equal to {2, 2, 2, ... }. Let us say that $\overline{A}_c$ = {1, 2, 3, 1, 2, 3, ... }.

But $\overline{A}_c$ can be approximated by the following infinite sequence of exponent sequences generated by non-counterexample infinite tuples.

{1, 2, 2, 2, ... },
(1, 2, 2, 2, ... },
{1, 2, 3, 2, 2, 2, ...}
{1, 2, 3, 1, 2, 2, 2, ... }

etc.

Thus for each $i \geq 2$, the i-level exponent sequence prefix $\overline{A}_c(i)$ generated by the counterexample tuple prefix $\bar{t}_c(i)$ = the i-level exponent sequence prefix $\overline{A}_{nc}(i)$ generated by some i-level non-counterexample tuple prefix $\bar{t}_{nc}(i)$, and yet $\overline{A}_c \neq \overline{A}_{nc}$ for any non-counterexample tuple.

Third Plausibility Argument

This Plausibility Argument is motivated by the error described in the previous sub-section.

1. We ask the reader to review the material from the beginning of “First Possible Proof of the 3x + 1 Conjecture Using the “Pushing Away” Strategy” on page 80, through the definition of “mark”.

2. Now assume counterexamples exist, and let x_c be a counterexample. Then x_c is the first element of an infinite tuple $\bar{t}_c$ which generates an infinite exponent sequence $A(\bar{t}_c) = \{a_2, a_3, a_4, ...$ }, where $A(t)$ for any tuple t, finite or infinite, denotes the exponent sequence generated by t. By definition of “counterexample”, we know that $A(\bar{t}_c)$ does not equal B *{2, 2, 2, 2, ... }, where B is any finite exponent sequence, and “*” denotes concatenation of exponent sequences.

3. By the material referenced in step 1, $\bar{t}_c$ has a mark at level $m \geq 2$ indicating the first level at which a prefix of $\bar{t}_c$, namely, the prefix $\bar{t}_c(m)$, becomes an anchor tuple. Thereafter, all prefixes, $\bar{t}_c(m+1)$, $\bar{t}_c(m+2)$, $\bar{t}_c(m+3)$, ... are anchor tuples, because every extension of an anchor tuple is an anchor tuple (Lemma 10.90 (“Once an anchor tuple, always an anchor tuple”)).

Now by Lemma 10.0, we know that there exists a sequence S_{nc} of non-counterexample infinite tuples $\bar{t}_{nc}$, $\bar{t}_{nc}'$, $\bar{t}_{nc}''$, $\bar{t}_{nc}'''$, ... having the following properties:

$A(\bar{t}_{nc}(2)) = \{a_2\}$,
$A(\bar{t}_{nc}'(3)) = \{a_2, a_3\}$,

$A(\bar{t}_{nc}''(4)) = \{a_2, a_3, a_4\}$,
$A(\bar{t}_{nc}'''(5)) = \{a_2, a_3, a_4, a_5\}$,
...

In words: the finite exponent sequences generated by the infinite sequence of non-counterexample infinite tuple *prefixes* of tuples in S_{nc}, generate a sequence of ever closer approximations to the sequence $A(\bar{t}_c)$ that is generated by our assumed infinite counterexample tuple $\bar{t}_c$.

4. Now it must be the case that an infinity U of tuples in S_{nc} has a prefix that generates the exponent sequence $\{a_2, a_3, a_4, ..., a_m\}$. We now assert that, for an infinity V of these tuples, the mark of each tuple is $> m$. ($V \subseteq U \subseteq S_{nc}$.)

Proof:
4.1 First, none of the marks can be $= m$, because if that were the case, then we would have, for two different infinite tuples, prefixes that are both anchor tuples and that both generate the same exponent sequence, which is impossible, by definition of "anchor tuple".

4.2 But an infinity of the marks can not be $< m$, because if that were the case, then, by the pigeon-hole principle, we would have, for two different infinite tuples, prefixes that are both anchor tuples and that both generate the same subsequence of $\{a_2, a_3, a_4, ..., a_m\}$, which is impossible by definition of "anchor tuple". (We would have an infinity of marks distributed over a finite range of marks, namely the range 2, 3, ..., m.)

4.3 Therefore the marks of tuples in U are all greater than m. □

5. We now repeat, and again prove, the assertion made in the material referenced in step 1, namely, that the range of the marks of tuples in U is unbounded.

Proof:
5.1 Assume, to the contrary, that the range is bounded — say the range is $\{2, 3, ..., m, ..., h\}$.

5.2 But then we simply apply the argument in step 4.2 above to show that this is not possible. □

Therefore, there will be a tuple in S_{nc} whose mark, *and that of each subsequent tuple in* S_{nc}, will be greater than the mark m of our given infinite counterexample tuple $\bar{t}_c$.

But this means that there will be an infinite tuple $\bar{t}_{nc}''^{\cdots}{}'$ in S_{nc} having a prefix $\bar{t}_{nc}''^{\cdots}{}'(m + k)$, $k \geq 0$, such that $A(\bar{t}_{nc}''^{\cdots}{}'(m + k)) = A(\bar{t}_c(m + k))$. But this means that an $(m + k)$-level non-counterexample anchor tuple and an $(m + k)$-level counterexample anchor tuple both generate the same $(m + k)$-level exponent sequence, which is impossible by definition of "tuple-set" and, in particular, of "anchor tuple", since the same i-level exponent sequence, $i \geq 2$, can not be generated by two different i-level anchor tuples, and, in particular, can not be generated by i-level counterexample and non-counterexample anchor tuples. (End of Third Plausibility Argument)

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Appendix D — A Curious Fact About the Inverse of the 3*x* + 1 Function

In 2003, we were struck by the fact that, whether or not counterexamples exist, the set of numbers that map to 1 in one iteration of the $3x + 1$ function, namely, the set $C^{(-1)}(1) = \{1, 5, 21, 85, 341, ... \}$ remains the same. And similarly for the set $C^{(-2)}(1)$ of numbers that map to 1 in two iterations, and the set $C^{(-3)}(1)$ of numbers that map to 1 in three iterations, and ... and the set $C^{(-n)}(1)$ of numbers that map to 1 in n iterations, $n \geq 1$. We felt that, if counterexamples exist, at least one of these sets should be different from what it would be if no counterexamples exist. We then tried to construct a proof that counterexamples do not exist by arguing that, if the the same set, namely, the set $B = C^{(-1)}(1) \cup C^{(-2)}(1) \cup C^{(-3)}(1) \cup ... \cup C^{(-n)}(1) \cup ...$ maps to 1 regardless of whether counterexamples exist or not, then counterexamples "have no effect" on the set of numbers that map to 1, "hence" they do not exist.

The mathematicians to whom we showed this argument did not agree with it. In the course of trying to repair the argument, we came up with an argument that counterexamples *do* exist, and this seemed, to us, to be more convincing that our previous argument that they do not exist. But that argument was equally fallacious.

After further struggles, we arrived at the following explanation of why each set $C^{(-n)}(1)$ is "the same" whether or not counterexamples exist.

1. **if** counterexamples exist, **then** by definition no counterexample can be an element of $C^{(-n)}(1)$ for any $n \geq 1$.
2. **if** counterexamples do not exist, **then**, trivially, no counterexample can be an element of $C^{(-n)}(1)$ for any $n \geq 1$.
3. Hence each $C^{(-n)}(1)$ is the same in both cases.

But the set $A - B$, where A is the set of odd, positive integers, and $B = C^{(-1)}(1) \cup C^{(-2)}(1) \cup C^{(-3)}(1) \cup ... \cup C^{(-n)}(1) \cup ...$ is definitely *not* the same whether or not counterexamples exist!

First Criticism of Our Argument, And Our Reply

Still not satisfied with this explanation, we constructed another possible proof that counterexamples do not exist using the same fact that each set $C^{(-n)}(1)$ is "the same" whether or not counterexamples exist. A mathematician wrote us as follows:

"[Your possible proof] is not a logical argument. There are no well-defined concepts of 'the set of integers that map to 1 if counterexamples exist' and 'the set of integers that map to 1 if counterexamples do not exist'. There is simply the set of integers that do map to 1. If counterexamples don't exist, that set is the set of all odd positive integers, while if counterexamples do exist then it is a proper subset thereof.

"A quick way to see that the argument is fallacious is to note that it could equally well be applied to the question of whether negative integers have the property that the $3x+1$ function eventually sends all of them to -1. (Note that on applying the $3x + 1$ function to $-n$, one forms $3(-n) + 1$ and divides it by the largest power of 2 dividing it, just as for positive integers, but one gets a negative integer as result.) The same argument, taken word-for-word, would 'prove' that all negative integers eventually give -1, but that is not true; starting with -5 we get $-5 \rightarrow -7 \rightarrow -5$, so these values cycle endlessly."

And yet we were bothered by the mathematician's statements that "there are no well-defined concepts of 'the set of integers that map to 1 if counterexamples exist' and 'the set of integers that map to 1 if counterexamples do not exist'. There is simply the set of integers that do map to 1."

We found it impossible to believe that, over the course of some 70 years of research on the $3x + 1$ Problem, competent mathematicians, in spoken conversation and even in published papers, had not often made the equivalent of the statements:

"the set of integers that map to 1 if counterexamples exist is a proper subset of the odd, positive integers" and

"the set of integers that map to 1 if counterexamples do not exist is the entire set of odd, positive integers".

We found the following argument intriguing.

Let B_c denote the set of integers that map to 1 if counterexamples exist;
Let B_{nc} denote the set of integers that map to 1 if counterexamples do not exist;
Let A denote the set of odd, positive integers;
Let $H = \varnothing$.

Then we can write:

(1) $B_{nc} \cup H = A$;

(2) $B_c \cup H' = A$.

We hope that the reader will agree that these two equations are correct. Now, as we have proved above, $B_c = B_{nc}$, and therefore we can solve for H'. We see immediately that H must $= H'$. But since $H = \varnothing$, we must conclude that $H' = \varnothing$.

But if our reasoning is valid (see below under "A Common Criticism of the Above Argument, and A Reply"), then we can let H denote the set of counterexamples if there are no counterexamples (in other words, the empty set), and we can let H' denote the set of counterexamples if there *are* counterexamples (necessarily not the empty set!).

Hence it would seem we have proved the $3x + 1$ Conjecture.

Is it possible that the fact that, as of this writing, unlike the $7x + 1$ case, we *do not know* if counterexamples exist in the $3x + 1$ case — that this fact can legitimately used in a proof?

Second Criticism of Our Argument, And Our Reply

The following criticism of the last of the above arguments is representative of several we received:

"Your B_c is *only* defined if the [$3x + 1$] Conjecture is not true. On the other hand B_{nc} is *only* defined if the Conjecture is true. As soon as you start working with B_{nc}, that means that you are already assuming that the Conjecture is true, an assumption you obviously can't make if you are trying to prove the Conjecture. What you are doing is even worse. You work with both B_c *and* B_{nc}, thereby assuming the Conjecture is true and false at the same time. With that contradiction you could prove anything you want!"

To which we replied as follows:

“First, you cannot escape the fact that $B_c = B_{nc}$. Second, a set is a set is a set. A set definition stands on its own. Nowhere in the textbooks on mathematical logic and set theory that we have read, have we come across a rule that says, in so many words, ‘The condition that defines a set must not be in contradiction to the condition that defines any other set’, because, if there had been such a rule, we (and most readers) would have said, ‘any other set — in the same sentence? In the same paragraph? On the same page? In the same proof? In the same chapter? In the same branch of mathematics? In the same universe?’

“In the vast literature on Fermat’s Last Theorem (FLT), for example, we would be surprised if the equivalent of the following statements have not occurred in the same paper:

“Let U = the set of four-tuples <x, y, z, n> for which FLT is true. This set includes, e.g., all such tuples in which $n < 125{,}000$. Let V = the set of counterexamples to FLT...”

“We do not see any logical inconsistency here, even though FLT cannot both be true and not true. If FLT is true, then V is empty, otherwise not.”

We attempted to make our point in another way:

Suppose a competent mathematician writes one or more paragraphs concerning the set D of odd, positive integers, and an unspecified proper subset E, of D. His text includes statements such as

$D \cup E = D$, and
$D \cap E = E$.

Other competent mathematicians, reviewing the text, find that all the statements it contains are correct. It is now revealed that D = the set of numbers that map to 1 if the $3x + 1$ Conjecture is true, and that E = the set of numbers that map to 1 if the $3x + 1$ Conjecture is false. Does the original mathematician’s text suddenly become full of logical inconsistencies? If it does, then this would seem to place a huge question mark over the entire body of mathematical literature, because suddenly the necessity has arisen of examining every page of this literature and asking if it is possible that any two more more of the sets discussed, can possibly be derived from contradictory assumptions!

Finally, we argued that following statements certainly seem legitimate:

Let U denote the set of roots of the equation in the statement of Riemann's Conjecture, i.e., of the equation:

$$\zeta(z) = 1 + 1/2^z + 1/3^z + \ldots = 0$$

Let V denote the set of roots lying on the line in the complex plane $a = 1/2$.

Let W denote the set of counterexamples to Riemann's Conjecture, i.e., the set of roots not lying on the line in the complex plane, $a = 1/2$.

Then $U = V \cup W$.

The Error in Our Argument

At the root of our argument is what might be called (at least in computer science circles) "pushing conditionals down into sets". The best example of this is the set *W* in the Riemann's Conjecture example, in the previous sub-section. Here, the conditionals — If the Riemann Conjecture is true, then..., If the Riemann Conjecture is false, then ... — are "pushed down" into the set *W*. But observe that here, the set is defined whether or not the Riemann Conjecture is true or false. Of course, the set is different in each case, but it is defined in each case.

This is not true of the sets B_c and B_{nc}, which we defined above in the sub-section "First Criticism of Our Argument, And Our Reply" on page 89. As the person who made the Second Criticism above pointed out, the set B_c is defined only if counterexamples exist, and the set B_{nc} is defined only if counterexamples do not exist.

As a result, as the mathematician quoted in the same sub-section stated, the fallacy in our argument based on equations (1) and (2) is that if counterexamples exist, then the set H' is not empty, hence, by our logic, so is *H* non-empty. Hence we get two different values for *H*, which is not possible.

If we let *B* = the set of numbers that map to 1 (regardless whether counterexamples exist or not) then we can legitimately write,

$A = B \cup H.$

But now we gain no new information: if counterexamples exist, then *H* is non-empty; otherwise it is empty.

So the lesson learned from the struggle described in this Appendix is simply this: if the technique of pushing conditionals down into sets is used, then each set must be defined for all possible conditions.

A Different, and Stronger, Argument

The Set *S*

It is easily shown that the set *S* of odd, positive integers mapping to 1 in one iteration of *C* is *S* = {1, 5, 21, 85, 341, ... }, where if x, x' are successive elements of *S*, then $x' = 4x + 1$. By Lemma 5.0, and the fact that 1 maps to 1 via an even exponent, namely, 2, the set of exponents by which the elements of *S* map to 1 is precisely the set of even positive integers. Thus, e.g., 1 maps to 1 via the exponent 2; 5 maps to 1 via the exponent 4; 21 maps to 1 via the exponent 6, etc.

The set S′ of odd, positive integers mapping to 1 in two iterations of *C* is the set of inverses of all elements of *S* that are non multiples-of-3, since no odd, positive integer maps to a multiple-of-3 via *C*. Thus, e.g., 13 is an element of S′ because 13 maps to 5 in one iteration of C, and then 5 maps to 1 in a second iteration of *C*. And similarly for the set S″ of odd, positive integers mapping to 1 in three iterations of C. Thus, e.g., 17 is an element of S″ because 17 maps to 13 in one iteration of *C*, and then 13 maps to 1 in two iterations of *C*. Etc.

The Set $\underline{S}$

We define the set $\underline{S}$ to be:

$\underline{S} = S \cup S' \cup S'' \cup \ldots$

We now make the following statement:

Lemma 50.0.

Whether or not counterexamples exist, $\underline{S}$ is the set of odd, positive integers that map to 1.

Proof:

If the Lemma is not true, then the laws of arithmetic are dependent upon whether certain odd, positive integers map to 1 or not, which is absurd. □

Remarks on Lemma 50.0.

Some readers have argued against the truth of Lemma 50.0 by saying that the "universe" in which the $3x + 1$ Conjecture is true is different from the "universe" in which it is false. These readers are apparently thinking of the fact, in physics, that it is very difficult to determine which universe, of several possible ones, we live in. The Anthropic Principle was developed as a response to this problem. It says that in any universe in which the fundamental physical constants differ by even a small amount from what they are for our universe, intelligent life such as ours would be impossible, and therefore there would be no one to ask the question, "What universe are we living in?"

Our response to the different universes objection is simply this: in mathematics, universes overlap. Thus, for example, the results that had been achieved by, say, 1990, in an attempt to prove Fermat's Last Theorem, were valid then and would remain valid regardless whether a proof or disproof of the Theorem was ever found. (As the reader no doubt knows, the Theorem was proved by Andrew Wiles in 1994.) It is highly unlikely that any mathematican around 1990 ever said, "Of course all our labors may be in vain, because if the Theorem is disproved, why, then it might be possible that for some $x, y, z, x^{41} + y^{41} = z^{41}$ (in 1990, the Theorem was known to be true for all prime exponents up to around 125,000).

Similarly, we believe that in both the universe in which the $3x + 1$ Conjecture is true, and the universe in which the $3x + 1$ Conjecture is false, the set $\underline{S}$ is the set of odd, positive integers that map to 1 via the function C.

Another objection that some readers have made is that it is "dangerous" to speak of certain facts being true "regardless whether counterexamples exist or not". But most of the elementary results concerning the $3x + 1$ Problem are true whether counterexamples exist or not. This is one of the things that make the $3x + 1$ Problem so tantalizingly difficult. Here are a few of these results that are well-known to researchers. These results apply to all domain elements x or all range elements y, of C.

Lemma 5.0 (Lemma 12.0 in [so] and [ar])

(a) If y is a range element of C, then y is mapped to, in one iteration of C, by all exponents of one parity only. (An instance of this result was cited above in "The Set S".)

(b) For each of the two parities, there exists a range element that is mapped to by every exponent of that parity.

Lemma 11.0. *If $U = \{x_1, x_2, x_3, \ldots\}$ is the set of odd, positive integers mapping to a range element y in one iteration of C, and x_i, x_j are successive elements of U, and $x_i < x_j$, then $x_j = 4x_i + 1$.* (An instance of this result was cited above in "The Set S".)

Lemmas 5.5, 5.7 *If x maps to y in one iteration of C, then:*
If $x \equiv 1 \mod 4$ then the exponent of 2 is ≥ 2;
If $x \equiv 3 \mod 4$ then the exponent of $2 = 1$;
If $y \equiv 1 \mod 3$ then the exponent of 2 is even;
if $y \equiv 2 \mod 3$ then the exponent of 2 is odd.

The Argument

Lemma 50.0 states that, whether or not counterexamples exist, the same set of odd, positive integers maps to 1. But this is impossible, since if counterexamples exist, the set of odd, positive integers that map to 1 is a proper subset of the set of odd, positive integers that map to 1 if counterexamples do not exist. Hence we have a contradiction and the $3x + 1$ Conjecture is true.

We will welcome comments from readers.

Appendix E — A Curious Fact About Tuple-sets

As we state under "Preliminary Discussion of Strategies" on page 19, "whether or not counterexamples exist, the set of all tuple-sets will remain unchanged. Every odd, positive integer, whether counterexample or non-counterexample, will occupy exactly the same place in every tuple of which the integer is a member." Not a single number in a single tuple will be different. Putting it in another, and somewhat more fanciful, way: suppose there are parallel universes, one in which the $3x + 1$ Conjecture is true, and another in which it is false. Then the set of all tuple-sets is the same in both. Thus we can consider the $3x + 1$ Conjecture to amount to no more than this: if the Conjecture is true, then all tuples in all tuple-sets are colored green; if it is false, then an infinity of tuples (the counterexample tuples) in each tuple-set are red, and an infinity green (Lemma 10.0). There is no difference in the actual numbers in the tuples.

Several readers have argued that the observation that "all tuple-sets will remain unchanged" is irrelevant and misleading. There is one and only set of tuple-sets. If counterexamples exist, then this set will reflect that fact. If not, then it won't. One reader has commented that the parallel universes argument would only apply if there were two axiom systems that could be used in constructing the $3x + 1$ function and the lemmas and theorems we prove about it.

And yet, for some of us, it is hard to get away from the fact that *finite* counterexample tuples are indistinguishable from *finite* non-counterexample tuples (as far as exponent sequences are concerned). Every finite tuple "remains the same" regardless whether counterexamples exist.

Let us see if we can clarify the difficulty here. We will attempt to do so by attempting to make the problem of proving Fermat's Last Theorem (FLT) as similar as we can to the problem of proving the 3x + 1 Conjecture.

We observe immediately that in both cases we are attempting to prove a conjecture, and that in both cases if the conjecture is false, there will be at least one counterexample to the conjecture. We can increase the similarity by defining the equivalent of tuple-set tuples, for FLT. For each x, y, z, we define an infinite tuple — which we will call an F-tuple — as follows:

the first element of the F-tuple is the value of $x^1 + y^1 - z^1$;
the second element of the F-tuple is the value of $x^2 + y^2 - z^2$;
the third element of the F-tuple is the value of $x^3 + y^3 - z^3$;
etc.

Now let us compare the nature of counterexamples in the two cases: for the $3x + 1$ Conjecture, a counterexample is an *infinite* entity, namely, an infinite tuple that does not end in an infinite sequence of 1s, i.e., an infinite tuple that is not $<x, y, ..., 1, 1, 1, ... >$.

For FLT, a counterexample in the usual parlance is a *finite* entity, namely an ordered 4-tuple $<x, y, z, n>$ such that $x^n + y^n - z^n = 0$.

But can we not declare the entire infinite F-tuple that contains an element = 0 to be a counterexample, say, an "F-counterexample"?

In any case, we can see how one might be inclined to say of all F-tuple *elements* that are not counterexamples in the usual parlance, that they "remain the same" whether or not a counterexample exists.

However, it is not true that all F-*tuples* "remain the same" whether or not counterexamples exist. If counterexamples exist, then at least one F-tuple will contain a 0. Otherwise not. Putting it another way, if counterexamples exist, then, in principle, a computer program exists that will determine that fact simply by first creating a linear ordering of all ordered 4-tuples <*x*, *y*, *z*, *n*> and then testing each one to see if $x^n + y^n - z^n = 0$. If a counterexample exists, then the program will eventually find that 4-tuple.

But no computer program is guaranteed to determine if a counterexample to the $3x + 1$ Conjecture exists, because for each odd, positive integer *x*, unless *x* gives rise to an infinite cycle of numbers other than 1, there is no iteration that will reveal that *x* is a counterexample: if, in iterating on *x*, we run out of computer resources (time, memory) before arriving at a 1, that will mean one of *two* things: that *x* is in fact a counterexample, or that *x* takes a very large number of iterations to yield 1.

Appendix F — Further Thoughts on the "Filling-in" Strategy

A Faulty Proof

We begin our discussion by presenting what we hoped was a proof of the $3x + 1$ Conjecture, but which contains a fundamental error.

Before reading the following, the reader should review the sub-section "Strategy of "Filling-in" of Intervals in the Base Sequence Relative to 1" on page 40 and, in particular, Lemma 17.0 and the discussion following it under "Three Important Lemmas" on page 43.

As set forth in the above sub-section, the "filling-in" strategy attempts to show that eventually every interval in the base sequence relative to 1, i.e., every interval in the sequence {1, 5, 21, 85, 341, ... } is filled in with a non-counterexample, i.e., a number that maps to 1.

1. Assume that counterexamples exist. Then since there are an infinite number of counterexamples, an infinity of intervals in the sequence {1, 5, 21, 85, 341, ... } must each contain at least one counterexample. More precisely, each interval s_k , $k \geq 1$, of an infinity of intervals in the base sequence relative to 1 must contain a fixed number $y_c(s_k)$ of counterexamples and a fixed number $y_{nc}(s_k)$ of non-counterexamples. ($y_c(s_k)$ is of course not necessarily equal to $y_c(s_{k'})$ for $k \neq k'$, and similarly for $y_{nc}(s_k)$ and $y_{nc}(s_{k'})$.)

2. In order to fix ideas, we make the following Claim:

Claim:

Suppose we have an infinity of intervals in the positive integers, subject to the sole condition that each interval (except the first) contains more elements than the previous. The intervals are numbered 1, 2, 3, ... Now consider the following procedure and assume that, prior to step 1, no element of any interval is marked.

Step 1. Mark one element of each interval beginning with interval 1.

The result is that an infinity of consecutive intervals (beginning with interval 1) each contains *one* marked element.

Step 2. Mark one unmarked element of each interval beginning with interval 10.

The result is that an infinity of consecutive intervals (beginning with interval 10) each contains *two* marked elements.

Step 3. Mark one unmarked element of each interval beginning with interval 100.

The result is that an infinity of consecutive intervals (beginning with interval 100) each contains *three* marked elements.

Step 4. Mark one unmarked element of each interval beginning with interval 1000.

The result is that an infinity of consecutive intervals (beginning with interval 1000) each contains *four* marked elements.

...

Our Claim is that, for each step $n \geq 1$, the result is an infinity of consecutive intervals (beginning with interval 10^{n-1}), each containing n marked elements.

3. Now Lemma 17.0 states:

Lemma 17.0. *For all $m \geq 2$, and for all $k \geq 2$, there exists an infinity of consecutive intervals in the base sequence relative to 1, i.e., in the sequence {1, 5, 21, 85, 341, ... }, such that each such interval contains $((m^k - 1)/(m - 1) - 1)$ numbers that map to 1.*

Thus, we can always add (or mark) at least one element in each of an infinity of consecutive intervals. Hence each interval does not contain a fixed number $y_c(s_k)$ of counterexamples, and we have a contradiction. (End of Possible Proof.)

The Error

The error lies in the fact that, as m, k grow larger, all the $((m^k - 1)/(m - 1) - 1)$ "spiral"s (and not just some of them) can always begin in larger intervals in the base sequence. And thus it is possible there is always room in each interval for counterexamples.

Attempts to Overcome the Error

If sets of $((m^k - 1)/(m - 1) - 1)$ "spiral"s advanced into larger intervals at a sufficiently slow rate as m, k increased, then they would necessarily have to fill up intervals as m, k increased. Let us investigate this possibility now.

Let y be any base element. Then, by Lemma 5.0, y is mapped to by either all positive odd exponents, or by all even positive exponents. In the first case the smallest exponent is 1 and we have $y = (3x + 1)/2^1$. Thus x is approximately $(2/3)y$. In the second case the smallest exponent is 2 and we have $y = (3x + 1)/2^2$. Thus x is approximately $(4/3)y$.

It is tempting to conclude that the first elements of an infinite "spiral" at level $k \geq 1$ relative to a given base element y, can be no larger than $(4/3)^k y$, but this is not correct. The reason is that in every "spiral" there are multiples-of-3, and nothing maps to a multiple-of-3. (However, it is easily shown that only every third element of a "spiral" is a multiple-of-3.) Thus 9 maps to 7 via the exponent 2, but 9 is a multiple-of-3, and so the smallest number at level 2 (relative to 7) that maps to 7 is 49, because 49 maps to 37 via the exponent 2 (37 maps to 7 via the exponent 4).

The worst case, for our purposes, is the following: a base element y is mapped to by all even exponents. The first element of the "spiral" mapping to y is a multiple-of-3, which necessarily maps to y via the exponent 2. Therefore we must proceed via the next element y' of the "spiral", which maps to y via the exponent 4. We find that the same situation applies: y' is mapped to by all even exponents. The first element of the "spiral" mapping to y' is a multiple-of-3, which necessarily maps to y' via the exponent 2. Therefore we must proceed via the next element y'' of the "spiral", which maps to y' via the exponent 4. Etc.

Therefore numbers mapping to our original y increase at most as $((2^4/3)^k)\,y$, $k \geq 1$ (actually, as less than this, since it is easily shown that no infinite sequence of exponents = 4 can map to y).

But we do not know how to go from this fact to a proof that sets of “spiral”s do not advance too rapidly.

Another possibility is based on the fact that Lemma 17.0 applies to counterexamples as well as to non-counterexamples. It is easily shown that the number of elements in the *k*th interval is given by

$$s_k = \frac{(2^2)^k - 1}{2^2 - 1}$$

Unfortunately, even if it could be proved that no interval can contain a sum of different $((m^k - 1)/(m - 1) - 1)$ terms (not all with the same m, k, of course), i.e., one term for non-counterexamples, and one or more for counterexamples, that would still not give us our proof, because the “spiral”s specified by these terms do not all appear simultaneously in a given interval. Some appear in earlier intervals than others. Hence we can not use our sum argument.

Appendix G — Results on the Minimum Counterexample

Tuple-sets make it easy to deduce some interesting results on the minimum counterexample, if counterexamples exist. To begin with, it is obvious that if x maps directly to a y such that $y > x$, then y cannot be the minimum counterexample. Such a mapping is only possible via the exponent 1. The tuple-set T_A, where $A = \{1\}$ is:

{<3, 5>, <7, 11>, <11, 17>, <15, 23>, ... }.

We conclude immediately that no second element y of these tuples can be the minimum counter-example. So we know that no element of the set

$A = \{5, 11, 17, 23, ... \}$

is the minimum counterexample.

But it is also obvious that if x maps directly to a y such $y < x$, then x cannot be the minimum counterexample. Such a mapping is possible for all exponents ≥ 2. For example, the tuple-set T_A, where $A = \{2\}$ is

{<1, 1>, <9, 7>, <17, 13>, <25, 19>, ... }

We conclude immediately that no first element x of these tuples can be the minimum counterex-ample. So we know that no element of the set

$B = \{1, 9, 17, 25, ... \}$

is the minimum counterexample.

So we know that no element of the set $A \cup B = \{x \mid x \equiv 5 \bmod 6 \text{ or } x \equiv 1 \bmod 8\}$ is the min-imum counterexample.

To take one more example, the tuples in the tuple-set T_A, $A = \{3\}$, are

{<13, 5>, <29, 11>, <45, 17>, <61, 23>, ...}

We conclude immediately that no first element x of these tuples can be the minimum counterex-ample. So we know that no element of the set

{13, 29, 45, 61, ...}

is the minimum counterexample.

In the space of a few minutes, we have computed three infinite sets none of whose elements is the minimum counterexample. Indeed, in any 2-tuple having different first and second elements — in other words, any 2-tuple except for <1, 1> — *the larger of the two elements cannot be the smallest counterexample*!

Clearly, with the help of the computer, and, in particular, known non-counterexamples already determined by computer test (far more than the first 10^{15} odd, positive integers are known to be non-counterexamples), plus the observations made under "Tuple-sets and Finite Stopping Times" on page 18 and "Strategy of Proving There Is No Minimum Counterexample" on page 26, we can further reduce (drastically!) the set of candidates for the minimum counterexample.

Open Questions

Open Question O1. We know by Lemma 3.057 that, for all $i \geq 2$, the set of all i-level (i.e., last) elements of first i-level tuples in i-level tuple-sets is the set of minimum residues of all reduced residue classes mod $2 \cdot 3^{i-1}$. (By Euler's theorem, the number of such minimum residues is $\varphi(2 \cdot 3^{i-1}) = 2 \cdot 3^{i-2}$. (Thus, e.g., there are $\varphi(2 \cdot 3^{3-2}) = 6$ such minimum residues for level $i = 3$. These residues are 1, 5, 7, 11, 13, 17. These are all the 3-level elements of all first 3-level tuples in all 3-level tuple-sets.)

Question: For all $i \geq 2$, what is the set of all j-level elements, $1 \leq j < i$, of first i-level tuples?

Table of Symbols and Terms

Table 7: Frequently Used Symbols and Terms in This Paper

Symbol	What Symbol or Term Typically (or Always) Denotes	Formal Definition
A	Finite exponent sequence (see also "T_A")	"Tuple" on page 5
$\overline{A}$	Infinite exponent sequence generated by an infinite tuple $\bar{t}$	"On Infinite Tuples" on page 66
$\underline{A}$	Infinite exponent sequence generated by an infinite tuple $\underline{t}$	"Some Infinite Inverse Exponent Sequences That are Not Generated by Any Range Element, y" on page 35
$A(t)$	The (finite) exponent sequence generated by the (finite) tuple t	"Third Plausibility Argument" on page 87
$A(\bar{t})$	The (infinite) exponent sequence generated by the (infinite) tuple $\bar{t}$	"Third Plausibility Argument" on page 87
anchor	An element of M_i	"Question 1. How Does the Existence of Counterexamples "Make a Difference" in the Set of All Tuple-Sets?" on page 20
anchor tuple	A tuple whose last element is an anchor; there is exactly one anchor tuple in every i-level tuple-set: it is the first i-level tuple	"Question 1. How Does the Existence of Counterexamples "Make a Difference" in the Set of All Tuple-Sets?" on page 20
$C(x)$	The $3x + 1$ function	"Statement of Problem" on page 2
cycle	An infinite tuple in which a sequence of elements repeats indefinitely	"Non-terminating Tuple (n-t-v-1, n-t-v-c)" on page 8
exponent sequence	A finite or infinite sequence of exponents in the denominators of successive interations of the $3x + 1$ function, C	"Tuple" on page 5
i	Usual symbol for a level in a tuple-set	"Level in a Tuple-set" on page 6

Table 7: Frequently Used Symbols and Terms in This Paper

Symbol	What Symbol or Term Typically (or Always) Denotes	Formal Definition
i_0	Frequently-used symbol to denote the first (smallest) level at which a counterexample is an anchor	"Question 1. How Does the Existence of Counterexamples "Make a Difference" in the Set of All Tuple-Sets?" on page 20
level	In an *i*-level tuple-set, the first elements of all tuples are on level 1; the second elements are on level 2; ...; the *i*'th elements are on level *i*.	"Level in a Tuple-set" on page 6
m	A mark, i.e., the level in an infinite tuple $\bar{t}$ at which the prefix $\bar{t}(m)$ of $\bar{t}$ is first an anchor tuple	"First Possible Proof of the 3x + 1 Conjecture Using the "Pushing Away" Strategy" on page 80
M_i	The set of anchors at level *i*, i.e., the set of minimum elements of the set of reduced residue classes mod $2 \bullet 3^{i-1}$	"On First Elements of Anchor Tuples" on page 70
n-t-v-1	non-terminating-tuple-via-1 (i.e., a non-counterexample infinite tuple)	"Non-terminating Tuple (n-t-v-1, n-t-v-c)" on page 8
n-t-v-c	non-terminating-tuple-via-c (i.e., a counterexample infinite tuple)	"Non-terminating Tuple (n-t-v-1, n-t-v-c)" on page 8
prefix	The first *j* elements of a sequence, where $1 \leq j \leq$ the number of elements in the sequence. Thus we speak of a prefix of a tuple or a prefix of an exponent sequence.	"Every Range Element Is Mapped to By Every (i - 1)-Level Exponent Sequence" on page 71;
R_i	The top row of an *i*-level tuple-set, i.e., the set of last elements of all *i*-level tuples in an *i*-level tuple-set	"Row" on page 7
suffix	The last *j* elements of a sequence, where $1 \leq j \leq$ the number of elements in the sequence. Thus we speak of a suffix of a tuple or a suffix of an exponent sequence.	"Every Range Element Is Mapped to By Every (i - 1)-Level Exponent Sequence" on page 71
t	A finite tuple, i.e., a tuple in a tuple-set	"Tuple" on page 5

Table 7: Frequently Used Symbols and Terms in This Paper

Symbol	What Symbol or Term Typically (or Always) Denotes	Formal Definition
t_{nc}	A (finite) non-counterexample tuple	
t_c	A (finite) counterexample tuple	
$\bar{t}$	An infinite tuple that is defined in the "upward" direction, i.e., a tuple $\bar{t} = <x, y, y', y'', ... >$	"On Infinite Tuples" on page 66
$\bar{t}_{nc}$	An infinite non-counterexample tuple that is defined in the "upward" direction	"On Infinite Tuples" on page 66
$\bar{t}_c$	An infinite counterexample tuple that is defined in the "upward" direction	"On Infinite Tuples" on page 66
$\bar{t}(i)$	The *i*-level prefix of the infinite tuple *t*	
$\underline{t}$	An infinite tuple that is defined in the "downward" direction, i.e., a tuple $\underline{t} = < ..., y'', y', y>$	
T_A	The tuple-set defined by the (finite) exponent sequence *A*	"Tuple" on page 5
tuple	The result of a sequence of consequtive iterations of the 3*x* + 1 function *C*(*x*).	"Tuple" on page 5
tuple-set	For a given *i*-level exponent sequence *A*, the set of all tuples that generate all 1-level, 2-level, ..., *i*-level prefixes of *A*	"Tuple" on page 5
$\{a_2, a_3, ..., a_i\}$	An *i*-level exponent sequence	See "*A*"
$<x, y, y', y'', ..., y''^{...'}>$	A finite tuple *t* defined in the "upward" direction.	"Tuple" on page 5
$<x, y, y', y'', ... >$	An infinite tuple $\bar{t}$ defined in the "upward" direction	"On Infinite Tuples" on page 66

Table 7: Frequently Used Symbols and Terms in This Paper

Symbol	What Symbol or Term Typically (or Always) Denotes	Formal Definition
*	Concatenation; thus, $A*A'$ denotes the concatenation of the exponent sequence A' onto the right-hand end of the exponent sequence A; $t*t'$ denotes the concatenation of the tuple t' onto the right-hand or upward end of t.	"Some Infinite Inverse Exponent Sequences That are Not Generated by Any Range Element, y" on page 35

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