Are We Near a Solution to the 3*x* + 1 Problem?

A Discussion of Several Possible Strategies

by

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"Very often in mathematics the crucial problem is to recognize and to discover what are the relevant concepts; once this is accomplished the job may be more than half done."[1]

"One of the greatest contributions a mathematician can make is to spot something so simple and powerful that everybody else has missed it."[2]

Note 1: *A* ***proof of the 3x + 1 Conjecture*** *is given in our paper, "A Solution to the 3x + 1 Problem" on occampress.com. That paper contains improved versions of some sections of this paper.*

Note 2 : This paper is being revised based on readers' comments and the author's further thought. If you disagree with, or have questions about, any part of it, you are encouraged to contact the author and, in any case, to re-visit the paper in a week or so.

Note 3: Readers can safely assume, *initially*, that all referenced lemmas are true, since their proofs have been checked and deemed correct by several mathematicians.

Key words: 3*x* + 1 Problem, 3*n* + 1 Problem, Syracuse Problem, Ulam's Problem, Collatz Conjecture, computational number theory, proof of termination of programs, recursive function theory

1. Herstein, I. N., *Topics in Algebra*, John Wiley & Sons, N.Y., 1975, p. 50.
2. Stewart, Ian, *The Problems of Mathematics*, Oxford University Press, N.Y., 1992, pp. 279-280.

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Abstract

We present several possible strategies for solving the $3x + 1$ Problem. The Problem asks if repeated iterations of the function $C(x) = (3x + 1)/(2^a)$ always terminate in 1. Here x is an odd, positive integer, and a is the largest positive integer such that the denominator divides the numerator. The $3x + 1$ Conjecture asserts that repeated iterations always do terminate in 1. The strategies are based on two structures underlying C: *tuple-sets*, which is the structure of the function in the "forward" direction, and *recursive "spiral"s*, which is the structure of the function in the "backward" or inverse direction.

Tuple-sets are a partition of the set of all finite sequences of iterations of C , each sequence being represented by a *tuple*. If a tuple is associated with the sequence $A = \{a_2, a_3, ..., a_i\}$ of exponents of 2, where $i \geq 2$, that is, if the tuple is generated by the sequence, then the tuple is an element of the tuple-set T_A. Each i-level tuple-set, where $i \geq 2$, has exactly one first i-level tuple, which is called the *anchor tuple*. The difference between the values of elements of successive tuples in each tuple-set is given by a set of simple functions called the *distance* functions ("Lemma 1.0" on page 12).

We show that if counterexamples exist, each tuple-set contains an infinity of counterexample tuples and an infinity of non-counterexample tuples ("Lemma 5.0" on page 16). We also show that each range element of C, including, for example, 1, is mapped to by every finite exponent sequence ("Lemma 18.0: Statement and Proof" on page 93).

Recursive "spiral"s are a representation of the function C in the "backward" or inverse direction. A fact that arises from our investigation of this structure is that exactly one set, J, of odd, positive integers maps to 1 regardless if counterexamples exist or not ("Lemma 8.8" on page 32).

These results then give rise to several possible strategies.

Introduction

Statement of Problem

For x an odd, positive integer, set

$$C(x) = \frac{3x+1}{2^{ord_2(3x+1)}}$$

where $ord_2(3x + 1)$ is the largest exponent of 2 such that the denominator divides the numerator. Thus, for example, C(17) = 13, C(13) = 5, C(5) = 1. The $3x + 1$ Problem, also known as the $3n$ + 1 Problem, the Syracuse Problem, Ulam's Problem, the Collatz Conjecture, Kakutani's Problem, and Hasse's Algorithm, asks if repeated iterations of $C(x)$ always terminate at 1. The conjecture that they do is hereafter called the $3x + 1$ Conjecture. We call C the $3x + 1$ function; note that $C(x)$ is by definition odd.

An odd, positive integer x such that repeated iterations of $C(x)$ terminate in 1, we call a *non-counterexample*. An odd, positive integer such that repeated iterations of $C(x)$ never terminate in 1, we call a *counterexample*.

Other equivalent formulations of the $3x + 1$ problem are given in the literature; we base our formulation on the C function (following Crandall) because, as we shall see, it brings out certain structures that are not otherwise evident.

Summary of Research on the Problem

As stated in [Lagarias 1985], "The exact origin of the $3x + 1$ problem is obscure. It has circulated by word of mouth in the mathematical community for many years. The problem is traditionally credited to Lothar Collatz, at the University of Hamburg. In his student days in the 1930's, stimulated by the lectures of Edmund Landau, Oskar Petron, and Issai Schur, he became interested in number-theoretic functions. His interest in graph theory led him to the idea of representing such number-theoretic functions as directed graphs, and questions about the structure of such graphs are tied to the behavior of iterates of such functions... In the last ten years [that is, 1975-1985] the problem has forsaken its underground existence by appearing in various forms as a problem in books and journals..."

As far as we have been able to determine, our approach to a solution of the Problem via the two structures, tuple-sets and recursive "spiral"s, is original.

Summary of Solution Strategies

A summary of solution strategies is given below under "Strategies to Prove the 3x + 1 Conjecture" on page 41. .

In Memoriam

Several of the most important lemmas in this paper were originally conjectured by the author and then proved by the late Michael O'Neill. He made a major contribution to this research, and is sorely missed.

Why Is the 3*x* + 1 Problem So Difficult?

At the time of this writing (May, 2016) the $3x + 1$ Problem is about 85 years old. Some of the world's best mathematicians have tackled it, including Paul Erdös, who remarked, "Mathematics is not yet ready for problems of this difficulty." We know of at least one veteran researcher who discourages graduate students from working on the Problem because "it is a waste of time".

We believe that one reason the Problem is so difficult is that (informally) the structure of counterexamples to the $3x + 1$ Conjecture, and the structure of non-counterexamples, are so similar. For example, the inverse of each range element y of the $3x + 1$ function, be that range element a counterexample or a non-counterexample, is an infinitary tree with y as root. (See "Recursive "Spiral"s: The Structure of the 3x + 1 Function in the "Backward", or Inverse, Direction" on page 26.). Furthermore, all the properties of these trees that we are aware of, are the same regardless whether the root is a counterexample or a non-counterexample.

Other results that we have obtained likewise apply to both counterexamples and non-counterexamples. Among these are Lemmas 1.0, 5.0, 6.0, 7.0, 11.0, 12.0, 13.0, 15.0, 18.0. Many, if not most, of the results in the literature seem to us equally applicable to both counterexamples and non-counterexamples. We have come to believe that, at the very least, future results about the $3x + 1$ function should be accompanied by clear statements as to whether the results apply to both types of integer.

A related reason why the Problem is so difficult is that the structure of the $3x + 1$ function is apparently the same as the structure of other functions in which counterexamples are known to exist. These functions include the $3x - 1$, $3x + 5$, and $3x + 13$ functions. (See "Appendix C — "3x + 1 - like" Functions" on page 99.)

Not to be overlooked is the fact that the $3x + 1$ Problem is what we might call a *global problem*, unlike, for example, the problem of finding a proof of Fermat's Last Theorem (FLT), which we might call a *local problem*. Here is what we mean. Given the expression $x^k + y^k - z^k$, where, x, y, z, k are specific positive integers, and $k \geq 3$, we can decide via a simple calculation if it represents a counterexample to FLT — if the expression = 0, then it is a counterexample. If not, it isn't. On the other hand, if we are given an odd, positive integer x, and are asked if it is a counterexample to the $3x + 1$ Conjecture, we cannot tell, unless (1) we have the $3x + 1$ function perform a calculation that may not halt — either because x is in fact a counterexample, or because, although it is a non-counterexample, our computing resources may be exhausted before the computation ends, or (2) unless we know from a prior calculation that it is a non-counterexample, or (3) an existing lemma states that it is a non-counterexample, So we say that FLT is a *local problem,* whereas the $3x + 1$ Problem is a *global problem*.

Tuple-Sets: The Structure of the 3x + 1 Function in the "Forward" Direction

In the first part of this paper, we describe a structure called *tuple-sets* that underlies all finite sequences of iterations of the $3x + 1$ function, C. We have placed virtually all definitions in this first part of the paper because the terms defined are used repeatedly in the lemmas and proofs given later.

A tuple-set can be briefly, and informally, described as follows. (A formal definition is given under "Tuple-set" on page 8.) Consider the sequence of two iterations of C: $C(17) = 13$ (via the exponent 2 in the definition of C) followed by $C(13) = 5$ (via the exponent 3 in the definition of C). This sequence of iterations can be represented by the tuple <17, 13, 5>. The tuple-set T_A defined by the 2-level exponent sequence $A = \{2, 3\}$ contains the tuple <17, 13, 5>. But in addition it contains all other tuples that are determined by the exponent sequence $\{2\}$ but not by $\{2, 3\}$ — in other words, all other tuples that are determined by "approximations" to, or prefixes of, A. For example, the tuples <33, 25> and <81, 61, 23> are in T_A, because <33, 25> is associated with the exponent sequence $\{2\}$ but 25 does not map to another odd positive integer via the exponent 3, and <81, 61, 23> is associated with the exponent sequence $\{2, 3\}$.

We then show that each i-level tuple-set, where $i \geq 2$, has a unique first i-level tuple (called an *anchor* tuple) that (like all tuples) must be either a non-counterexample tuple or a counterexample tuple, but cannot be both.

We now proceed with our definitions.

Iteration

An *iteration* takes an odd, positive integer, x, to another odd, positive integer, y, via one application of the $3x + 1$ function, C. Thus, in one iteration C takes 17 to 13 because $C(17) = 13$.

Trajectory

A *trajectory* (sometimes called an *orbit*) is a sequence of one or more successive iterations of C, that is, if the sequence is finite,

$$(C^k(x))_{k \geq 0} = (x, C(x), C^2(x), \ldots C^k(x))$$

or, if the sequence is infinite,

$$(C^{\infty}(x)) = (x, C(x), C^2(x), \ldots)$$

The last element of the finite sequence need not be 1 and it need not be an infinity of successive 1's in the case of an infinite sequence.

A trajectory or orbit is the same as a *tuple*, which is defined below.

Non-Counterexample and Counterexample

If x is the first element of an infinite tuple <x, ..., 1, 1, 1, ... > , then x is called a *non-counterexample*. Otherwise, x is called a *counterexample*. Thus, a counterexample never yields 1 under repeated iterations of the $3x + 1$ function.

Exponent

If $C(x) = y$, with $y = (3x + 1)/2^a$, we say that *x maps under iteration to y* (or *x maps directly to y*) *via the exponent a*, and that *a is the exponent associated with x*. By abuse of language, we sometimes speak of a as *mapping directly to y.* We sometimes omit the word *directly* when context makes clear that it is implied. The sequence $\{a_2, a_3, ..., a_i\}$, where $a_2, a_3, ..., a_i$ are the exponents associated with x, $C(x)$, ..., $C^{(i - 1)}(x)$ respectively, is called an *admissible vector* in (Wirsching 1998). We call the sequence an *exponent sequence*. We define the function $e(x)$ to be the exponent associated with x. We sometimes refer to y as a *range element*. It is easily shown that y cannot be a multiple of 3 (see "Lemma 10.0: Statement and Proof" on page 87). An element x of the domain of the $3x + 1$ function, whether multiple of 3 or not, we sometimes refer to as a *domain element*.

I Clearly, an exponent is a positive integer.

Tuple

A *tuple* is a sequence of one or more successive iterations of C, that is, if the sequence is finite,

$$(C^k(x))_{k \geq 0} = (x, C(x), C^2(x), \ldots C^k(x))$$

or, if the sequence is infinite,

$$(C^{\infty}(x)) = (x, C(x), C^2(x), \ldots)$$

A finite sequence is not required to end with a 1, and an infinite sequence is not required to end with an infinity of successive 1's. If an infinite sequence does not end with an infinity of successive 1's, then it consists of counterexamples to the $3x + 1$ Conjecture.

A finite tuple is denoted[1] <$x, y, y', ..., y^{(n)}$>. We say that *x maps to $y^{(n)}$*. For example, <5, 1> and <11, 17, 13> are finite tuples. An infinite tuple, which represents an infinite trajectory, is denoted <x, y, y', ... >. For example, <5, 1, 1, 1, ... > and <11, 17, 13, 5, 1, 1, 1, ... > are infinite tuples.

Let t = <$x, y, y', ..., y^{(n)}$> be a finite tuple. Then the tuple t' = <$x, y, y', ..., y^{(n)}, y^{(n+1)}$> is an *extension* of t. An extension of an extension of t we likewise call an extension of t, etc. By definition of the function C, every finite tuple has an infinite number of extensions. In the case of a sequence of iterations of C that eventually yield 1, the corresponding infinite tuple is <x, y, y', ..., 1, 1, 1, ...>. A tuple consisting of an infinite number of extensions is an *infinite tuple*. We denote an infinite tuple by $\bar{t}$.

1. In a tuple, "$x^{(n)}$", "$y^{(n)}$", etc., denotes x with n primes, y with n primes, etc.

Clearly, since the domain of *C* consists of the odd, positive integers, every odd, positive integer is the first element of an infinite tuple.

If $\overline{t}$ is an infinite tuple, we denote the first *i* levels of $\overline{t}$ (that is, the first *i* elements of $\overline{t}$), by $\overline{t}(i)$, and we call $\overline{t}(i)$ a *prefix* of $\overline{t}$. For example, if $\overline{t}$ = <17, 13, 5, 1, 1, 1, ... >, then $\overline{t}(1)$ = 17, and $\overline{t}(4)$ = <17, 13, 5, 1>. Thus every finite tuple is a prefix of an infinite tuple and every prefix of an infinite tuple is a finite tuple. The term *tuple* standing alone, without the qualifier "infinite", denotes a finite tuple, that is, the prefix of an infinite tuple, unless context clearly indicates the reference is to an infinite tuple.

In the literature on the 3*x* + 1 Problem, tuples are sometimes called "trajectories" or "orbits".

Each tuple element except, possibly, the first, is an odd, positive integer that is not a multiple of 3. The element is odd by definition of the 3*x* + 1 function, *C*, and is not a multiple of 3 by "Lemma 10.0: Statement and Proof" on page 87.

Exponent Sequence Associated With a Tuple

As we established under "Exponent" on page 7, associated with every non-empty finite sequence of iterations of the function *C* — hence with every tuple — is an exponent sequence. We speak of the exponent sequence *associated with* a finite tuple. If *t* is a tuple, then we denote the exponent sequence associated with *t* by *A*(*t*). Thus, for example, if *t* = <17, 13, 5, 1> then *A*(*t*) = {2, 3, 4} because 17 maps directly to 13 via the exponent 2, 13 maps directly to 5 via the exponent 3, and 5 maps directly to 1 via the exponent 4.

Extension of an Exponent Sequence

Let $A = \{a_2, a_3, ..., a_i\}$ be a finite sequence of exponents, where $i \geq 2$. Then an exponent sequence $A' = \{a_2, a_3, ..., a_i, a_{i+1}\}$ is an *extension* of *A*. An extension of *A*′ is also an extension of *A*, etc.

Tuple-set

(The reader might find it helpful to refer to Fig. 1 in this sub-section while reading the following.)

Let $A = \{a_2, a_3, ..., a_i\}$ be a finite sequence of exponents, where $i \geq 2$. The *tuple-set* T_A consists of all and only the following tuples:

all tuples <*x*> such that *x* does not map to an odd, positive integer via a_2;

all tuples <*x*, *y*> such that *x* maps to *y* via a_2 (that is, $e(x) = a_2$) but *y* does not map to an odd, positive integer via a_3;

all tuples <*x*, *y*, *y*′> such that *x* maps to *y* via a_2 (that is, $e(x) = a_2$) and *y* maps to *y*′ via a_3 (that is, $e(y) = a_3$) , but *y*′ does not map to an odd, positive integer via a_4;

...

all tuples <$x, y, y', ..., y^{(i-3)}, y^{(i-2)}$> such that x maps to y via a_2 (that is, $e(x) = a_2$) and y maps to y' via a_3 (that is, $e(y) = a_3$) and ... and $y^{(i-3)}$ maps to $y^{(i-2)}$ via the exponent a_i (that is, $e(y^{(i-3)}) = a_i$). (The longest tuple in an i-level tuple-set has i elements.)

In other words, for each i-level exponent sequence A:

there are tuples <x> whose associated exponent sequence is a prefix of A for no exponent of A, and
there are other tuples <x, y> whose associated exponent sequence is a prefix of A for the first exponent of A, and
there are other tuples <x, y, y'> whose associated exponent sequence is a prefix of A for the first two exponents of A, and
...
there are other tuples <$x, y, z, ..., y^{(i-2)}$> whose associated exponent sequence is a prefix of A for all $i - 1$ exponents of A.

Tuples are ordered in the natural way by their first elements.

The set of first elements of all tuples in a tuple-set is the set of odd, positive integers (see proof under "The Structure of Tuple-sets" on page 9). Thus, there is a countable infinity of tuples in each tuple-set.

For each $i \geq 2$, tuple-sets are a *partition* of the set of all i-level tuples.

The Structure of Tuple-sets

It is important for the reader to understand that the structure of each tuple-set is unchanged by the presence or absence of counterexample tuples. Regardless if counterexample tuples exist or not, the set of first elements of all tuples in each tuple-set is always the same, namely, the set of odd, positive integers. *Proof:* Let x be any odd, positive integer and let $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, be any exponent sequence. Then there are exactly two possibilities:

(1) x maps to a y in a single iteration of the $3x + 1$ function, C, via the exponent a_2, or
(2) x does not map to a y in a single iteration of C via the exponent a_2.

But if (1) is true, then a tuple containing at least two elements, with x as the first, is in T_A; if (2) is true, then the tuple <x> is in T_A. There is no third possibility. □

For each tuple-set, the first element of the first tuple is 1, the first element of the second tuple is 3, the first element of the third tuple is 5, etc.

It can never be the case that, if counterexample tuples exist, then somehow there are "more" tuples in a tuple-set than if there are no counterexample tuples[1].

Furthermore, the distance functions defined in "Lemma 1.0" on page 12 are the same regardless if counterexample tuples exist or not.

1. To make this statement more precise: in no tuple-set does there ever exist a first element of a tuple, regardless how large that first element is, such that there are more tuples in that tuple-set having smaller first elements if counterexamples exist, than if counterexamples do not exist.

If $A = \{a_2, a_3, ..., a_i\}$ is a finite exponent sequence, then of an i-level tuple t in the tuple-set T_A, we say that *t is generated by* the exponent sequence A and that A is *associated with t*. Finally, we say that the tuple-set T_A *is generated by* the sequence A. To review: for each tuple, we speak of the exponent sequence *associated with* it; for each exponent sequence, we speak of the tuple or tuple-set it *generates*.

As an example of (part of) a tuple-set: in Fig. 1, where $A = \{a_2, a_3, a_4\} = \{1, 1, 2\}$ and where we adopt the convention of orienting tuples vertically on the page, the tuple-set T_A includes:

the tuple <1>, because $e(1) \neq a_2$;
the tuple <3, 5> , because $e(3) = a_2 = 1$, but $e(5) = 4 \neq a_3 = 1$;
the tuple <15, 23, 35>, because $e(15) = a_2 = 1$, and $e(23) = a_3 = 1$, but $e(35) = 1 \neq a_4 = 2$.

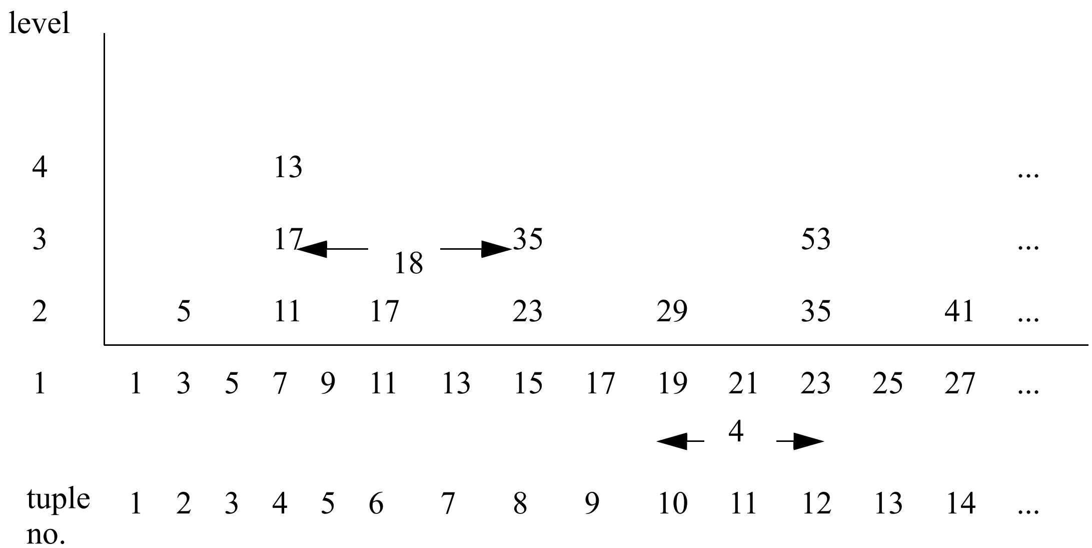

Fig. 1. Part of the tuple-set T_A associated with the sequence $A = \{1, 1, 2\}$

The numbers 18 and 4 between the arrows are values of the distance functions established by Lemma 1.0 (see "Lemma 1.0" on page 12).

In each i-level tuple-set T_A, where $i \geq 2$, for each odd, positive integer x there exists a tuple whose first element is x. The tuple may be one-level (<x>), or two-level (<x, y>), or ... or i-level (<$x, y, y', ..., y^{(i-3)}, y^{(i-2)}$>). Thus each tuple-set is non-empty.

Lemma 4.0 (see "Lemma 4.0: Statement and Proof" on page 84) establishes that a tuple-set T_A exists for each exponent sequence A.

Note: *our proofs will almost always involve only i-level tuples in i-level tuple-sets. We have included j-level tuples, where $2 \leq j \leq i$, in our definition of tuple-set because we feel that these tuples are necessary to fully describe the structure of tuple-sets. If the level of a tuple in an i-level tuple-set is not specified, the reader should assume that the tuple is i-level.*

Ordering of Tuples in a Tuple-set

Tuples in a tuple-set T_A are linearly ordered by the natural order of their first elements. We denote a specific tuple in a tuple-set by $t_{(r)}$, where $r \geq 1$. If T_A is an i-level tuple-set, where $i \geq 2$, we denote the jth element of $t_{(r)}$ (if it exists in T_A) by $t_{(r)(j)}$, where $1 \leq j \leq i$.

The reader may find it helpful to imagine an i-level tuple-set, where $i \geq 2$, as a "picket fence" infinite to the right, with the tuples serving as the pickets, as suggested by Fig. 1 under "Tuple-set" on page 8.

Level in a Tuple-set

A *level j* in a tuple-set is defined as follows. If $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, is a finite sequence of exponents, the subscript j in a_j, $2 \leq j \leq i$, denotes the *level j* in the sequence, that is, in the tuple-set T_A. Subscripts of exponents in an exponent sequence are numbered beginning with 2 instead of with 1 so that the last subscript then indicates the number of levels in the corresponding tuple-set. Thus, for example, if $A = \{a_2\}$, then T_A is a 2-level tuple-set; if $A = \{a_2, a_3\}$, T_A is a 3-level tuple-set, etc. Level 1 is then the level containing the set of all possible tuple first elements {1, 3, 5, 7, ... } in T_A, that is, the set of odd, positive integers. Thus, for example in the tuple <17, 13, 5, 1>, 17 is at level 1, 13 is at level 2, 5 is at level 3, and 1 is at level 4. We denote the element at level j in the nth tuple in a i-level tuple-set, where $i \geq 2$, by $t_{(n)(j)}$, where $1 \leq j \leq i$. (The element at level j is the jth element in the tuple.)

If a tuple has an element at level j, but none at level $j + 1$, we refer to the tuple as a *j-level tuple*. If the tuple also has an element at level $j + 1$, we sometimes refer to the tuple as a ($\geq j$)-level tuple. The longest tuple in a tuple-set generated by an i-level exponent sequence is an i-level tuple.

In the case that $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, we refer to A as an *i-level exponent sequence*. An i-level exponent sequence consists of (i - 1) exponents.

Tuples Consecutive at Level j

Tuples *consecutive at level j*, $j \geq 2$, are defined as follows. Let t_k, t_n be ($\geq j$)-tuples in some i-level T_A, where $i \geq 2$. If there is no ($\geq j$)-tuple between t_k and t_n, we say that t_k and t_n are *tuples consecutive at level j*. Here, "between" means relative to the natural linear ordering of tuples based on their first elements.

Thus, for example, in Fig. 1, the tuples numbered 4 and 8 are consecutive at level 3.

Extension of a Tuple-set

Let T_A be a tuple-set, where $A = \{a_2, a_3, ..., a_i\}$. Then a tuple-set $T_{A'}$, where $A' = \{a_2, a_3, ..., a_i, a_{i+1}\}$ is an *extension* of T_A. A proof that there exists such an extension for each exponent a_{i+1} is given in Lemma 3.0 (see "Lemma 3.0: Statement and Proof" on page 82).

Tuple-sets and Infinite Tuples

Tuples in a tuple-set are oriented vertically in accordance with our convention (see "Tuple-set" on page 8). Each tuple is a prefix of an infinite tuple (see "Tuple" on page 7). Therefore the infinite tuples whose prefixes constitute the finite tuples in a tuple-set, are likewise oriented vertically.

The infinite tuples having prefixes in a tuple-set thus occupy a single, vertical plane P_A that is infinite in the upward direction and to the right.

If T_A is an i-level tuple-set, where $i \geq 2$, then *each tuple-set that is an extension of* T_A *is contained, as a set of prefixes, in the set of infinite tuples whose i-level prefixes constitute the tuples in* T_A. Putting it another way, each tuple-set that is an extension of T_A — each tuple in each such tuple-set — is contained in the single, vertical plane P_A .

Tuple-sets: Infinities of Arbitrarily Long Tuples

What one might call the *grandeur* of the 3*x* + 1 function is represented by the fact that, for each arbitrarily long but finite sequence of positive integers (exponents) there exists a tuple-set containing a countable infinity of 1-level tuples, plus a countable infinity of 2-level tuples, plus ...,, plus a countable infinity of i-level tuples, where $i - 1$ is the number of exponents in the sequence.

Thus, for example, given an exponent sequence of length, say, 10,000,000,000, there nevertheless exists a countable infinity of 10,000,000,001-level tuples in the tuple-set defined by that exponent sequence, in addition to a countable infinity of each shorter tuple. Furthermore, as we shall prove in "Lemma 5.0" on page 16, if counterexamples exist, there also exists a countable infinity of counterexample tuples in the same tuple-set.

Distance Functions on Tuple-sets

Lemma 1.0

(a) Let $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, *be a sequence of exponents, and let* t_k, t_n *be tuples consecutive at level i in* T_A. *Then d(i, i), the distance between* t_k *and* t_n *at level i, is defined to be the absolute value of the difference between the level i elements of* t_k *and* t_n*, that is, it is defined to be* $|t_{k(i)} - t_{n(i)}|$, *and is given by*:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) Let t_k, t_n *be tuples consecutive at level i in* T_A. *Then d(1, i), the distance between* t_k *and* t_n *at level* 1*, is defined to be the absolute value of the difference between the level 1 elements of* t_k *and* t_n*, that is, it is defined to be* $|t_{k(1)} - t_{n(1)}|$*, and is given by:*

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3})\dots(2^{a_i})$$

Thus, in Fig. 1 under "Tuple-set" on page 8, the distance $d(3, 3)$ between $t_{8(3)} = 35$ and $t_{4(3)} = 17$ is $2 \cdot 3^{(3-1)} = 18$. The distance $d(1, 2)$ between $t_{12(1)} = 23$ and $t_{10(1)} = 19$ is $2 \cdot 2^1 = 4$.

Proof: see "Lemma 1.0: Statement and Proof" on page 77.

Remarks About the Distance Functions

(1) Strictly speaking, we should include the sequence A of exponents as arguments of $d(1, i)$, $d(i, i)$, but this notation would be cumbersome and, since typically this sequence is known, unnecessary.

(2) The distance functions make clear that, for each finite sequence of exponents, there exists an infinity of tuples produced by that sequence. (The equivalent of this statement is made in [Wirsching 1998] (p. 48).) The following table shows the distance relationships for $(i - j)$-level elements of tuples consecutive at level $(i - j)$ in an i-level tuple-set, where $0 \le j \le (i - 1)$. The distances are easily proved using Lemma 1.0. (An example is given following the table.) We only use the distances at levels 1 and i in this paper.

Table 1: Distances between elements of tuples consecutive at level *i*

Level	**Distance between $(i - j)$-level elements of tuples consecutive at level $(i - j)$, where $0 \le j \le (i - 1)$**
i	$2 \cdot 3^{i-1}$
$i - 1$	$2 \cdot 3^{i-2} \cdot 2^{a_i}$
$i - 2$	$2 \cdot 3^{i-3} \cdot 2^{a_{i-1}}2^{a_i}$
$i - 3$	$2 \cdot 3^{i-4} \cdot 2^{a_{i-2}}2^{a_{i-1}}2^{a_i}$
...	...
2	$2 \cdot 3 \cdot 2^{a_3} \ldots 2^{a_{i-1}}2^{a_i}$
1	$2 \cdot 2^{a_2}2^{a_3} \ldots 2^{a_{i-1}}\ 2^{a_i}$

For example, consider the tuple-set T_A, where $A = \{2, 1, 1\}$. The first two 3-level tuples in T_A are <9, 7, 1> and <25, 19, 29>. There is no 2-level tuple between these. Now19 is an $i - 1 = 3 - 1 =$ level 2 element. Our table tells us that the distance from the previous level 2 element, namely, 7, to 19 is given by

$$2 \bullet 3^{(i-2)} \bullet 2^{a_i} = 2 \bullet 3^{(3-2)} \bullet 2^1 = 12$$

and indeed 7 + 12 = 19.

(3) Lemma 1.0 makes clear that no two i-level tuples in an i-level tuple-set have the same last element. In fact, the values of the last elements of i-level tuples in an i-level tuple-set always increase as one proceeds along the sequence of i-level tuples.

(4) For each $i \ge 2$, the set of all i-level elements of all i-level tuple-sets is the set of all odd, positive integers mod $2 \bullet 3^{i-1}$. That is, each i-level element is an element of a reduced residue class mod $2 \bullet 3^{i-1}$. (A reduced residue class is one having no multiples of 2 or multiples of 3.)

There are $2 \bullet 3^{i-2}$ such classes. If we think of the positive integers mod $2 \bullet 3^{i-1}$ in accordance with our "lines-and-circles" model[1], then the first three levels (circles) become the first

1. See Part (4) of the paper, "Is There a 'Simple' Proof of Fermat's Last Theorem?" , on occampress.com.

level (circle) mod $2 \cdot 3^{(i-1)+1}$, the second three levels (circles) become the second level (circle) mod $2 \cdot 3^{(i-1)+1}$, etc.

We now state the two lemmas that are required for our proof that tuple-sets exist as defined.

Every Possible 2-Level Tuple-set Exists

Lemma 2.0

For each exponent a_2, a tuple-set T_A, where $A = \{a_2\}$, exists.

Proof: See "Lemma 2.0: Statement and Proof" on page 82.

Every Possible Extension of Each *i*-Level Tuple-set Exists

Lemma 3.0

Each i-level tuple-set T_A, where $A = \{a_2, a_3, ..., a_i\}$ and $i \geq 2$, has an extension via each odd or even exponent a_{i+1},

Proof: See "Lemma 3.0: Statement and Proof" on page 82.

How Tuple-sets "Work"

Each *i*-level tuple-set, where $i \geq 2$, can be extended by any positive integer, *m* (Lemma 3.0). For each *m*, there is a countable infinity of *i*-level tuples in the tuple-set that are extended by *m*. If *m* is that by which the first *i*-level tuple is extended, then the extended tuple remains the first (i + 1) - level tuple in the resulting (i + 1) - level tuple-set. If not, then the first tuple in the (i + 1) level tuple-set is the first one, in the linear ordering of i-level tuples in the *i*-level tuple-set, that is extended by *m*. An infinite tuple results from infinite extensions of a tuple, each of which establishes the tuple-set that the tuple is in. The "distance" between (i + 1)-level elements in successive (i + 1)-level tuples in each (i + 1)-level tuple-set, is $2 \cdot 3^{i+1-1}$ (part (a) of Lemma 1.0).

Proof That Tuple-sets Exist as Defined

Lemma 4.0

For each exponent sequence $A = \{a_2, a_3, a_4, ..., a_i\}$, where $i \geq 2$, there exists a tuple-set T_A.

Proof: See "Lemma 4.0: Statement and Proof" on page 84.

Lemmas 2.0, 3.0 and 4.0 establish, as part of their proofs, that there are an infinite number of tuples in each tuple-set. A plausible question at this point is: Why should there be? The answer is given in the next section.

On the Number of Tuple-sets

Lemma 4.5

(a) For each $i \geq 2$, the number of i-level tuple-sets is countably infinite.
(b) The number of all tuple-sets is countably infinite.

Proof of (a): See "Lemma 4.5: Statement and Proof" on page 84.
Proof of (b): A countable infinity of countable infinities is a countable infinity. □

On the Set of All i-Level Elements of All i-Level Tuple-sets

Lemma 4.75

For each $i \geq 2$, the set of all i-level elements of all i-level tuples in all i-level tuple-sets is the set of all range elements of the 3x + 1 function.

Proof: See "Lemma 4.75: Statement and Proof" on page 84.

A Recursive Description of Any Tuple-set

Let $\boldsymbol{x}$ denote the set of odd, positive integers. Let $\boldsymbol{y} = \mathbf{C}\{a_2 \bmod 2 \cdot 3^{(1-1)}\}(\boldsymbol{x})$ denote the set of range elements of the $3x + 1$ function produced by the exponent $a_2 \bmod 2 \cdot 3^{(1-1)}$ operating on all the elements of $\boldsymbol{x}$. As we know from Lemma 1.0, $\boldsymbol{y}$ is one of two sets, namely, the set of all $y \equiv 1 \bmod 2 \cdot 3^{(1-1)}$ (if a_2 is even) or the set of all $y \equiv 5 \bmod 2 \cdot 3^{(1-1)}$ (if a_2 is odd).

We can repeat the process recursively, so that, if $A = \{a_2, a_3, ..., a_i\}$, then

(1)
$$T_A = \mathbf{C}\{a_i \bmod 2 \cdot 3^{((i-1)-1)}(... \mathbf{C}\{a_3 \bmod 2 \cdot 3^{(2-1)}\}(\mathbf{C}\{a_2 \bmod 2 \cdot 3^{(1-1)}\}(\boldsymbol{x}))...).$$

The reason that this is a recursive description of the tuple-set T_A is that it is precisely the sequence of tuple-set extensions,

$$T_{\{a_2\}}, T_{\{a_2, a_3\}}, T_{\{a_2, a_3, a_4\}}, \ldots, T_{\{a_2, a_3, a_4, \ldots a_i\}}$$

The reason we only need to consider the indicated finite set of exponents at each level is established by Lemmas 7.0 and 7.1 in the first part of the second file of the paper, "The Structure of the 3x + 1 Function: An Introduction" on the web site occampress.com.

We remind the reader that if $\boldsymbol{y}''^{\cdots}''$ is a set mapped to by $\mathbf{C}\{a_i...\}(\boldsymbol{y}''^{\cdots}')$, then we know by "Lemma 1.0" on page 12 that $\boldsymbol{y}''^{\cdots}''$ is a reduced residue class mod $2 \cdot 3^{((i+1)-1)}$.

Equation (1) describes the behavior of the $3x + 1$ function over its entire domain, namely, the set of all odd, positive integers, regardless if counterexamples exist or not.

Why There Are An Infinite Number of Tuples in Each Tuple-set

Every finite exponent sequence — that is, every finite sequence of positive integers — generates an i-level tuple-set ("Lemma 4.0: Statement and Proof" on page 84), where $i \geq 2$. The last element (that is, the i-level element) of each tuple maps directly to one and only one odd, positive integer via one and only one exponent. Consider the tuple-set T_A generated by the exponent sequence $A = \{a_2, a_3, a_4, ..., a_i\}$ where $i \geq 2$. T_A has an extension for *each* positive integer a_{i+1} ("Lemma 3.0: Statement and Proof" on page 82). But since the last element of each tuple in T_A maps directly to one and only one odd positive integer, and since by Lemma 2.0 (see "Lemma 3.0: Statement and Proof" on page 82) each tuple-set $T_{A'}$, $A' = \{a_0, a_1, a_2, ..., a_i, a_{i+1}\}$, likewise has an extension for each positive integer a_{i+2}, etc., it follows that, for *each* a_i, there exists an *infinity* of tuples in T_A whose last elements directly map to their respective odd, positive integers *via* a_i. In short, the reason there are an infinite number of tuples in each *i*-level tuple-set is that (1) each *i*-level tuple-set has an infinity of extensions, namely, one for each exponent a_{i+1}, but (2) each tuple maps directly to one and only one odd, positive integer via one and only one exponent.

Thus, in each *i*-level tuple-set T_A, where $i \geq 2$, the countable infinity of *i*-level non-counterexample tuples consists of:

an infinity that have an extension via the exponent 1, and
an infinity that have an extension via the exponent 2, and
an infinity that have an extension via the exponent 3, and
...
If counterexamples exist, the same is true for *i*-level counterexample tuples.

The Merging of All Tuple-sets into a Single Row of Tuples

For each odd, positive integer, an infinite tuple is generated by endlessly repeated iterations of the $3x + 1$ function. So we can order these infinite tuples by the natural order of their first elements. As with tuple-sets, we adopt the convention that the tuples are vertical relative to the horizontal axis containing the first elements.

For each *i*, where $i \geq 2$, and for each infinite tuple *t*, we connect, via a square bracket, the *i*-level element of *t* to the *i*-level element of the next infinite tuple t' such that the *i*-level prefixes of both tuples are associated with the same exponent sequence. The bracket is *not* meant to enclose a set of tuples, we merely want the two ends of the bracket, which are perpendicular to the *i*-level elements we have just described, to indicate that the two *i*-level prefixes are associated with the same exponent sequence.

Of course, we must make sure that the line parts of brackets are not on top of each other.

The result is a compression of all tuple-sets to a single row of (infinite) tuples.

.

On Non-Counterexample and Counterexample Tuples in a Tuple-set

Lemma 5.0

Assume a counterexample exists. Then for all i ≥ 2, each i-level tuple-set contains an infinity of i-level counterexample tuples and an infinity of i-level non-counterexample tuples.

Proof: see "Lemma 5.0: Statement and Proof" on page 85.

Remark 1

This lemma establishes that there is no way to distinguish counterexamples from non-counterexamples on the basis of the *finite exponent sequences* associated with each. Of course, if a non-trivial cycle exists, then an infinite tuple $<x_1, x_2, ..., x_1, x_2, ..., x_1, x_2, ... >$ exists, and thus the finite tuple $<x_1, x_2, ..., x_1>$ immediately tells us that a counterexample exists. But there is no requirement that a counterexample be the source of a non-trivial cycle. A counterexample can simply give rise to an infinite tuple in which no element recurs, and which has no element = 1.

To repeat: there is no way of telling from a *finite exponent sequence* that it is associated with a counterexample. For example, the sequence $\{a_2, a_3, ..., a_2, a_3, ..., a_2, a_3, ... \}$, in which $\{a_2, a_3, ..., a_2\}$ is repeated, say, a trillion times, does not imply the existence of a counterexample cycle.

Remark 2

Lemma 5.0 implies that the set of all i-level non-counterexample tuples, where $i \geq 2$, is associated with the set of all i-level exponent sequences and, if counterexamples exist, then the set of all i-level counterexample tuples is likewise associated with the set of all i-level exponent sequences.

Lemma 9.7

(a) If counterexamples do not exist, then for all i-level tuple-sets $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, if x is the first element of an i-level (necessarily non-counterexample) tuple in T_A, then the first element of the next i-level (necessarily non-counterexample) tuple is

(1)

$$(x + (2 \cdot (2^{a_2})(2^{a_3})\ldots(2^{a_i})))$$

(b) If counterexamples exist, then in each i-level tuple-set $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, there exists an x which is the first element of an i-level non-counterexample tuple in T_A such that the first element of the next i-level non-counterexample tuple in T_A is greater than the value in (1).

Proof:

Part (a) follows directly from part (b) of the distance function lemma, namely,"Lemma 1.0" on page 12. Part (b) follows from the fact that, if counterexamples exist, then, by "Lemma 5.0" on page 16, each tuple-set contains an infinity of counterexample tuples and an infinity of non-counterexample tuples. Hence there must exist at least one non-counterexample tuple that is followed by at least one counterexample tuple. Hence the distance to the next non-counterexample tuple is greater than (1).

Remark: The Lemma shows that, informally, if counterexamples exist, non-counterexamples are "farther apart" from each other than if counterexamples do not exist.

Effect of the Existence of Counterexamples on the Set of All Tuple-sets

Consider the set of all tuple-sets in the two cases that (1) there are no counterexamples and (2) that counterexamples exist. It is natural to say that there are "no differences", because if x is an element of a tuple, then x maps to a certain y in one iteration of the $3x + 1$ function, and this y is the same whether or not counterexamples exist. If (contrary to fact) counterexamples and only counterexamples were negative numbers, then the set of all tuple-sets if no counterexamnples existed would be different (no negative numbers in tuples) from the set of all sets of tuple-sets if counterexamples existed (negative numbers in some tuples).

As we have pointed out on several occasions in this paper, there is no way of distinguishing counterexamples locally, meaning, by examining a single odd, positive integer, or even by examining the odd, positive integers produced by several iterations of the $3x + 1$ function (it is known that the shortest cycle, if a cycle exists, would be thousands of elements long).

When we consider the structure of the inverse of the $3x + 1$ function (see "Section 2. Recursive 'Spiral's", in the first file of our paper "The Structure of the 3x + 1 Function: an Introduction" (www.occampress.com)) we see that there definitely *is* a difference in this structure if counterexamples exist as opposed to if counterexamples do not exist. Specifically, if counterexamples do not exist, then there is no infinite set of "spiral"s whose set of elements is disjoint from the set of elements in the infinite set of "spiral"s having base element 1. If counterexamples exist, on the other hand, then there exists at least one infinite set of "spiral"s whose set of elements is disjoint from the set of elements in the infinite set of "spiral"s having base element 1.

For a long time, we did not realize that exactly the same kind of difference holds for the set of all tuple-sets, namely, that if counterexamples do not exist, then the elements of all tuples in all tuple-sets are connected in the sense that for each element in each tuple, we can proceed through extensions of that tuple until we arrive at 1, and then from 1 we can proceed "backwards" through some other tuple until we arrive at any pre-selected element in another tuple.

If counterexamples exist, this is not possible. In that case, we can partition the set of tuples in the set of all tuple-sets into a set of (partial) tuple-sets whose tuples contain only non-counterexamples, and one or more other (partial) tuple-sets whose tuples contain only counterexamples.i

Infinite Exponent Sequences <u>Not</u> Associated With Counterexamples

Lemma 5.5.

Let a be a finite exponent sequence such that if x maps to y via a, then y > x. Then there does not exist a counterexample x such that the infinite tuple <x, ... > is associated with the exponent sequence {a, a, a, ... }.

Proof: See "Lemma 5.5: Statement and Proof" on page 85.

Lemma 5.6

No 3x + 1 infinite counterexample tuple can be associated with the same exponent sequence as the negative of the 3x – 1 infinite counterexample tuple.

Proof: We can extend each $3x + 1$ tuple-set into the odd, negative integers. The result is the negative of the $3x – 1$ function. Since each infinite tuple would have a fixed first element, they would reach a length that would cause a violation of the Distance Function defined in part (b) of "Lemma 1.0" on page 12 .□

Examples

Examples of infinitely-repeating exponent sequences in the odd, negative integers are {1, 1, ... }, {1, 2, 1, 2, ..., } and {1, 1, 1, 2, 1, 1, 4,1, 1, 1, 2, 1, 1, 4,...}.

For details, see Lemma 1.5 in the first part of our paper, "The Structure of the $3x + 1$ Function: An Introduction" on occampress.com.

Anchor and Anchor Tuple

Since tuples in a tuple-set are linearly ordered by the natural order of their first elements, in every i-level tuple-set, where $i \geq 2$, there is a unique first i-level tuple, which we call the *anchor tuple* of the tuple-set. The last element, that is, the i-level element, of the anchor tuple we call the *anchor* of the anchor tuple, sometimes referring to it as the *i-level anchor.*

Each anchor tuple element (like the elements of all tuples) is an odd, positive integer that is not a multiple of 3. The element is odd by definition of the $3x + 1$ function, C, and is not a multiple of 3 by "Lemma 10.0: Statement and Proof" on page 87.

Lemma 6.0

Let t be the i-level anchor tuple in an i-level tuple-set, where $i \geq 2$. *Then the last element y of t, that is, the i-level element of t (which is the anchor), is a number* less than $2 \cdot 3^{(i-1)}$.

Proof: see "Lemma 6.0: Statement and Proof" on page 86.

Definition of "Reduced Residue Class" and of "Complete Set of Reduced Residue Classes"

If a residue class mod m is such that each element of the class is relatively prime to m, then we call the class a *reduced residue class mod m*. Thus, for example, the residue class mod 6 whose minimum element is 5 is a reduced residue class mod 6. The set of all reduced residue classes mod m we call a *complete set of reduced residue classes mod m*.

Lemma 7.0

(a) For each i-level tuple-set T_A, *where* $A = \{a_2, a_3, ..., a_i\}$, *the set of all i-level elements of all i-level tuples is a reduced residue class mod* $2 \cdot 3^{(i-1)}$.

(b) The set of all such reduced residue classes, over all i-level tuple-sets T_A, *is a complete set of reduced residue classes mod* $2 \cdot 3^{(i-1)}$.

Proof: see "Lemma 7.0: Statement and Proof" on page 86.

Anchors and Reduced Residue Classes

For each $i \geq 2$, there are $2 \cdot 3^{i-2}$ reduced classes. If we think of the positive integers mod $2 \cdot 3^{i-1}$ in accordance with our "lines-and-circles" model[1], then the the first level (circle) consists of the set of all i-level anchors. (This level contains all range elements less than $2 \cdot 3^{i-1}$.) The first *three* levels (circles) become the *first* level (circle) mod $2 \cdot 3^{(i-1)+1}$ (that is, the set of all $(i + 1)$-level anchors), the second three levels (circles) become the second level (circle) mod $2 \cdot 3^{(i-1)+1}$, etc.

1. See Part (4) of the paper, "Is There a 'Simple' Proof of Fermat's Last Theorem?" , on occampress.com.

Mark

Lemma 8.0

For each odd, positive integer x there exists a minimum $i = i_0$ *such that for each* $i \geq i_0$*, x is the first element of the first i-level tuple in some i-level tuple-set, that is, x is the first element of an i-level anchor tuple in some i-level tuple-set. In terms of infinite tuples, this lemma states: if x is an odd, positive integer, then in the infinite tuple* $\bar{t} = <x, y, y', \ldots>$*, there exists a minimum level* i_0 *such that:*

$\bar{t}(i_0)$ is the i_0-level anchor tuple in an i_0-level tuple-set;
$\bar{t}(i_0 + 1)$ is the $(i_0 + 1)$-level anchor tuple in an $(i_0 + 1)$-level tuple-set;
$\bar{t}(i_0 + 2)$ is the $(i_0 + 2)$-level anchor tuple in an $(i_0 + 2)$-level tuple-set;
etc.

(Of course, the $(i_0 + k + 1)$-level tuple-set, where $k \geq 0$, must be an extension of the $(i_0 + k)$-level tuple-set by the same exponent by which the anchor tuple is extended.)

Proof: see "Lemma 8.0: Statement and Proof" on page 87.

Remark

To describe the infinite sequence of anchor tuples in the lemma, we sometimes say, informally, "Once an anchor tuple, always an anchor tuple".

Definition of "Mark"

We call the level i_0 in Lemma 8.0 the *mark* of the infinite tuple $\bar{t}$. We denote the mark i_0 by m. We write $m(\bar{t})$ to denote the mark of $\bar{t}$, and we write $\bar{t}(m)$ to denote the prefix (that is, finite tuple) corresponding to the mark m. This prefix is an anchor tuple.

For example, the mark of the infinite tuple <3, 5, 1, 1, 1, 1, ... > is at level 2 (namely, at 5) because 5 is the first element of the tuple that is less than $2 \bullet 3^{(i-1)}$ for some $i \geq 2$. Specifically, for $i = 2$, $2 \bullet 3^{(i-1)} = 6$, and $5 < 6$. As another example, consider the infinite tuple <433, 325, 61, 23, 35, ..., 1, 1, 1, 1, ... >. The mark is not at 325 (level 2) because for level 2, $2 \cdot 3^{(i-1)} = 6$ and 325 is not less than 6. The mark is not at 61 (level 3) because for level 3, $2 \cdot 3^{(i-1)} = 18$ and 61 is not less than 18. The mark *is* at 23 (level 4) because for level 4, $2 \cdot 3^{(i-1)} = 54$ and 23 *is* less than 54.

Infinite Tuples, Marks, and Tuple-sets

We here summarize the pertinent facts concerning infinite tuples, marks, and tuple-sets, because it is crucial that the reader understand these facts and their relationships.

By definition, an i-level tuple-set T_A, where $i \geq 2$, *includes* all i-level tuples t such that $A(t) = A$, that is, such that the exponent sequence associated with t is A. We emphasize *includes* because, by definition of "tuple-set", the tuple-set also includes 1-level, 2-level, 3-level, ..., $(i-1)$-level tuples (see "Tuple-set" on page 8). Another way of saying what we have just said regarding i-level tuples is: a tuple-set T_A, where $i \geq 2$, includes all prefixes $\bar{t}(i)$ of infinite tuples $\bar{t}$ such that $A(\bar{t}(i)) = A$. Thus by abuse of language we may say that a tuple-set consists of a set of infinite tuples.

At this point it is appropriate that we describe the relationship between *successive* prefixes of an infinite tuple $\bar{t}$ (counterexample or non-counterexample) and the tuple-sets in which the prefixes appear. Let $\overline{t}$ = <x_1, x_2, x_3, x_4, ... > and let {a_2, a_3, a_4, a_5, ... } be the associated exponents. That is,

x_1 maps to x_2 in one iteration of the 3*x* + 1 function via a_2;
x_2 maps to x_3 via one iteration of the 3*x* + 1 function via a_3;
etc.

Then, by definition of *tuple-set*:
in each tuple-set T_A determined by the exponent sequence $A = \{b_2, b_3, b_4, b_5, ... b_i\}$ such that $b_2 \neq a_2$, the tuple <x_1> is an element;
in each tuple-set T_A determined by the exponent sequence $A = \{b_2, b_3, b_4, b_5, ... b_i\}$ such that $b_2 = a_2$, but $b_3 \neq a_3$, the tuple <x_1, x_2> is an element;
in each tuple-set T_A determined by the exponent sequence $A = \{b_2, b_3, b_4, b_5, ... b_i\}$ such that $b_2 = a_2$, $b_3 = a_3$, but $b_4 \neq a_4$ the tuple <x_1, x_2, x_3> is an element;
...
in the one tuple-set T_A determined by the exponent sequence $A = \{b_2, b_3, b_4, b_5, ... b_i\}$such that $b_2 = a_2$, $b_3 = a_3$, $b_4 = a_4$, ..., $b_i = a_i$ the tuple <x_1, x_2, x_3, ..., x_i> is an element;

Let $\overline{t}$ be an infinite tuple. It has a mark, *m*. Each prefix $\overline{t}\,(m + j)$ of $\overline{t}$, where $j \geq 0$, is an anchor tuple. But then, by abuse of language, we allow ourselves to say that each prefix $\overline{t}\,(i)$, where $i \geq 2$, is *a prefix of an anchor tuple* (namely, the anchor tuple $\overline{t}\,(m + j)$). Thus each prefix $\overline{t}\,(i)$, where $i \geq 2$, is a prefix of an *infinity o*f anchor tuples.

Each infinite tuple $\overline{t}$ is an independent entity. By this we mean that an infinite tuple $\overline{t}$ is determined solely by its first element. Thus, informally, an infinite tuple does not somehow "acquire" properties depending on the tuple-set in which it has a prefix.

In an *i*-level tuple-set there is exactly one infinite tuple with a mark that is less than or equal to *i* , namely, the infinite tuple whose prefix is the anchor tuple. All other infinite tuples having *i*-level prefixes in the tuple-set must have marks greater than *i* (otherwise there would be two or more anchor tuples in a tuple-set, which is impossible). It may well be the case, however, that an $(i - j)$-level tuple (prefix), where $1 \leq j \leq (i - 1)$, in the tuple-set has a mark! The following is an example:

The infinite tuple $\overline{t}$ = <7, 11, 17, 13, 5, 1, 1, 1, ... > has its mark at level 3 (namely at 17) because 17 is the first element of the tuple that is less than $2 \cdot 3^{(i-1)}$ for some $i \geq 2$. Here, $i = 3$, so $2 \cdot 3^{(i-1)} = 18$, and 17 < 18. So <7, 11, 17> = $\overline{t}(3)$ is an anchor tuple: specifically, it is the anchor tuple of the tuple-set T_A, where $A = \{1, 1\}$ (7 maps to 11 via the exponent 1; 11 maps to 17 via the exponent 1). By our rule (see under "Mark" on page 20) expressed informally as "once an anchor tuple, always an anchor tuple", we know that <7, 11, 17, 13> = $\overline{t}(4)$ is also an anchor tuple: specifically, it is the anchor tuple of the 4-level tuple-set $T_{A'}$, where $A' = \{1, 1, 2\}$ (7 maps to 11 via the exponent 1, 11 maps to 17 via the exponent 1, 17 maps to 13 via the exponent 2).

But <7, 11, 17> = $\overline{t}(3)$ is also present in the 4-level tuple-set $T_{A''}$, where $A'' = \{1, 1, 1\}$ The reason is that, since 17 maps to 13 via the exponent 2, not via the exponent 1, the tuple <7, 11, 17> is associated with merely an "approximation", namely {1, 1}, to the exponent sequence {1, 1, 1}. But therefore, by definition of "tuple-set" (see under "Tuple-set" on page 8), it belongs in the tuple-set $T_{A''}$.

We conclude our preparation for a possible proof of the $3x + 1$ Conjecture with the definition of “sufficiently long extension of a tuple” and “sufficiently long extension of an exponent sequence”.

“Sufficiently Long” Extensions of Tuples and Exponent Sequences

“Bottom Up” Sufficiently Long Extensions

We begin with two definitions. First we recall that each infinite tuple has a mark m that denotes the smallest prefix of the tuple that is an anchor tuple (see “Mark” on page 20).

Definition of “Sufficiently Long” Extension of a Tuple

Definition: Let $\bar{t}$ be an infinite tuple with mark m. Let $\bar{t}(i)$ be a prefix of $\bar{t}$, where $i < m$. Then there exists an extension $\bar{t}(i + j)$ of $\bar{t}(i)$, where $m = i + j$. We say that $\bar{t}(i + j)$ is a *sufficiently long extension of* $\bar{t}(i)$ *that is an anchor tuple*. (All longer extensions are likewise anchor tuples, by our rule, “once an anchor tuple, always an anchor tuple”.)

It follows (trivially) that:

For each tuple (that is, for each prefix of an infinite tuple) there exists a sufficiently long extension of the tuple that is an anchor tuple.

Definition of a “Sufficiently Long” Extension of an Exponent Sequence

Definition: Let $\bar{t}$ be a *non-counterexample* infinite tuple with mark m. Let $\bar{t}(i)$ be a prefix of $\bar{t}$, where $i < m$. Let $A(\bar{t}(i))$ denote the exponent sequence associated with $\bar{t}(i)$. Let the extension $\bar{t}(i + j)$ of $\bar{t}(i)$ be a sufficiently long extension of $\bar{t}(i)$ that is an anchor tuple. Then we say that $A(\bar{t}(i + j))$ *is an extension of* $A(\bar{t}(i))$ *that is sufficiently long to be associated with a non-counterexample anchor tuple*.

An Erroneous Objection to the Definition

Several readers have challenged the definition of a “sufficiently long” extension of an exponent sequence with the following argument. Let us imagine, they say, a “demon” who presents us with an i-level exponent sequence, A, where $i \geq 2$. The demon has before him all the non-counterexample infinite tuples $\bar{t}_{nc}$ having i-level prefixes that are associated with the exponent sequence A. In other words, he has before him all the non-counterexample infinite tuples $\bar{t}_{nc}$ whose prefixes constitute all the i-level non-counterexample tuples in the tuple-set T_A. He now proceeds to concatenate exponents onto A, taking care that, as soon as the resulting exponent sequence equals $A(\bar{t}_{nc} + (m - 1))$ for some infinite non-counterexample tuple $\bar{t}_{nc}$ whose mark is m, the next exponent in his sequence will make the resulting exponent sequence *not* equal to $A(\bar{t}_{nc}(m))$. He repeats this indefinitely. It is clear, then, that his exponent sequence will never be that of a non-counterexample anchor tuple.

The error in this objection is that the demon is not creating a sequence of exponent sequences that are associated with a sequence of extensions of a single tuple. Rather, he is in effect switching tuples in order to create his sequence of exponent sequences.

Complete Sets of Tuples

Definition of a "Complete" Set of Tuples

Let S be a set of i-level tuples, where $i \geq 2$. Then we say that S is *complete* if S is associated with the set of all i-level exponent sequences. Otherwise, we say that S is *incomplete*.

Lemma 8.5

Assume counterexamples exist. Let $\bar{t}_{nc}$, $\bar{t}_c$ be non-counterexample and counterexample infinite tuples, respectively, with marks m_{nc}, m_c respectively.
Then for all levels $i \geq max(m_{nc}, m_c) = i_0$, $A(\bar{t}_{nc}(i)) \neq A(\bar{t}_c(i))$, where max(u, v) denotes the maximum of u, v, and A(t) denotes the exponent sequence associated with the tuple t.

Proof: Assume the contrary. Then for some $i \geq i_0$, $A(\bar{t}_{nc}(i)) = A(\bar{t}_c(i))$, which implies that a tuple-set exists having both a non-counterexample and a counterexample anchor tuple, which is impossible. □

Lemma 8.7

If counterexamples do not exist, then
(a) *For each $i \geq 2$, the set of i-level non-counterexample anchor tuples is complete.*

(b) *Each non-counterexample infinite tuple has a prefix, namely, that determined by its mark, such that that prefix, and all larger prefixes, are elements of* ***complete*** *sets of non-counterexample anchor tuples.*

If counterexamples exist, then

(c) F*or each $i \geq$ some i_0, the set of i-level non-counterexample anchor tuples is incomplete, so that a complete set of i-level non-counterexample tuples must include tuples other than anchor tuples.*
(d) *Each non-counterexample infinite tuple has a prefix, namely, that determined by its mark, such that that prefix, and all larger prefixes, are elements of* ***incomplete*** *sets of non-counterexample anchor tuples.*

Proof

(a) Follows trivially from the fact that if counterexamples do not exist, all tuples in all tuple-sets are non-counterexample tuples.

(b) Follows trivially from the fact that the mark determines the smallest prefix of an infinite tuple that is an anchor tuple.

(c) By "Lemma 8.5" on page 23, if counterexamples exist, then for all $i \geq max(m_{nc}, m_c) = i_0$, there exist i-level exponent sequences with which i-level anchor tuples are not associated. These are the exponent sequences with which i-level *counterexample* anchor tuples are associated. But by "Lemma 5.0" on page 16, each i-level tuple-set, regardless whether the anchor tuple is non-counterexample or counterexample, contains an infinity of non-counterexample tuples and an infinity of counterexample tuples. Thus to obtain a complete set of i-level non-counterexample tuples, it is necessary to include a non-counterexample tuple from each tuple-set having a counterexample anchor tuple.

(d) Follows directly from "Lemma 8.5" on page 23. □

Challenging Questions About Anchor Tuples and Tuple-sets

Regardless of the success of the proof strategies described in this paper, and of the implementations of some of these strategies that are given in the paper, "A Solution to the $3x + 1$ Problem" on the website www.occampress.com, this research will not be completed until the questions described in this section are satisfactorily answered. They lie at the heart of the tantalizing difficulty of discovering valid proofs of the $3x + 1$ Conjecture.

Question 1: "Why Are There an Infinite Number of Tuples in Each Tuple-set?"

This question we believe has been satisfactorily answered in the section "Why There Are An Infinite Number of Tuples in Each Tuple-set" on page 15.

Question 2 "What Is the Difference Between Anchor Tuple Extensions and Others?"

This question arises from an error in one of our early attempts at a proof of the 3x + 1 Conjecture. We had made the following argument: if counterexamples exist, then beginning at some level $i_0 \geq 2$, there must be both non-counterexample and counterexample anchor tuples. But since for all $i \geq 2$ the set of all i-level anchor tuples must be associated with the set of all i-level exponent sequences, this means that some i-level exponent sequences, where $i \geq i_0$, will not be associated with non-counterexample anchor tuples (these exponent sequences will be "missing" from the set of exponent sequences associated with i-level non-counterexample anchor tuples), and similarly for counterexample anchor tuples. Furthermore this fact holds for all levels greater than i . But then, we argued, this contradicts "Lemma 5.0" on page 16, hence we have our proof.

Readers pointed out that Lemma 5.0 states that, if counterexamples exist, each tuple-*set* contains a countable infinity of non-counterexample and a countable infinity of counterexample tuples, so the "missing" exponent sequences are not really missing. An infinity of non-counterexample tuples are associated with them, and similarly for the "missing" exponent sequences for counterexample tuples.

So our question is: "What is the difference between the sequence of exponent sequences associated with the sequence of extensions of an anchor tuple, and the sequence of exponent sequences associated with other tuples in the corresponding sequence of tuple-set extensions?"

One answer is the following: the sequence of exponent sequences associated with the sequence of extensions of an anchor tuple are all associated with extensions of *one* tuple, namely, the anchor tuple. But the sequence of exponent sequences associated with other tuples in the corresponding sequence of tuple-set extensions are *not* all associated with extensions of one tuple. That is, in order for the tuples in a sequence of tuple-set extensions always to be associated with the sequence of anchor tuple extensions, it is necessary that some tuples "fall away" and that the remaining ones have the required extensions. In some of our papers, we refer to this phenomenon as the "pushing away" phenomenon, because tuples whose exponent sequence matches that of the anchor tuple, are always farther and farther away (as measured by the difference between first elements) from the anchor tuple.

Thus, an arbitrarily long exponent sequence can only be associated with the arbitrarily long extension of *one* anchor tuple, not with arbitrarily long extensions of more than one tuple (anchor or non-anchor).

Question 3: "What Would Happen If We Removed Just One Non-Counterexample Anchor Tuple from All Tuple-sets?"

In Question 2, we pointed out that, if counterexamples exist, then for all levels greater than or equal to some minimum level i_0, there will be both non-counterexample and counterexample anchor tuples. We would like to get a clearer understanding of the implications of this fact.

We begin with the case that counterexamples do not exist. We ask (and this is Question 3), "What would happen if we removed just one anchor tuple (necessarily a non-counterexample anchor tuple) from the set of all tuple-sets?" (We know from our discussion in Question 2 that at least one non-counterexample anchor tuple (in fact an infinity) would be removed from the set of all tuple-sets if counterexamples existed.)

To remove one non-counterexample anchor tuple is to remove one non-counterexample infinite tuple $\bar{t}_{nc}$. But since each element of each infinite tuple except possibly the first element is a range element, then by "Lemma 13.0: Statement and Proof" on page 89 each element is mapped to by an infinity of odd, positive integers, and so on, recursively. And indeed, as the reader can confirm by checking Fig. 4 in "Section 2. Recursive 'Spiral'" in the first file of our paper, "The Structure of the $3x + 1$ Function: An Introduction" on the web site www.occampress.com, it appears that if we remove just one non-counterexample infinite tuple, and all tuples having a last element that is a range element in $\bar{t}_{nc}$, and all tuples having a last element that is a range element in each of these tuples, and ..., *we remove all non-counterexample tuples,* because one of the elements in each non-counterexample infinite tuple is 1.

If the reader argues that we are not justified in removing all tuples having a last element that is a range element in $\bar{t}_{nc}$, then we must ask what becomes of these tuples if $\bar{t}_{nc}$ is replaced by a counterexample infinite tuple?

Of course, if, in fact, the removal of just one non-counterexample anchor tuple would constitute the removal of all non-counterexample anchor tuples, then it would seem that we have a proof of the $3x + 1$ Conjecture, since the removal of all those tuples would contradict "Lemma 5.0" on page 16.

Another way of answering our question is this: if counterexamples exist, there are nevertheless non-counterexample infinite tuples having prefixes that are anchor tuples. Each non-counterexample ultimately contains 1. Therefore the set of all odd, positive integers that map to 1 must be present, eventually, as anchors. But this is precisely the case if no counterexamples exist. In short, it does not seem possible for there to be counterexample anchor tuples.

Question 4: "Why, In the 3x– 1 Function, Is the Set of All Non-Counterexample Anchor Tuples Inomplete, When This Is Not the Case in the 3x + 1 Function?"

This question is simple but so far tantalizingly difficult to answer. It is: "Why, in the $3x - 1$ function, for all levels $i \geq 2$, is the set of all i-level non-counterexample anchor tuples incomplete, whereas for at least the first 35 levels of the $3x + 1$ function, the set of non-counterexample anchor tuples at each of these levels is complete?"

An answer to this question would be an inductive proof of the $3x + 1$ Conjecture based on the completeness of the set of non-counterexample anchor tuples for each of these first 35 levels.

Question 5: "What Is the Relationship Between Tuple-sets and Recursive 'Spiral's?"

See "Relating Tuple-sets and Recursive "Spiral"s" on page 39.

Recursive "Spiral"s: The Structure of the 3x + 1 Function in the "Backward", or Inverse, Direction

Recursive "spiral"s are a graphical description of the inverse of the $3x + 1$ function.

The 1-Tree

We define a tree called the *1-tree* as follows (see Fig. 4 below):

The set of all odd, positive integers that map to 1 in a *single* iteration of the $3x + 1$ function is Level 1 in the tree; This set is {1, 5, 21, 85, 341, ... }.

The set of all odd, positive integers that map to 1 in *two* iterations of the $3x + 1$ function is Level 2 in the tree;

The set of all odd, positive integers that map to 1 in *three* iterations of the $3x + 1$ function is Level 3 in the tree;

...

Thus the 1-tree contains all odd, positive integers that map to 1 in a finite number of iterations of the function.

The tree is not strictly an infinitary tree, because if a node is a not a range element of the $3x + 1$ function, that is, if a node is a multiple-of-3, then it has no descending nodes (no odd, positive integers map to a multiple-of-3).

We call the set of odd, positive integers that map to a range element y in one iteration of the function, a *recursive "spiral"*, or just a *"spiral"* for short. Thus , for example, {1, 5, 21, 85, 341, ... } is a "spiral". It maps to 1 in one iteration of the 3x + 1 function. It is easily shown that the elements of a "spiral" map to y either by all odd exponents, or by all even exponents (which is the case with {1, 5, 21, 85, ... }). It is likewise easily shown that if x is an element of a "spiral", then the next "spiral" element is $4x + 1$.

Each node in the 1-tree is a "spiral" element. From now on, we will use the term *"spiral" element*.

Each "spiral" element y that is a range element of the $3x + 1$ function is the root of a y-tree. The definition of the y-tree is the same as that of the 1-tree, with y replacing 1.

The odd, positive integers between successive "spiral" elements we call an *interval* in the "spiral". Thus, for example, in the "spiral" {1, 5, 21, 85, 341, ... }, 3 is the only element in the first interval; 7, 9, 11, 13, 15, 17, 19, are the elements of the second interval, etc.

If x is an element of a "spiral", then $4x + 1$ is the next element; thus {odd, positive integers y | y maps to 1 in *one* iteration of the $3x + 1$ function} = {1, 5, 21, 85, 341, ... }.

The "spiral" contains a countable infinity of multiples of 3. These cannot be range elements of the $3x + 1$ function (by "Lemma 10.0: Statement and Proof" on page 87), that is, cannot be mapped to;

The "spiral" also contains a countable infinity of range elements of the function: each in turn is mapped to by another "spiral", which yields, recursively, the set of odd, positive integers that map to 1 in two, three, four, ... iterations.

It is therefore clear that no odd, positive integer can be added to or removed from a "spiral". Hence the 1-tree is unique, regardless whether counterexamples exist or not.

If a counterexample exists, then it is an element of an interval, but it is never a "spiral" element. The tree or trees generated by counterexamples are of course separate (disjoint) from the 1-

tree. There are "spiral"s, "spiral" elements, and intervals in counterexample trees just as there are in the 1-tree.

Let y denote a "spiral" element in {1, 5, 21, 85, ... } that is a range element. Then if x is a non-counterexample, it must be an element of a y-tree.

So, to summarize: In the case of the 1-tree, all "spiral" elements are non-counterexamples. Some interval elements we know are non-counterexamples. If a counterexample exists, then it is an element of an interval, but never of a "spiral" in the 1-tree. In the case of a counterexample tree, all spiral elements are counterexamples. Some interval elements are counterexamples, others are non-counterexamples.

There is exactly one 1-tree. This means that an odd, positive integer maps to 1 regardless if counterexamples exist or not. Thus, for example, 13(an element of the 5-tree) maps to 1 (in two iterations of the function) today. If a proposed proof of the $3x + 1$ Conjecture is accepted as valid tomorrow, 13 will still map to 1. And if the Conjecture is proved false tomorrow, 13 will *still* map to 1. (A proof that there is only one 1-tree is the proof of "Lemma 8.8" on page 32.

Understanding the 1-Tree Relative to Successive Non-Counterexamples Beginning With 1

1. It is known by computer test[1] that all successive odd, positive integers less than approximately $5.76 \cdot 10^{18}$ are non-counterexamples. Call this set W. So let us consider the "spiral" $S =$ {1, 5, 21, 85, ... }, all of whose elements map to 1 in one iteration of the $3x + 1$ function, and proceed as follows.

2. Obviously, 1 maps to 1, and so we color 1 green in S.

Moving to the next odd, positive integer to the right of 1,which is 3, we can easily find that 3 maps to 1. So we color 3 green om the first interval of S.

Moving to the next odd, positive integer to the right of 3, which is 5, we know that 5 maps to 1, because all elements of S map to 1. So we color 5 green in S. Since 5 is a range element, a "spiral" maps to it in one iteration of the function. That "spiral" is {3, 13, 53, ... }. We color all elements of the "spiral" green in intervals of S. We also color all other elements in the 5-tree green. We then color green in intervals of S, all elements of the 5-tree.

Moving to the next odd, positive integer to the right of 5, which is 7, we proceed similarly.

Etc.

If we come to an odd, positive integer that is already green (because it is a member of an earlier y-tree) we move to the next odd, positive integer on the right.

3. If all odd, positive integers that we encounter (including those that are greater than the largest odd, positive integer in W (see step 1), are either already green, or are elements of S, then we will have a proof of the $3x + 1$ Conjecture if the number of successive green integers is greater

---

1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers to at least $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$, which is greater than $3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35 - 1)} - 1$, have been tested and found to be non-counterexamples.

than or equal to the minimum number *G* of successive green integers that guarantee that all subsequent odd, positive integers are green. At present, as far as we know, *G* is not known.

The reader is urged to contemplate the fact that the portion of the 1-tree (including intervals) that is green is *exactly* what the portion of the 1-tree looks like if counterexamples do not exist — or if they do exist (see previous sub-section regarding fact that there is exactly one 1-tree).

One fact that seems promising for a proof of the $3x + 1$ Conjecture is that each successive element of each 'spiral" is an element of an infinity of successive intervals in the "spiral" $S = \{1, 5, 21, 85, 341, ... \}$. Thus, consider the elements of the "spiral" {3, 13, 53, 213, ... }.

If we can prove that there exists an infinity of intervals — no matter how far apart — that contain solely non-counterexamples, we will have a proof of the Conjecture, because that will mean there are intervals that cannot contain a counterexample, hence counterexample "spiral"s cannot exist.

Many relevant results and ideas will be found in our paper, "Are We Near a Solution to the $3x + 1$ Problem?", on occampress.com, in the sub-section, " 'Spiral's, Intervals and Levels" of the section, "Strategy of 'Filling-in' of Intervals in the Base Sequence Relative to 1".

In the case of the 1-tree for the $3x – 1$ function, the first counterexample, 5, occurs in the first interval in the "spiral" {1, 3, 11, 43, ... } that maps to 1 in one iteration of the $3x – 1$ function — namely, in the interval whose elements are 5, 7, 9. (The element 7 is also a counterexample, because <5, 7, 5, ... > is an infinite cycle that never yields 1.)

Graphical Representation of the Set *J* as Recursive "Spiral"s

The set *J* (which we sometimes refer to as the *1-tree*) is an infinite set of recursive "spiral"s whose base element is 1. These infinite sets are defined in "Section 2. Recursive 'Spiral's" in the first file of the paper "The Structure of the $3x + 1$ Function: An Introduction" on the web site www.occampress.com. The following is a diagram of part of *J* :

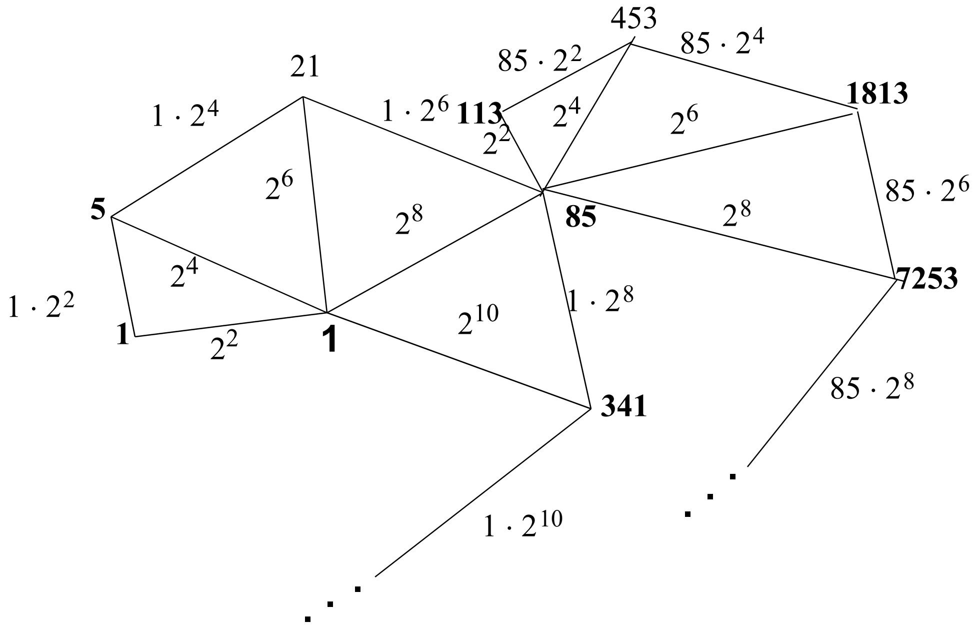

Fig. 4. Recursive "spirals" structure of odd, positive integers that map to 1.

Bold-faced numbers are range elements (21 and 453 are multiples of 3, hence not range elements). Partial "spirals" surrounding the base elements 1 and 85 are shown. The line connecting 1813 to 85 is marked with a 2^6 because $(3 \cdot 1813 + 1)/ 2^6 = 85$. The line connecting 453 to 1813 is marked $85 \cdot 2^4$ because $453 + 85 \cdot 2^4 = 1813$. The quantity $85 \cdot 2^4 = 3 \cdot 453 + 1$, and similarly for the difference between successive elements of a "spiral" in all "spiral"s. These facts follow from the fact that if x, y are consecutive elements of a "spiral", with $x < y$, then $y = 4x + 1$.

The exponents of 2 are not even in all "spiral"s, of course. For example, the "spiral" of numbers (not shown) mapping to 341 has odd exponents.

Partial List of Facts Pertaining to the 1-Tree

It is a major challenge to find a proof of the $3x + 1$ Conjecture based on the 1-tree. We believe we have made good progress toward that goal (see Possible Strategies in Appendix C of our paper, "A Solution to the 3x + 1 Problem" on occampress.com. In this section, we set forth a list of facts that may enable the creative mathematician to improve our strategies, or discover additional ones.

- There is exactly one 1-tree, whether or not counterexamples exist (Lemma 8.8 in our paper, "Are We Near a Solution to the 3x + 1 Problem?", on occampress.com). The question is, Does the 1-tree contain all odd, positive integers (in which case the Conjecture is true) or does it contain only a proper subset of the odd, positive integers (in which case the Conjecture is false).

- If x is an element of a "spiral", then $4x + 1$ is the next element.

Each "spiral" maps to a range element y in one iteration of the $3x + 1$ function either by all odd exponents, or by all even exponents. Thus, for example, in the "spiral" {1, 5, 21, 85, ... }, 1 is mapped to by all even exponents, 5 is mapped to by all odd exponents, 21 is not mapped to because it is a multiple-of-3, 85 is mapped to by all even exponents, ...

- The successive elements of a "spiral" are mapped to in accordance with a rule that can be expressed as ... 2, 1, 3, 2, 1, 3, ..., where "2" means "is mapped to by all even exponents", "1" means "is mapped to by all odd exponents", and "3" means "is not mapped to because element is a multiple-of-x, hence not a range element. (See proof under "Properties of the 1-Tree" in our paper, "A Solution to the 3x + 1 Problem", on occampress.com.)

- The elements of each "spiral" other than *S*, occupy an infinity of successive intervals in *S*, one element per interval. Thus, consider the "spiral" {3, 13, 53, 213, ... }.

- Each "spiral" element *x*, and the range element *y* it maps to in one iteration of the $3x + 1$ function, is a tuple $<x, y>$ in a tuple-set. Since the set of elements of a "spiral" map to their range element either by all odd exponents or by all even exponents, it is easy to see that the set of all elements of all "spiral"s in the 1-tree is the set of all first elements of all non-counterexample 2-tuples in all 2-level tuple-sets.

The set of all (finite) upward paths (that is, paths in the direction of the root, 1, of the 1-tree, or away from 1 in the upward direction) is the set of all non-counterexample tuples in the set of all tuple-sets.

The following table shows the first three 2-tuples in the 2-level tuple-sets T_A for $A = \{1\}, \{3\}$, and $\{5\}$. (The reader might find it more natural to imagine the exponent 1 tuple-set as being foremost, with the exponent 3 tuple-set parallel and directly behind the exponent 1 tuple-set, and the exponent 5 tuple-set as parallel and directly behind the exponent 3 tuple-set, etc.

Observe the first three elements of the "spiral" that maps to 5 in one iteration of the $3x + 1$ function, namely, the elements 3, 13, 53 running vertically.

Observe the first three elements of the "spiral" that maps to 11 in one iteration of the $3x + 1$ function, namely, the elements 7, 29, 113 running vertically.

Observe the first three elements of the "spiral" that maps to 17 in one iteration of the $3x + 1$ function, namely, the elements 11, 45, 181 running vertically.

In keeping with the rule governing successive elements of "spiral"s, namely, that if *x* is a "spiral" element, then $4x + 1$ is the next element, we observe that $3(4) + 1 = 13$, etc.; $4(7) + 1 = 29$, etc., and $11(4) + 1 = 45$, etc.

But observe also, in keeping with "Lemma 1.0" on page 12,

the sequence of level-2 elements in each 2-level tuple-set is 5, 6, 11, ...

The difference between level-1 elements in the tuple-set T_A where $A = \{1\}$ is $2(2^1)$,

The difference between level-1 elements in the tuple-set T_A where $A = \{3\}$ is $2(2^3)$,

The difference between level-1 elements in the tuple-set T_A where $A = \{1\}$ is $2(2^5)$,

Table 2: Partial View of Relationship Between 2-Level Tuple-sets and Odd-Exponent "Spiral"s

Exponent				
5	5	11	17	...
	53	117	181	...
3	5	11	17	...
	13	29	45	...
1	5	11	17	...
	3	7	11	...

Similar 2-level tuple-sets exist for even exponents, that is, for 2-level tuple-sets T_A where A = {2}, {4}, {6},,, .

Whether a proof of the 3x + 1 Conjecture can be derived from these numerical relationships remains to be seen. However, the following may offer a basis for a proof.

1. We must remember that there is exactly one 1-tree, whether or not counterexamples exist. If the set of all "spiral" elements in the 1-tree is the set of all odd, positive integers, then counterexamples do not exist. Otherwise, counterexamples exist.

2. If a counterexample exists, it is the first element of a 2-tuple in a 2-level tuple-set. However, by what we have said regarding the table above, this means that the counterexample is an element of a counterexample "spiral". The "spiral" maps to a range-element y of the $3x + 1$ function in a single iteration of the function, and therefore y is the root of a y-tree. The y-tree contains an infinity of counterexample "spiral"s.

If a counterexample exists, there is an infinity of counterexamples ("Lemma 5.0" on page 16, and the fact that, by an item in the above list (the one concerning the ... 2, 1, 3, 2, 1, 3, ..., pattern in "spiral" elements) we know that there is an infinity of counterexample range-elements. Hence there is an infinity of y-trees, where y is a counterexample range element).

3. There are no counterexample tuples in the set of all tuple-sets containing only non-counterexample tuples.

It is easily shown that in order for there to be an infinity of tuples associated with each finite exponent sequence there must be an infinity of non-counterexample tuples in each tuple-set that are extended by the exponent 1, and an infinity that are extended by the exponent 2, etc. (See "Extension of a Tuple-set" on page 11.) However, this property cannot survive the omission of a countable infinity of "spiral"s.

It seems, therefore, that we must conclude that the existence of a counterexample results in the failure of non-counterexample tuples to be associated with *each* finite exponent sequence, contradicting "Lemma 5.0" on page 16.

If this reasoning can be made precise and valid, we have a proof of the $3x + 1$ Conjecture.

(*Note*: we must reconcile the omission of "spiral"s from tuple-sets, with the fact that the 1-tree is the same, whether or not counterexamples exist.)

Counterexample Trees

Unlike the 1-tree, no counterexample tree has a single root like 1, from which all other subordinate trees are descended. Consider, for example, the tree containing the minimum counterexample, y_c, which maps to an infinity of successive tuple elements, none of which can be less than y_c. Each such element that is a range element, is the root of a tree that has the same properties as the tree having a non-counterexample range element as its root. These properties include the existence of the "spiral" containing all odd, positive integers that map, in one iteration of the $3x + 1$ function, to the range element that is the root. If x, y are successive elements of the "spiral", and $y > x$, then $y = 4x + 1$.

Lemma 8.8

Motivation

The odd, positive integer 13 maps to 1, as the reader can verify. We ask: if the $3x + 1$ Conjecture were proved false tomorrow, would 13 map to 1 thereafter? We reply yes. Let y be any odd, positve integer that is known to map to 1. We ask: if the $3x + 1$ Conjecture were proved false tomorrow, would y map to 1 thereafter? Again we reply yes. So it seem plausible that exactly one set J of odd, positive integers maps to 1, regardless whether counterexamples exist or not. This is the gist of Lemma 8.8 It is certainly a counter-intuitive statement, but not, we believe, a false one . (One reason that some readers regard Lemma 8.8 as false seems to be that they confuse the statement of this Lemma with the statement, "The *range* of the $3x + 1$ function is the same regardless whether counterexamples exist or not.." Now this statement is clearly false, because if counterexamples do not exist, then the range of the $3x + 1$ function is $\{1\}$. If they do exist, then the range is a larger set that contains 1.)

Lemma 8.8 means that the $3x + 1$ Conjecture can be expressed as: Are there any odd, positive integers besides those that map to 1? If the answer is yes, then counterexamples exist. If the answer is no, then counterexamples do not exist. The following should make the matter even clearer:

(1) There is exactly one set, J, of odd, positive integers that map to 1, regardless whether counterexamples exist or not.

(2) Let S_1 denote the singleton set containing the set of all odd, positive integers. Let S_2 denote the set containing all proper subsets of the odd, positive integers. Then if counterexamples do not exist, $J \in S_1$; if counterexamples exist, then $J \in S_{2.}$

We can express in a similar way the question whether there are any odd perfect numbers. Let P denote the set of perfect numbers. (These are numbers that are equal to the sum of their proper factors. Thus, for example, $6 = 3 + 2 + 1$ is a perfect number, as is $28 = 14 + 7 + 4 + 2 + 1$.) Let

P_E denote the subset of P consisting of even perfect numbers, and let P_O denote the subset of P consisting of odd perfect numbers. Then the question, Do odd, perfect numbers exist? (the answer is not yet known) can be expressed as, Are there any perfect numbers besides those that are in P_E? If the answer is yes, then odd, perfect numbers exist. If the answer is no, then odd, perfect numbers do not exist. In either case, observe that the following statement is true: There is exactly one set of even, perfect numbers, regardless whether odd, perfect numbers exist or not.

(The equivalent of the $3x + 1$ function in the perfect number case is a function f that, for the positive integer n, returns "yes" if n is a perfect number, "no" otherwise. It is sufficient if the program that implements f does so by simply determining the proper factors of n, then adding them and determining if the result is n. Clearly, f cannot be Euler's well-known formula for even perfect numbers, $2^{k-1}(2^k - 1)$, where $2^k - 1$ is a Mersenne prime, because the formula returns only even perfect numbers.)

Lemma 8.8: Statement and Proof

Exactly one set J of odd, positive integers maps to 1, whether or not counterexamples exist.

Proof:

The set J =

{odd, positive integers y | y maps to 1 in *one* iteration of the $3x + 1$ function} ∪
{odd, positive integers y | y maps to 1 in *two* iterations of the $3x + 1$ function} ∪
{odd, positive integers y | y maps to 1 in *three* iterations of the $3x + 1$ function} ∪
...

The set of odd, positive integers that map to 1 in one iteration of the $3x + 1$ function is {1, 5, 21, 85, 341, ...}. This set is called a "spiral" in "Section 2. Recursive 'Spiral's" in the first file of the paper "The Structure of the 3x + 1 Function: An Introduction" on the web site www.occampress.com. In that Section it is shown that:

If x is an element of the "spiral", then $4x + 1$ is the next element;

The "spiral" contains a countable infinity of multiples of 3. These cannot be range elements of the $3x + 1$ function (by "Lemma 10.0: Statement and Proof" on page 87), that is, cannot be mapped to;

The "spiral" also contains a countable infinity of range elements of the function: each in turn is mapped to by another "spiral", which yields, recursively, the set of odd, positive integers that map to 1 in two, three, four, ... iterations.

It is therefore clear that no odd, positive integer can be added to or removed from a "spiral". Hence the set J is unique, regardless whether counterexamples exist or not. .□

Definition of "Fixed-Set"

We call the set J the *Fixed-Set* because it is the set of all odd, positive integers each of whose values (namely, 1), under the $3x + 1$ function, is the same regardless if counterexamples exist or not. Thus, for example, 13 maps to 1 today, and it will map to 1 if the $3x + 1$ Conjecture is proved true tomorrow, and it will *still* map to 1 if the Conjecture is proved false tomorrow. (Clearly, no counterexample can be an element of the Fixed-Set.) We will at times speak of proper sub-sets of the Fixed-Set, and, by abuse of language, the tuples of which they are elements. Thus, for example, in a specified tuple-set, the set of all tuples whose first elements are in the set of consecutive odd positive integers, beginning with 1, that are known to map to 1 — this set of tuples we will say is in the Fixed-Set. At the time of this writing, all consecutive odd, positive integers beginning with 1 and less than a *quadrillion* (10^{15}) are known by computer test to be non-counterexamples.

Ways of Understanding the Meaning of Lemma 8.8

Readers who have difficulty believing that Lemma 8.8 is valid might be helped by considering that, e.g., 13 maps to 1 today, and if the Conjecture is proved true tomorrow, it will map to 1, and if the Conjecture is proved false tomorrow it will *still* map to 1. The same holds for each odd, positive integer that maps to 1.

The 1-tree described under "Recursive "Spiral"s: The Structure of the 3x + 1 Function in the "Backward", or Inverse, Direction" on page 26 is the tree of all odd, positive integers that map to 1. It should be clear that the set of all such integers is not affected by the existence or non-existence of counterexamples.

Another possible aid to readers' understanding is the following:

Let S_1 denote the set whose only element is the set of all odd, positive integers. Let S_2 denote the set containing all proper subsets of the odd, positive integers. Then if counterexamples do not exist, $J \in S_1$; if counterexamples exist, then $J \in S_2$.

We can think of the $3x + 1$ Problem as asking if J is an element of S_1 or of S_2, and the $3x + 1$ Conjecture as asserting that J is an element of S_1.

The following analogy might be of help to readers.

Assume there is a board with n holes, and a bag, J', containing $\leq n$ marbles. Each hole in the board can contain exactly one marble. The number of marbles in the bag J' is *fixed*. Then the equivalent of the question, "Do all odd, positive integers map to 1 via the $3x + 1$ function?" is, "Will all the marbles in the bag occupy all the holes in the board?"

How to Avoid Faulty Proofs Based on Lemma 8.8

It is all-too-easy to create faulty proofs based on Lemma 8.8. Here is how those proofs can arise.

Consider the $3x - 1$ Conjecture, which we know is false, 5 and 7 being the smallest counterexamples. Let us reason as follows.

1. Assume counterexamples exist. (They do.)

2. Then *J* must be a proper subset of the odd, positive integers. (It is.)

3. Now assume that counterexamples do not exist. By Lemma 8.8, *J* is still a proper subset of the odd, positive integers. But that is a contradiction, since if counterexamples do not exist, *J* must obviously be the entire set of odd, positive integers.

4. If we now infer from this contradiction that our initial assumption – that counterexamples exist – must have been wrong, we are clearly contradicting the fact that counterexamples do exist.

What the contradiction in step 3 tells us is that our assumption that counterexamples do not exist is false. That is all. And similarly for cases where we begin with the assumption that counterexamples do not exist when we do not know that for a fact.

Our error really arose from our violating the protocol of the Comparison Strategy (see "Description of the Comparison Strategy" on page 46). That protocol consists of three parts:

1. We begin with a line of reasoning that starts from the assumption that *p* is true. The line of rasoning must not contain any phrase that is equivalent to "if *p* is false". At a point of our choosing, we end the line of reasoning.

2. We begin with a line of reasoning that starts from the assumption that *not- p* is true. The line of reasoning must not contain any phrase that is equivalent to "if *not-p* is false". At a point of our choosing, we end the line of reasoning.

(It doesn't matter if we begin with step 2, and then proceed to step 1.)

3. We then compare (if possible) a statement in the first line of reasoning, with a statement in the second line of reasoning, that gives us a statement that yields our desired concluding statement.

In the case of the above proof errors, we began with *p* (counterexamples exist), but then, in step 3, we interjected "if *not-p*" (counterexamples do not exist) which is not allowed by the protocol.

It appears to us, on the basis of our experience, that the Comparison Strategy can only work if there is a Fixed-Set from which the reasoning can begin — that is, a set that is the same whether or not counterexamples exist.

Meaning of "Same" When Referring to the Set *J*

We occasionally say that the set *J* of odd, positive integers that are non-counterexamples is the *same* whether or not counterexamples exist. Our justification for using the word *same* in this context is given in the section, "Meaning of the Word "Same" When Applied to Sets in a Comparison" on page 47.

Statement of 3x + 1 Problem in Terms of the Set *J*

We see, therefore, that the $3x + 1$ Problem can be expressed as follows: a set J of odd, positive integers maps to 1, regardless whether counterexamples exist or not. Obviously, the set of odd, positive integers is the same, regardless whether counterexamples exist or not. So the question is: Are there any other odd, positive integers (namely, counterexamples) in the set of odd, positive integers besides those that map to 1? This is certainly a counter-intuitive expression of the Problem. Initially, at least, it is natural for us to assume that, if counterexamples exist, then some of the odd, positive integers that map to 1 if counterexamples do not exist, "become" counterexamples if counterexamples exist. But that is not correct.

We can express the $3x + 1$ Problem in terms of the set J as follows Let S_1 denote the singleton set containing the set of all odd, positive integers. Let S_2 denote the set containing all proper subsets of the odd, positive integers. Then the $3x + 1$ Problem asks whether J is an element of S_1 or of S_2.

Lemma 8.9: Statement and Proof

Each element of the 1-tree is the first element of an anchor tuple.

Proof:

Let y be an element of the 1-tree, where $y \neq 1$. Then y is the first element of a tuple t whose last element is 1. This tuple t is associated with an i-level exponent sequence $A = \{a_2, a_3, a_4, ..., a_i\}$, where $i \geq 2$. Since the last element of the tuple is 1, and since 1 is the anchor of each i-level tuple-set in which it occurs, t is the anchor tuple of the tuple-set T_A., and hence y is the first element of an anchor tuple. □

Corollary

Let y be the first element of a spiral s. Then y maps to 1 via an i-level exponent sequence $A = \{a_2, a_3, a_4, ..., a_i\}$, where $i \geq 2$. And hence successive elements y', y'', y''', ... of s map to 1 via the exponent sequences $A' = \{a_2', a_3, a_4, ..., a_i\}$, $A'' = \{a_2'', a_3, a_4, ..., a_i\}$, $A''' = \{a_2''', a_3, a_4, ..., a_i\}$, ..., respectively, where $a_2' = a_2 + 2$, $a_2'' = a_2' + 2$, $a_2''' = a_2'' + 2$, ... By Lemma 8.9, each of y', y'', y''', ... is the first element of an i-level anchor tuple. □

Proof:

Follows from Lemma 13.0 (see "Lemma 13.0: Statement and Proof" on page 89).

Remark 1

We call to the reader's attention "Lemma 18.0: Statement and Proof" on page 93. This Lemma states that for each range element y (counterexample or non-counterexample), each possible finite exponent sequence maps to y, although the exponent sequence might be followed by an additional "buffer" exponent. Obviously, 1 is a range element.

Remark 2

By "Lemma 5.0" on page 16, if counterexamples exist, there exists an infinity of counterexample tuples in each tuple-set having an element y of the 1-tree as the first element of its anchor tuple. Since each tuple-set has exactly one anchor tuple, Lemma 8.9 and its Corollary, plus what we have said in Remark 1, suggest that there exist counterexamples that are never anchors. If this can be shown to be true, then we have a proof of the $3x + 1$ Conjecture, for ever range element must eventually be an anchor, and each counterexample in each tuple in each tuple-set is a range element except, possibly, in the case that the counterexample is the first element of a tuple.

Computer Tests of the 3*x* + 1 Conjecture

As a result of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html, all odd, positive integers up to at least $20 \bullet 2^{58} \approx 5.76 \cdot 10^{18}$ are known to be non-counterexamples. If the reader solves $2 \bullet 3^{i-1} = 5.76 \cdot 10^{18}$, he or she will find that $i > 39$. This means that all 39-level anchors, hence all 39-level anchor tuples, are non-counterexamples. (In this and our other 3*x* + 1 papers, we have said, conservatively, that all 35-level anchors, hence all 35-level anchor tuples, are non-counterexamples.) The reader can also confirm that, since the distance between level-2 elements of successive 2-level tuples in any 2-level tuple-set is, by part (a) of "Lemma 1.0" on page 12, $2 \bullet 3^{2-1} = 6$, the number of consecutive 2-level tuples in any 2-level tuple-set that are non-counterexamples is at least 3^{33}.

The set *J* (the 1-tree) in the previous section gives rise to a question: why test all those odd, positive integers when the set *J* tells us all odd, positive integers that map to 1? We cannot believe that Prof. Oliveira e Silva is unaware of the 1-tree or its equivalent. Yet we can't help wondering why he didn't simply proceed through the 1-tree, using the computer to give closed-form representations of infinite sets (for example, all elements of a "spiral" whose first element is known to be a non-counterexample[1]). (We remind the reader that if *y* is a non-counterexample range element then we immediately know a countable infinity of non-counterexamples, namely, the elements of the "spiral" that map to *y*. Furthermore, each "spiral" contains an infinity of other range elements, etc.) The computer could be programmed to specify all candidates for counterexamples as of the current depth of penetration of the 1-tree.

Perhaps the changes in the set of candidates as the depth of 1-tree penetration increases, might give an insight as to whether counterexamples can really exist. For example, if the smallest counterexample candidate keeps increasing, then if we can prove that this increase is inevitable, we would have a proof of the 3*x* + 1 Conjecture.

1. Such a closed form representation is possible because, as we state in the previous section, if x , x' are successive elements of a "spiral", then $x' = 4x + 1$.

Relating Tuple-sets and Recursive "Spiral"s

A fundamentally important question is the following: "What is the relationship between the two structures underlying the $3x + 1$ function, namely, tuple-sets and recursive 'spiral's?" By "the relationship" we mean a closed-form function that takes an i-level non-counterexample tuple as input, and shows where this tuple is located in (a) its i-level tuple-set (easy), and (b) where it is located in the infinite set of recursive "spiral"s with base element 1 or with base element some counterexample (hard).

Relevant Facts

Let J be the set of odd, positive integers that map to 1, structured as described under "Lemma 8.8: Statement and Proof" on page 33. Then the following is equivalent to the $3x + 1$ Conjecture:

Conjecture: If x is an element of a tuple in a tuple-set, then x is an element of J.

(*Note*: we occasionally refer to J as *the 1-tree*.)

Relating the set of all tuple-sets to the set J has been very difficult, but we now believe we have figured out how to do it. See below under "Mechanism of the Relationship Finally Discovered" on page 40. In this section we will merely point out some facts.

• Each non-counterexample tuple in each tuple-set represents a path in J.

• In each i-level tuple-set T_A, where $i \geq 2$, and A is an exponent sequence that maps to 1 — that is, the exponent sequence of a tuple whose last element is 1, in other words, a tuple in the set J — 1 is an anchor, that is, 1 is the last element of the anchor tuple, which is the first i-level tuple in the tuple-set. So for each i, the set of all i-level tuple-sets contains the set of all i-level tuples that map to 1. (There are, of course, exponent sequences that do not map to 1.)

• In each tuple-set, each of the countable infinity of tuples that map to 1, is a tuple in the set J.

We can say more:

It is easily shown that, for all $i \geq 2$, the set of all i-level elements in all i-level tuples in all i-level tuple-sets is the set of range elements of the $3x + 1$ function, C. In fact, the set of all these elements consists of the union of the reduced residue classes mod $2 \bullet 3^{i-1}$, by part (a) of "Lemma 1.0" on page 12.

Let y be an i-level element in an i-level tuple t in an i-level tuple-set. If the first element x of t is not a multiple of 3 — in other words, if the first element of t is a range element — then x is the i-level element of an i-level tuple t' in an i-level tuple-set. This process continues without end unless a first level element is arrived at that is a multiple of 3. We call this process the *down, up, down...* process.

The down, up, down... process allows us to state the following:

• There is nothing in the set of all i-level tuple-sets, where $i \geq 2$, that is not in the set of all 2-level tuple-sets.

Proof:

Let t = <x, y, y', ..., $y'^{\cdots}{}'$, z> be an i-level tuple in an i-level tuple-set. Then in the set of all 2-level tuple-sets there is a tuple <x, y>, and a tuple <y, y'>, and ... and a tuple <$y'^{\cdots}{}'$, z>. □

Thus, for example, the 5-level tuple <11, 17, 13, 5, 1> in the 5-level tuple-set T_A, where A = {1, 2, 3, 4} is composed of the following sequence of 2-level tuples in 2-level tuple-sets: <11, 17>, <17, 13>, <13, 5>, <5, 1>.

• Let y be an i-level element in an i-level tuple t in an i-level tuple-set. Then y is the base element of an infinite set of recursive "spiral"s.

Mechanism of the Relationship Finally Discovered

After a great deal of effort, we believe we have finally discovered the mechanism of the relationship between tuple-sets and recursive "spiral"s — in particular, between tuple-sets and the infinite set of recursive "spiral"s representing the set J (the 1-tree)[1]. It is based on the down, up, down... process described in the previous sub-section, and is as follows:

Let T_A be any i-level tuple-set, where $i \geq 2$, having 1 as an anchor. (That is, such that 1 is the i-level element of the first i-level tuple.) Let x be the first element of the anchor tuple. Then if x is a range element (that is, not a multiple-of-3) then x is an i-level element in some i-level tuple-set in the set of all i-level tuple-sets (by "Lemma 4.75" on page 15). We now repeat the process for x. That is, for each i-level tuple having x as last element, let x' be the first element of the tuple. Then if x' is a range element (that is, not a multiple-of-3) then x' is an i-level element in some i-level tuple in the set of all i-level tuple-sets. Etc.

For each i, there is a countable infinity of i-level tuple-sets having 1 as anchor. The set of all infinitely long tuples mapping to all the 1 anchors via the mechanism we have just described, is the set of all paths to 1 in the 1-tree. Thus we see how the 1-tree is contained in the set of all tuple-sets.

We should emphasize that we can start with an arbitrarily large (though finite) i. In any case, we always obtain all upward paths in the 1-tree in concatenations of i-level tuples.

The tantalizing question is, "Can this relationship, and the fact that all range elements less than $2 \cdot 3^{35-1}$ — that is, all anchors for all 35-level tuple-sets — are known, by computer test, to be non-counterexamples, give us a proof of the $3x + 1$ Conjecture?"

We seem, inevitably, to find ourselves confronting the idea that, *because* all successive odd, positive integers up to a large number — at least $2 \cdot 3^{35-1}$ — are non-counterexamples, there is *no difference* between non-counterexamples and counterexamples. The reader is urged to read "First Proof" and "Second Proof" in the paper, "A Solution to the $3x + 1$ Problem" on occampress.com.

1. At present, we have not figured out how to make this mechanism give us the closed-form function described in the first paragraph of this section.

Strategies to Prove the 3*x* + 1 Conjecture
Preliminary Remarks

To properly understand our approaches to a possible proof of the Conjecture, it is essential that the reader be aware of:

(1) a common misconception about the nature of the $3x + 1$ function, and

(2) a common misconception about "$3x + 1$-like function" tests, and

(3) common misconceptions about the nature of comparison of mutually-exclusive cases

Furthermore, in order for the proofs to be evaluated with minimum chance for misunderstandings, it is essential that they be read *one sentence at a time*, with the reader asking each time, "Is this sentence clear?" and "Is this sentence correct?" If the answer to either question is no, the reader is urged to stop reading and contact the author, so that he can try to repair the problem before the reader proceeds.

If a proof is incorrect, then there is a first sentence in it that is incorrect.

We now provide a few details on points (1) through (3).

(1) A Common Misconception About the Nature of the 3x + 1 Function

We will use an analogy to explain this misconception.

Suppose there is a black box that contains a marble. The marble is either white or black. The Marble Conjecture states that the marble is white. It is clear that, in a proposed proof of the Marble Conjecture, any comparison of the case that the marble is white with the case that the marble is black would almost certainly be illegitimate (but in any case fruitless).

Some readers imagine that the $3x + 1$ function is like what we can call the "marble function". That is, they imagine that either all odd, positive integers map to 1, or none of them do, and furthermore that at present we do not know which case is true.

However, *the 3x + 1 function is fundamentally different from the marble function*. The reason is that, by "Lemma 8.8" on page 32, if an odd, positive integer maps to 1, then it does so *regardless if counterexamples exist or not*. It is an element of what we are calling the *Fixed-Set* (see "Definition of "Fixed-Set"" on page 34). Computer tests have shown that all odd, positive integers up to at least 10^{15} map to 1. These integers are elements of the Fixed-Set. In fact it is easy to show, using the 1-tree described in "Lemma 8.8: Statement and Proof" on page 33, and the fact, also easily shown, that a countable infinity of odd, positive integers map to each range element of the $3x + 1$ function, hence that a countable *infinity* of odd, positive integers map to 1. These constitute all the elements of the Fixed-Set.

The fact that a large number (indeed an infinity) of integers map to 1 r*egardless if counterexamples exist or not*, makes the $3x + 1$ function *fundamentally different* from the marble function (which has only one domain element (the marble) and one value (black or white)) . The marble function is a trivial example of a function that has *no* Fixed-Set, that is, a function such that no domain element has a fixed value, regardless if counterexamples exist or not. Our proofs of the $3x + 1$ Conjecture in this paper cannot be applied to such a function. On the basis of our communications with readers, it seems clear that many readers imagine that the $3x + 1$ function is a function without a Fixed-Set. Many of their objections to our proofs would make perfect sense if that were the case. But it is not.

(2) A Common Misconception About "3x + 1-Like Function" Tests

The $3x - 1$ Test is the application of a proposed proof of the $3x + 1$ Conjecture to the $3x - 1$ function. If the proposed proof also proves the $3x - 1$ Conjecture, then the proof may be faulty, because counterexamples to the $3x - 1$ Conjecture are known (5 and 7 are two of them). We say "may be faulty" rather than "is faulty" because in order for the Test to be valid, it must be the case that all the lemmas supporting the $3x + 1$ proof are also valid in the $3x - 1$ case.

There is an infinite class of what we have called "$3x + 1$-like functions" (they are defined in Appendix *C* of our paper, "Are We Near a Solution to the $3x + 1$ Problem?" on occampress.com), and include the $3x - 1$, $3x + 5$ and $3x + 29$ functions). Some readers have felt that a proof of the $3x + 1$ Conjecture is not valid unless it can be shown not to apply to the 3x + 5 function as well as to the $3x - 1$ function. Upon being convinced that the $3x + 1$ proof passes the $3x + 5$ Test, some of these readers have felt that the proof must pass the $3x + 29$ Test as well (counterexamples to the $3x + 29$ Conjecture are known), before the $3x + 1$ proof can be considered valid.

Since there is an infinite number of $3x + 1$-like functions, and since at present there is no known property of all of them such that if a $3x + 1$ proof passes the Test for one of them, it passes the test for all, this demand that a $3x + 1$ proof pass successive $3x + 1$-like function Tests, amounts to a declaration that the $3x + 1$ Conjecture is undecidable.

In any case, a proof must stand or fall on its own. We feel that if a reader believes our proof has failed a Test, then he or she must show the error in our proof.

In reply, some readers have gone so far as to claim that, even if all steps of a proof are correct, the proof as a whole can nevertheless be invalid. Our reply to this is that if the reader can get a paper published that proves the validity of that statement, then he or she will become world famous, because the statement contradicts one of the fundamental theorems of foundations of mathematics, namely, that if a proof is correct, then the correctness can be confirmed by machine (computer program).

In fact, that fundamental theorem gives us another rebuttal to those who claim that our proof of the $3x + 1$ Conjecture cannot be considered valid unless we can show that it does not also apply to the possible countable infinity of similar conjectures that contain counterexamples, namely, the conjectures associated with 3x + 1-like functions (and possibly others!). For no program at present can (1) determine all "similar" conjectures, and (2) for each one, determine if a counterexample exists, and then (3) verify that our proof does not also apply to a proof of the (false) conjecture.

(Full details on $3x + 1$-like functions, and our arguments against the demand for unlimited Tests, and against excessive reliance on even one or two of the Tests, are contained in the above-mentioned Appendix C.)

(3) Common Misconceptions About the Nature of Comparison of Mutually-Exclusive Cases

We Do Not Claim That the Existence of a Large Number of Consecutive Non-Counterexamples Implies No Counterexamples!

Despite the simplicity of "First Proof" and "Second Proof", below, we have found that there are readers who believe that the proofs argue that *because* a very large number of odd, positive integers map to 1, *therefore* all odd, positive integers map to 1, or that somehow the distance functions ("Lemma 1.0" on page 12) are able to discriminate between counterexamples and non-counterexamples. These beliefs are false. Our proof is based on a *comparison* of two 2-level tuple-sets: one under the assumption that counterexamples do not exist, and the other (defined by the same exponent sequence as the first) under the assumption that counterexamples exist. An elementary inductive argument yields the fact that both tuple-sets have exactly the same contents, which in turn implies that counterexamples do not exist.

Comparison of Mutually-Exclusive Cases Is Made Frequently Inside of And Outside of Mathematics

Some readers claim that the comparison implies that the two possibilities somehow exist simultaneously, which, of course, would be absurd. In fact, such comparisons are made every day, both inside of, and outside of, the mathematical culture. For example,

"If the Yankees win the pennant this year, then ... but if they don't, then ...", or

"If the stadium is built on the north side of campus, then ... but if it is built on the south side of campus, then ..", or.

"If the *abc*-conjecture is true, then ... but if it is not, then ...", or

"If an odd, perfect number exists, then ... but if an odd, perfect number does not exist, then..." or,

"If a counterexample to the $3x + 1$ Conjecture exists that results from an infinitely-repeating cycle of odd, positive integers, then there is a computer program that, in principle, will find the counterexample and halt. But if there is no such counterexample, then the program will run forever," or,

(Prior to the confirmation of the existence of the Higgs boson), "If the Higgs boson exists, then ... but if it does not exist, then ..."

Another refutation of the claim that comparison implies simultaneous existence, is the following: suppose an architect designs a skyscraper. His client looks at the plans, then suggests a change, though one that does not affect the basic structure of the building. The architect prepares a second set of plans, this set showing the change. He places both sets of plans side by side on a table so that the client can compare them.

Surely something like this process takes place routinely in the field of architecture! We are confident that neither the architect nor the client ever says words to the effect, "Our comparing the two sets of plans unfortunately implies that the change both exists and does not exist, which of course is a contradiction, and therefore the comparison cannot be made."

(The unchanged drawings are analogous to the set of all tuple-sets if counterexamples do not exist ; the changed drawings are analogous to the set of all tuple-sets if counterexamples exist.)

The analogy of the claim that two mutually-exclusive possibilities cannot be compared because only one of them actually exists, would be the claim that, when the architect's client wanted to look at the drawings showing the changes, the architect would be required to remove from the room the drawings without the changes , and only then bring in the drawings with the changes. If the client later wanted to view the drawings without the changes, the architect would be required to remove from the room the drawings with the changes, then bring in the drawings with the changes.

A variation of the claim that comparison of mutually-exclusive cases implies that both cases exist simultaneously, is the following (I quote the words of the critic):

> When you refer to "the set of non-counterexample tuples if counterexamples exist" and "the set of non-counterexample tuples if counterexamples do not exist", you are assuming that there is a well-defined such set in each case; in other words, unique completions to the statements "If counterexamples do not exist, then the set of non-counterexample tuples is —" and "If counterexamples exist, then the set of non-counterexample tuples is —".
>
> But I maintain that though one of those statements (the one whose hypothesis is true, whichever that is) does have a well-defined completion (i.e., a unique set) the other does not.

This criticism betrays a complete misunderstanding of the nature of a comparison. When we compare two possibilities, we are not concerned with questions of existence! We are concerned solely with the properties of the entities being compared. As we said above, we can compare two entities both of which exist, only one of which exists, or neither of which exists.

The above critic continues:

> "If" sometimes means "In those cases where". Then a statement of the form "If p is true, then the value of X is Y" can be true for a unique Y if there are some cases where p is true, and if in all those cases, X has the same value Y.
>
> (But if the value of X is different in different cases where p is true, or at the opposite extreme, if there are no cases where p is true, then there is not a unique value one can assign to Y that makes the statement true.)

We reply as follows. First, in the context of the Comparison Strategy, "if" *always* means "in those cases where". Second, it is entirely possible, in a given application of the Comparison Strategy, that a Y may have several possible values. It is then up to the writer to decide if that blocks the application of the Strategy in this particular instance, or if one of the values will, along with statements from the "if not-*p*, then ..." side, lead to the desired proof. But the fact that there are no guarantees in the Comparison Strategy in no way nullifies its legitimacy any more than the fact that there are no guarantees in the use of inductive proof or proof by contradiction nullifies the legitimacy of these proof methods!

The above critic seems not to understand implications like the following.

Suppose we write:

(1)
If Fermat's Last Theorem (FLT) is false, then there exists a computer program that, in princi-

ple, will find the counterexample.

Proof:

1. Place in lexicographical order all expressions of the form, $a^k + b^k - c^k$, where a, b, c, k are positive integers, and $k > 2$.

2. Starting at the beginning ot the order, compute each $a^k + b^k - c^k$. Eventually, one will be reached having the value 0, which will mark it as a counterexample. □

Now, the critic would no doubt argue that the antecedent of (1) is now known to be false, and since false implies anything the statement is meaningless. But that overlooks the fact that if I precede (1) with: "Statement (1) was true before Wiles proved FLT", then the critic's objection is removed. Or, we could simply replace "is" in the antecedent with "were", and "then there exists" in the consequent with "then there would exist".

Fundamentally, the critic seems unable to understand the meaning of "if".

A Truth-Table Argument

We now believe that the claim, "Comparison implies simultaneous existence", is in fact a logical fallacy. Apart from the informal refutations of the claim given above, there is one that follows directly from a truth-table argument.

Let p denote "Counterexamples to Conjecture X exist". Now consider:

(1)
$(p \Rightarrow r)$ and $(\sim p \Rightarrow s) \Rightarrow (p$ and $\sim p)$,

where "$\Rightarrow$" denotes "implies",
"$\sim$" denotes "not",
r is a true statement describing properties that exist if p is true, and
s is a true statement describing properties that exist if $\sim p$ is true.

The truth table for (1) yields (true $\Rightarrow$ false), which is a false implication. So it is false that the comparison of the two cases, p and $\sim p$, implies that both exist simultaneously.

Another Purely Logical Argument

Some readers feel that it is only legitimate to "consider" one case at a time, because to consider both implies both cases exist simultaneously. In other words, one must somehow blot out from one's mind, all thought of the case not being considered (a task not greatly different from that of not thinking of a white bear all day), just as the architect's client, in the above analogy, could require the architect to show only one set of drawings at a time. But these readers are wrong. The truth of the following sentence confirms the validity of our Comparison Strategy.

If a mathematician writes, on a sheet of paper, "If p, then ..." and below that, on the same sheet of paper, he then writes, "If *not-p*, then ..." he has *not* thereby written a contradiction.

Description of the Comparison Strategy

The Strategy is to compare two possibilities for a statement *p*:

If *p*, then ...
If *not-p*, then ...

The two if-statements are *not* a conjunction!

Whether *p* or *not-p* is true is assumed unknown at the application of the Strategy.

"If *not-p*" or equivalent must *not* appear in "..." in "If *p*, then ..." Similarly, "If *p*" or equivalent must *not* appear in "..." in "If *not-p*, then ...".

The Strategy has been deemed logically valid by a leading authority on modern symbolic logic.

The Strategy is applicable *only* when *p* is a statement about a set *X* — in our case, this is typically the set of all odd, positive integers, or the set of all tuples. Thus, attempts to dismiss the Strategy by arguing that *p* might be 2 + 2 = 5, are illegitimate. It is necessary that there be a Fixed-Set[1] which is a large initial proper subset of *X* ("initial" under some appropriate ordering of elements in the set *X*) that contains the same elements whether or not *p* is true. In the case of the $3x + 1$ Problem, this large initial proper subset is the set of all odd, positive integers from 1 to at least $10^{15} - 1$, since all these integers are known, by computer test, to be non-counterexamples.

In an implementation of the Strategy, one reasons in the first case from the assumption that *p* is true ("if *p*, then ...") to a certain statement s_1, and in the second case from the assumption that *not-p* is true ("if not-*p*, then ...") to a certain statement s_2, such that s_1 and s_2 can then be used to show that *p* is true. We do not aim for s_1 or s_2 being the conclusion of a proof of the $3x + 1$ Conjecture! We aim for these two statements taken together to lead to a proof.

As we said at the start of this section, it is crucial in any application of the Strategy, that in the line of reasoning beginning with "If *p*", no expression equivalent to "If *p* is false" appear. And similarly, it is crucial in any application of the Strategy, that in the line of reasoning beginning with "If *not-p*", no expression equivalent to "If *not-p* is false" appear.

And yet a few readers have argued that in "If *p,* then ..." , it is necessary to consider the case not-*p*. And since, if *p* is true, not-*p* is false, and since "false implies anything", the reasoning must always be meaningless. The argument is, of course, absurd. A common technique for proving statements of the form, "If *p*, then *q*" is to begin with the assumption that *p* is true and then use known facts (lemmas and theorems) to arrive at a proof that *q* is true. The readers' argument would nullify all these proofs, and hence would nullify countless proofs in the body of mathematics.

But some readers still insist that in "If *p*, then ..." it is necessary to consider the case *not-p*, and similarly in "If *not-p*, then ..." it is necessary to consider the case *p*. A counterargument to their claim is the following:

1. See "Definition of "Fixed-Set"" on page 34.

1. Suppose that, in one classroom on campus, a mathematician begins a sequence of deductions beginning "If *p*, then ...". The deductions begin with the assumption that *p* is true. Furthermore, suppose he nowhere mentions the possibility not-*p*. This is perfectly legitimate. To deny it, is to call into question numerous proofs in mathematics that follow the rule that is taught in courses on symbolic logic as one way to prove statements of the form, "If *p*, then *q*". The rule says, "Assume *p* is true, and then use known facts (lemmas, theorems) to construct a proof that *q* is true." The rule nowhere mentions that the possibility that *not-p* is true must somehow be considered.

2. Now suppose that in another classroom on campus, another mathematician begins another sequence of deductions. This sequence begins "If *not-p*, then ... " The deductions begin with the assumption that *not-p* is true. Furthermore, suppose he nowhere mentions the possibility *p*. This is perfectly legitimate for the reason given in step 1.

3. We can assume that neither mathematician knows what the other is doing. To deny that that is legitimate is to claim that any mathematician wanting to carry out deductions beginning "If *p*, then ..." (or "If *not-p*, then ...") must somehow first contact all mathematicians in the world and find out which ones are about to carry out deductions based on the negation of the antecedent that the mathematician wants to carry out deductions from.

4. Suppose, now, that each mathematician's sequence of deductions has been recorded on video. The next day, in another classroom, the video of the first mathematician's deductions is played for persons in a different classroom from that in which the previous day's deductions took place. There is nothing illegitimate about this.

5. Suppose, next, that, following that video, the second video is played in the same classroom before the same audience. There is nothing illegitimate about this.

6. Finally, suppose that one of the mathematicians, or a member of the audience, compares the final statements in each video, and makes an observation based on the comparison. There is nothing illegitimate about this.

We hope this scenario will convince skeptics that it is perfectly legitimate for the two sequences of deductions to exist in "two universes", and not the one universe as skeptics demand.
.

For our proofs of the $3x + 1$ Conjecture, the Fixed-Set[1] (see "Definition of "Fixed-Set"" on page 34) consists of all odd, positive integers $\leq 10^{15} - 1$, these having all been determined by computer test to be non-counterexamples. Our proofs most certainly are *not* based on the invalid argument that *because* this large subset exists, *therefore* all odd, positive integers are non-counterexamples. The large subset is just the point of departure for our proofs.

Meaning of the Word "Same" When Applied to Sets in a Comparison

We sometimes say that the set J of non-counterexamples is t*he same* whether or not counterexamples exist. The meaning of *the same* in this context is as follows.

1. This is actually a proper subset of the Fixed-Set, which is the set of all non-counterexamples.

If, in the previous sub-section, our mathematician writes,

"If p, then... the set of elements having property q we denote by A...", and then, below this, on the same sheet of paper, he writes, "if *not-p*, then ... the set of elements having property q we denote by B...", he has not thereby written a contradiction.

Furthermore, it is perfectly legitimate for him (or us) to *compare* sets A and B. There are two possibilities: (1) $A = B$ or (2) $A \neq B$.

We can express possibility (1) in language such as "The set of elements having property q is *the same* whether or not p". This the case with the set J, above. In that case, p is "counterexamples exist" and "*not-p* is "counterexamples do not exist". Of course, strictly speaking, we are abusing language when we say "the set J is the same whether or not counterexamples exist". We should say words to the effect, "Let J_{nc} denote the set of odd, positive integers if counterexamples do not exist, and let J_c denote the set of odd, positive integers if counterexamples exist. Then, $J_{nc} = J_c = J$".

Comparison of Mutually-Exclusive Cases Does Not Necessarily Involve Questions of Existence

We must point out that comparison of two entities does not require that even one of them exists! Recall Kant's refutation of the ontological proof for the existence of God. That proof asserted that a perfect Being that does not exist is less perfect than a perfect Being that does exist, therefore God exists. Kant's reply was "Existence is not a predicate!" That is, existence is not a property.

Thus, we can compare two unicorns in a painting or cartoon film (as to, say, size), or we can compare two characters in a novel, or two different sets of drawings for a proposed building (neither plan might represent a future building, or only one might, or both might, if separate buildings are built). In short, we can compare two things: both of which exist, or only one of which exists, or neither of which exists.

On the Phrase, "Whether or Not Counterexamples Exist"

Some readers question the validity of statements of the form, "Whether or not counterexamples exist, q." They argue (correctly) that such statements are equivalent to the two statements,

(A)
"If counterexamples exist, then q" and
"If counterexamples do not exist, then q".

But some readers then argue (1) that the two statements are logically ambiguous, hence meaningless, or (2) that because the antecedent in one of these statements is false, and "false implies anything", the statements, hence the original statement, "Whether or not counterexamples exist, then q", are meaningless. We will now reply to these arguments.

Reply to (1): If q could be true or false, then the readers' argument would be correct, because, on the one hand, if q is true, then the two statements are true, by the truth-table for implication. But if q is false, then one of the two statements is false, again, by the truth-table. So the two statements are logically ambiguous, hence meaningless.

However, in this paper, *q is always true* because it is the statement of a lemma, and so the two statements are always true, by the truth-table for implication, and hence there is no ambiguity.

Reply to (2): Argument (2) ultimately rests upon a fundamental misunderstanding of, or failure to accept, the Comparison Strategy. We try to clear up at least the misunderstanding in the next section.

On Informational Implication: it is important for the reader to understand that the two implications, "if counterexamples exist, then *q*" and "if counterexamples do not exist, then *q*", are cases of what has been called "informational implication". For example, prior to the confirmation of the existence of the Higgs boson, statements of the form, "Whether or not the Higgs boson exists, the law [name of accepted physical law} will continue to hold" must have been made frequently. This statement is equivalent to "If the Higgs boson exists, then the law ... will continue to hold; if the Higgs boson does not exist, then the law... will continue to hold."

These implications provide information about the relationship between the existence of the Higgs boson and a certain law. It is highly unlikely that a physicist ever made the (illegitimate) reply "But the antecedent in one of the implications is false, and since false implies anything, the original statement is meaningless."

An informational implication has the property: the antecedent is true and the consequent is true. Thus the consequent provides information about the antecedent. Therefore, "if counterexamples exist, then 2 + 2 = 4" is not an informational implication, because the consequent does not provide information about the antecedent. And, of course, the same distinction applies to statements of the form, "whether or not *p*, *q*", where *p*, *q* are statements. If *q* does not provide information about *p*, then "whether or not *p*, *q*" is not informational.

We can place all this on a rigorous logical basis. Recall that, as we stated above, "whether or not *p*, *q*" is the equivalent of

(I)
(if *p* then *q*) **and** (**if not**-*p* then *q*)

In all the cases that we are concerned with in this paper, *q is true* — that is, *q* is a fact that provides information in a certain context, namely, the context of *p* being true or of *not-p* being true. And so, as the reader can easily verify by examining the truth-table for (I) when *p* is true, (I) is true. It is not "meaningless".

We must emphasize that, since informational implication is the only kind of implication that is found in this paper, the statements (A) above should be written:

(A′)
"If it is true that counterexamples exist, then *q*" and
"If it is true that counterexamples do not exist, then *q*".

The phrase "whether or not" occurs in and outside of mathematics. For example, in mathematics: "Whether *X* is orientable or not, [a certain cap product between $H^p(X; \mathbf{Z}/2)$ and $H_{n-p}(X; Z/2)$] is an isomorphism[1]". Outside of mathematics, for example in everyday life:

"Whether or not it rains tomorrow, we will leave for San Diego."

[Professor to student]: "Whether or not you pass this course, you will not be able to graduate in June."

"Whether or not state taxes are increased, the extension of the stadium will be completed."

"Whether or not a Democrat is elected president in the 2016 election, the U.S. will continue to have a national debt."

Some readers have argued that, because "whether or not counterexamples exist, 2 + 2 = 4" is trivially true and therefore unimportant, *all* statements "whether or not counterexamples exist, q", where q is any true statement, are trivially true and therefore unimportant. But these readers are ignorant of the difference between trivial (true) implications and informational (true) implications. The statement, "whether or not counterexamples exist, each tuple-set contains an infinity of non-counterexample tuples", is an informational (true) statement. It is by no means obvious, in the way that 2 + 2 = 4 is obvious, that each tuple-set should contain an infinity of non-counterexample tuples. It would be perfectly reasonable if a person just beginning his or her study of this paper, wondered if the existence of counterexamples might reduce the number of non-counterexample tuples in at least one tuple-set to a finite number.

Possible Explanation for Readers' Difficulty With Comparison of These Cases

Some readers nevertheless continue to believe that comparison of mutually-exclusive cases implies simultaneous existence of the cases. Through patient questioning, we have come to the conclusion that there are two reasons for this belief: (A) the readers' belief that the $3x + 1$ function is a function without a Fixed-Set (see "(1) A Common Misconception About the Nature of the 3x + 1 Function" on page 41), and (B) the readers' imagining that there is a "realm" in which the cases are to be discussed or written about. This realm, they feel, only has room for one case at a time. So if we want to discuss or write statements about the case that counterexamples exist, then we must first remove from this realm the case that counterexamples do not exist. And if we want to discuss or write statements about the case that counterexamples do not exist, then we must remove from this realm, the case that counterexamples exist. To compare the two cases is to place both of them in the one realm at one time, and that results in contradictions.

However, it is perfectly legitimate to imagine the realm as being big enough to hold the two cases simultaneously — "side-by-side". There are then no contradictions in saying, for example, "assume x in the one case is a counterexample, and that x in the other case is a non-counterexample." (The realm that is big enough to hold the two cases simultaneously is an example of the use of increased logical "space" to avoid contradictions.)

So we encourage the reader who is skeptical about the validity of the Comparison Strategy to get a piece of paper, draw a few lines descending from a point (the root of the tree representing the set of all tuple-sets) and below the top of the paper to write, "Set of All Tuple-Sets if Counterexamples Exist.) We then ask the reader to get a second piece of paper, and do the same, with the title at the top reading "Set of All Tuple-Sets if Counterexamples Do Not Exist."

We hope that it is clear that if the reader were to write, somewhere on the first sheet, "Let x be a counterexample," and somewhere on the second sheet, "Let x be a non-counterexample", there

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1. Munkres, James R., *Elements of Algebraic Topology*, Addison-Wesley Publishing Company, Menlo Park, California, 1984, p. 394. Our word-processor does not have all the symbols used to represent the cap product in the actual text, hence the phrase in brackets.

would be no contradiction!

Other refutations of the claim that comparison implies simultaneous existence, are given in our short paper, "Is It Legitimate to Begin a Sentence With 'If Counterexamples Exist, Then...' ", on occampress.com.

The Most Important Fact About the 3*x* + 1 Function

After a great deal of struggle, and many failed proofs, we now believe that the single most important fact about the $3x + 1$ function as far as a proof of the $3x + 1$ Conjecture is concerned, is that for at least the first 35 levels, all anchor tuples are non-counterexample tuples (see "Computer Tests of the 3x + 1 Conjecture" on page 38).

By contrast, in the case of the $3x - 1$ function not even at level 2 are all anchor tuples non-counterexample tuples. (See the next sub-section.)

Our avoiding of thinking through the immediate implication of this contrast between the two functions cost us years of wasted effort in trying to make a proof of the Conjecture out of the following argument.

1. "Lemma 18.0: Statement and Proof" on page 93 states that, for each range element y (for example, the range element 1), and for each exponent sequence A, there exists an x that maps to y via A possibly followed by a single "buffer" exponent.

2. "Lemma 8.8" on page 32 states that the set of odd, positive integers that directly or indirectly map to 1 is the same regardless if counterexamples exist or not (the laws of arithmetic are not subject to the truth or falsity of the $3x + 1$ Conjecture).

3. Therefore (we reasoned) the first i-level tuple in each i-level tuple-set (in other words, the anchor tuple), for all $i \geq 2$, must be a non-counterexample tuple. But if counterexamples exist, there must be anchor tuples in some i-level tuple-sets that are counterexample tuples. This contradiction implies that counterexamples do not exist.

The $3x - 1$ function provides a direct counter to this reasoning. For, although Lemma 18.0 and Lemma 8.8 apply to the $3x - 1$ function as well, for each $i \geq 2$, there are counterexample anchor tuples and non-counterexample anchor tuples!

The Most Important Test of Possible Strategies

The most important test of a possible strategy is: Does it also apply to other $3x + 1$-like functions — in particular, to $3x + 1$-like functions in which counterexamples are known. These functions are defined in "Appendix C — "3x + 1 - like" Functions" on page 99. The 3x – 1 function is such a function, and the most important one to date. Clearly, if the strategy also applies to such a function that has a known counterexample, then the strategy has a flaw.

The test is all the more important because many of the $3x + 1$ lemmas also hold for these functions. For example, in the $3x + 13$ function, which is a $3x + 1$-like function, the following are successive 3-level tuples in the 3-level tuple-set T_A, where $A = \{2, 2\}$: <13, 13, 13>, <45, 37, 31>. (Thus 13 is a counterexample to the $3x + 13$ Conjecture because 13 gives rise to an infinite cycle.) Observe that $13 + 2 \cdot 3^{3-1} = 13 + 18 = 31$, and that $13 + 2 \cdot 2^2 \cdot 2^2 = 13 + 32 = 45$. Thus the distance, 18, from 13 to 31 is exactly as specified by part (a) of "Lemma 1.0" on page 12 and the distance, 32, from 13 to 45 is exactly as specified by part (b) of the Lemma.

A way of increasing the chances that a possible strategy will pass the test is to concentrate on strategies that employ unique properties of the $3x + 1$ function, that is, properties that are not shared by other $3x + 1$-like functions. These properties include: (1) the $3x + 1$ term itself in calculations; (2) the distance function between successive elements, x, x' of a "spiral", namely $x' = 4x + 1$ (if a $3x + C$ function is a $3x + 1$-like function, then this distance function is $x' = 4x + C$); and (3) the fact that a very large number of consecutive odd, positive integers are known (as a result of computer tests) to be non-counterexamples for the $3x + 1$ Conjecture. The number is at least $2 \bullet 3^{35-1}$. In the $3x + 1$-like functions known to have counterexamples (for example, the $3x - 1$, $3x + 5$ and $3x + 13$ functions) counterexamples appear in the first 18 odd, positive integers.

Complete List of All Our Results

A complete list of all results (lemmas) we have obtained so far in our $3x + 1$ research is contained in "Appendix A — Statement and Proof of Each Lemma" on page 77, amd in the first part of the second file of our paper, "The Structure of the $3x + 1$ Function: An Introduction" on the web site occampress.com.

Possible Strategies Based on Tuple-sets

(See also, "Possible Strategies for Proving the $3x + 1$ Conjecture Using Tuple-sets" in the first part of our paper, "The Structure of the $3x + 1$ Function: An Introduction", on occampress.com.)

Tuple-sets have one important advantage over recursive "spiral"s, namely, that if a counterexample exists, then by "Lemma 5.0" on page 16, each tuple-set contains a countable infinity of counterexample tuples, as well as a countable infinity of non-counterexample tuples. On the other hand, an infinite set of recursive "spiral"s with base element 1 cannot contain *any* counterexample tuples, and an infinite set of recursive "spiral"s with base element a counterexample, cannot contain *any* non-counterexample tuples.

Most Promising Possible Strategy At Present

The most promising possible strategy is the one we use for our current proof of the 3x + 1 Conjecture in our paper, "A Solution to the 3x + 1 Problem", on occampress.com. The strategy shows that if counterexamples exist, then they are the same as non-counterexamples, which is absurd.

Strategy Showing There is No Difference Between Non-Counterexample and Counterexample Tuples

This is the strategy we use in our proof of the $3x + 1$ Conjecture. See our paper, "A Solution to the 3x + 1 Conjecture", on occampress.com.

The "Pushing Away" Strategy

In the "Pushing Away" Strategy we attempt to show that every tuple containing an assumed counterexample is "pushed away" from tuples whose elements map to 1, i.e., every tuple containing a counterexample must always be the second, or third, or fourth, or ... tuple in any tuple-set, but never the first. Thus counterexample tuples never become anchor tuples, hence counterexample tuples do not exist, because if an odd, positive integer exists, it must eventually be an element of an anchor tuple.

For further details, see "The 'Pushing Away' Strategy In Brief", and following sections, in the first part of oru paper, "The Structure of the 3x + 1 Function: An Introduction", on occampress.com.

Current Status of the "Pushing Away" Strategy

We believe it will not work, because the "pushing away" behavior fails in the case of the $3x - 1$ function, where counterexample tuples are in fact the first i-level tuples in some i-level tuple-sets, for each $i \geq 2$.

The Tantalizing Strategy: Induction on Non-Counterexample Anchor Tuples

Certainly one of the most obvious strategies, and yet so far at least the most tantalizingly difficult to implement[1], is induction on non-counterexample anchor tuples. It begins with the observation that, as a result of computer tests, we know that for all levels i, where $2 \leq i \leq 35$, all i-level anchor tuples are non-counterexample tuples. So why is an inductive proof so difficult? At present we do not know, although we are convinced that the fact that all anchor tuples up to at least level 35 are non-counterexample tuples, is of fundamental importance. Perhaps we can obtain some insight by investigating why the first counterexample anchor tuple in the case of the $3x - 1$ function, occurs already at level 2.

An Implementation Derived from the Induction Strategy

1. It is easily shown that, for each $i \geq 2$, the set E_i of i-level elements in *first* i-level tuples in all i-level tuple-sets is the set of odd, positive integers less than $2 \cdot 3^{(i-1)}$ that are not divisible by 3 (by part (a) of "Lemma 1.0" on page 12). Thus, for example, E_2 is the set of all the odd, positive integers less than $2 \cdot 3^{(2-1)} = 6$,, that are not divisible by 3, and these integers are 1 and 5.

2. By computer test, it is known that $E_2, E_3, E_4, \ldots$, up to at least E_{35} each consists solely of non-counterexamples[2].

3.

(1) For each 35-level tuple-set T_A, the sequence S of 35-level elements in the sequence of 35-level tuples is given by $x + n(2 \cdot 3^{(35-1)})$, where $n \geq 0$ and x is the 35-level element of the first 35-level tuple in the tuple-set T_A. The sequence S is the sequence if counterexamples do not exsit, and it is also the sequence if counterexamples exist. As we stated in step 2, x is a non-counterexample element,

Proof: Follows from part (a) of "Lemma 1.0" on page 12. The Distance Functions are not themselves functions of the truth or falsity of the $3x + 1$ Conjecture. □

Note: the fact that all elements of E_{35} are non-counterexamples is emphatically *not* the case

1. Not any longer: see our proof of the $3x + 1$ Conjecture in "A Solution to the $3x + 1$ Problem", on occampress.com.

2. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All consecutive odd, positive integers to at least $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$, which is greater than $3.33 \cdot 10^{16} \approx 2 \cdot 3^{(35-1)} - 1$, have been tested and found to be non-counterexamples. .

for the 3x – 1 function, where one of the elements, 5, of E_2 is already a counterexample,. Thus there exists a first 2-level tuple, namely <7, 5>, in a 2-level tuple-set that is a counterexample tuple. Each subsequent E_i contains counterexamples, each of which is the *i*-level element of the first *i*-level tuple in an *i*-level tuple-set. Each of these tuples is therefore a counterexample tuple. *So our proof cannot be used to prove the false 3x – 1 Conjecture.*

4. We infer from (1) that if counterexamples exist in *S*, then some elements of *S* are both non-counterexamples and counterexamples, which is absurd.

5. We must now ask if counterexamples can exist in T_A in *j*-level tuples, where $j < 35$. The answer is No, because each *j*-level tuple in T_A is a 35-level tuple in some other 35-level tuple-set, and T_A is any 35-level tuple-set.

So we must conclude, from the contradiction in step 4, that counterexamples do not exist.. □

Remark

The reason why we say the above possible proof is *derived* from the Induction Strategy, is that originally, we argued that the fact that the sequence *S* is precisely the sequence that would exist if counterexamples did not exist, allows us to assert that each extension of the tuple-set T_A — and there is an extension for each of the exponents 1, 2, 3, ... — allows us to say that the 36-level tuple-sets resulting in each case, also have the property that *S* is precisely the sequence if countrexamples do not exist, and so on, and from that conclude that counterexamples do not exist. But then we realized the above proof, which is limited to the original *S*, suffices.

Current Status of the Tantalizing Possible Strategy

This possible strategy is the strategy we use in our proof of the 3*x* + 1 Conjecture. See our paper, "A Solution to the 3x + 1 Conjecture", on occampress.com.

Possible trategy Based on Idea There is "Not Enough Room" for Counterexamples

Description of Strategy

Our strategy is to show that, if counterexamples exist, there is (informally) not enough "room" for all the non-counterexample anchor tuples *and* for all the counterexample anchor tuples that are required by "Lemma 5.0: Statement and Proof" on page 85.

Most Promising Implementation of the Strategy At Present

1. Regardless if counterexamples exist or not, the structure of all tuple-sets remains the same, in accordance with the definition. In particular, if counterexamples exist, some tuple-sets do not somehow acquire an "extra" anchor tuple that is a counterexample tuple.

2. We know from "Lemma 8.8: Statement and Proof" on page 33 that exactly one set of odd, positive integers (all those contained in the 1-tree) maps to 1 whether or not counterexamples exist. So if counterexamples exist, it is definitely not the case that some elements of the 1-tree "disappear" because they have become counterexamples.

Therefore, exactly the same set of non-counterexample tuples (anchor and non-anchor) exists whether or not counterexamples exist.

3. We know, as a result of computer tests, that for each i, $2 \le i \le 35$, all i-level anchor tuples are non-counterexample tuples.

If counterexamples do not exist, then each tuple-set of level greater than 35 (as well as level ≤ 35) has a non-counterexample anchor tuple.

We now ask about the case that counterexamples exist. If that is so, then we know, by "Lemma 5.0" on page 16, that each i-level tuple-set, where $i \ge 2$, contains an infinity of non-counterexample tuples.

Each infinite tuple having a counterexample as first element has a mark that denotes the first level i at which a prefix t_c of the infinite tuple is an i-level anchor tuple in an i-level tuple-set T_A. But then there is no non-counterexample i-level anchor tuple t_{nc} associated with the exponent sequence A, because that would imply that two different tuple-sets are determined by the same exponent sequence, which is impossible.

Now if counterexamples do not exist, then, for all $i \ge 2$, each i-level anchor tuple is a non-counterexample tuple. But if counterexamples exist, then some i-level anchor tuples are counterexample tuples.

By "Lemma 8.8: Statement and Proof" on page 33, we know that there is exactly one set J of non-counterexamples whether or not counterexamples exist. This set J is the set of nodes of the 1-tree (see "Graphical Representation of the Set J as Recursive "Spiral"s" on page 28). Hence there is exactly one set of non-counterexample tuples, whether or not counterexamples exist.

So we must ask where the i-level non-counterexample anchor tuples "went" that were replaced by i-level counterexample anchor tuples? The answer is, by Lemma 8.8, that they didn't go anywhere, and thus we have the contradiction of two i-level anchor tuples being associated with the same i-level exponent sequence, which is impossible. If our reasoning is correct, this contradiction gives us a proof of the $3x + 1$ Conjecture.

Other Possible Implementations of the Strategy

A Fact That We Must Keep In Mind

Lemma 5.0 states that if counterexamples exist, then each tuple-set contains an infinity of counterexample tuples and an infinity of non-counterexample tuples. (The proof of the Lemma has been checked and deemed correct by several mathematicians.) The $3x – 1$ function is the negative of the $3x + 1$ function over the negative integers (see "Definition of 3x – 1 Function" on page 104), and so is represented by the extension of each $3x + 1$ tuple-set into the negative integers. There are counterexamples to the $3x – 1$ Conjecture (5 and 7 are two of them), and yet we have no reason to doubt that counterexamples and non-counterexample tuples exist in each tuple-set representing the 3x – 1 function exactly as Lemma 5.0 requires. So we must always keep in mind what we have called the $3x – 1$ Test when considering any implementation. Note that "Most Promising Implementation of the Strategy At Present" on page 54 passes this Test.

Show That There Are No Finite Marks In Counterexample Infinite Tuples, An Impossibility

That is, show that, by an iterative argument, counterexample marks are "pushed up" without limit. (In other papers, we have sometimes expressed this as: counterexample tuples are "pushed away" from the set of anchor tuples for all i-level tuple-sets, as i increases.) This is equivalent to showing that counterexample tuples are always tuples in tuple-sets having non-counterexample anchor tuples. But if no counterexample is an element of a counterexample anchor tuple, then no counterexample is less than $2 \bullet 3^{(i-1)}$ for any i. Hence there are no counterexamples.

This is a particularly tantalizing implementation. We know that, by "Lemma 5.0" on page 16, and by the section, "Infinite Tuples, Marks, and Tuple-sets" on page 20, for each i-level exponent sequence A, where $i \geq 2$, the set of i-level prefixes of all non-counterexample infinite tuples is complete. Furthermore, each non-counterexample infinite tuple is an infinity of successive anchor tuples.

Let $\bar{t}_c$ be a counterexample infinite tuple. It has a mark m_c. Let $\bar{t}_{nc}$ be a non-counterexample infinite tuple. It has a mark m_{nc}. Assume $m_{nc} < m_c$. (This is a legitimate assumption, since we know, as a result of computer tests, that an infinity of non-counterexample infinite tuples have marks less than $2 \bullet 3^{(35-1)}$, whereas no counterexample infinite tuples do.) Let $A(\bar{t}(j))$ denote the exponent sequence associated with the *prefix* $\bar{t}(j)$ of an infinite tuple $\bar{t}$.

Then for each $j \geq 0$, if $A(\bar{t}_{nc}(m_{nc}+j) = A(\bar{t}_c(m_{nc}+j)$, m_c must be greater than $m_{nc}+j$. Otherwise there would be two anchor tuples in the same tuple-set, an impossibility. Here m_c is pushed up for all j. Hence m_c is not finite, which implies $\bar{t}_c$ does not exist. .

The problem with this implementation is that $m_{nc} < m_c$ only applies at the highest level at which the set of non-counterexample anchor tuples is complete. (By computer test, we know that that level is greater than 35.) At higher levels, we must deal with the possibility that $m_c < m_{nc}$.

Show That Non-counterexample and Counterexample Infinite Tuples Have the Same Exponent Sequences, An Impossibility

The reason for the impossibility is that two identical infinite sequences imply that for an infinity of consecutive levels, namely all those greater than the maximum of the marks of the two infinite tuples, there are two anchor tuples in the same tuple-set, a contradiction. See "Lemma 5.6" on page 18.

A Crucially Important Fact About the Exponent Sequences of Infinite Tuples

A fact that we have attempted to exploit in various ways is: if the exponent sequences of two different infinite tuples differ, then there is a first level at which the exponent sequences differ. All longer sequences therefore must differ as well, even if, after the level at which they differ, they are once again the same.

Our idea has been to derive a contradiction from the fact that the exponent sequences associated with counterexample infinite tuples must differ from the exponent sequences associated with non-counterexample infinite tuples. (No exponent sequence associated with a counterexample infinite tuple can terminate with an infinite sequence of 2's (1 maps to 1 via the exponent 2).)

However, we have overlooked the crucially important fact that **the exponent sequences of every pair of infinite tuples must differ**, **regardless** if one tuple is non-counterexample and the other is counterexample, or if both tuples are non-counterexamples, or both tuples are counterexamples! The reason is that if the exponent sequences are the same, then in the infinite sequence of extensions of the corresponding tuple-sets, eventually the distance between first elements of the corresponding tuples would violate part (b) of "Lemma 1.0" on page 12.

Show That The Existence Of Counterexamples Implies That No Prefix Of A Non-counter-example Infinite Tuple Is Associated With A Certain Finite Exponent Sequence

This is a contradiction to “Lemma 5.0” on page 16.

Show That a Certain “Completeness” Property of Infinite Tuples Makes Counterexamples Impossible

(*Note*: we now believe that this implementation will not work. The reason is that it assumes that “Lemma 5.0” on page 16 makes it impossible for counterexample and non-counterexample anchor tuples to exist at any level or succession of levels. But this is disproved by the fact that this occurs in the $3x - 1$ function (which is equivalent to the negative of the $3x + 1$ function over the odd, negative integers). For details, see “Definition of 3x – 1 Function” on page 104.

We describe the implementation just in case a reader might see a way to overcome the faulty assumption.

Let $\bar{t}_c$ be a counterexample infinite tuple, and let $\bar{t}_{nc}$ be a non-counterexample infinite tuple. *Definition*: a set of j-level prefixes of infinite tuples is *complete* if the set is associated with the set of all j-level exponent sequences. Then by “Lemma 5.0” on page 16 we know that:

(1)

The set $\{\bar{t}_c(2)\}$ of all 2-level prefixes of all infinite tuples in $\{\bar{t}_c\}$ is complete.
The set $\{\bar{t}_c(3)\}$ of all 3-level prefixes of all infinite tuples in $\{\bar{t}_c\}$ is complete.
The set $\{\bar{t}_c(4)\}$ of all 4-level prefixes of all infinite tuples in $\{\bar{t}_c\}$ is complete.
...

(2)

The set $\{\bar{t}_{nc}(2)\}$ of all 2-level prefixes of all infinite tuples in $\{\bar{t}_{nc}\}$ is complete.
The set $\{\bar{t}_{nc}(3)\}$ of all 3-level prefixes of all infinite tuples in $\{\bar{t}_{nc}\}$ is complete.
The set $\{\bar{t}_{nc}(4)\}$ of all 4-level prefixes of all infinite tuples in $\{\bar{t}_{nc}\}$ is complete.
...

We emphasize that the statements in (1) and (2) concern *prefixes* of infinite tuples. A prefix of an infinite tuple is not necessarily an anchor tuple, although it is a prefix of an infinity of successive anchor tuples.

We offer the following thoughts, which might lead to other proofs.

Recall that if t is a prefix of an infinite tuple (that is, if t is a finite tuple), then we denote the exponent sequence associated with t by $A(t)$. We can now make the following statements:

Let $\bar{t}_{nc}$ be a fixed non-counterexample infinite tuple with mark m_{nc}. We now consider all pairs $< \bar{t}_{nc}, \bar{t}_c >$, where $\bar{t}_c$ is any counterexample infinite tuple. The following statements hold:

If $A(\bar{t}_{nc}(2)) = A(\bar{t}_c(2))$, and $m_{nc} > 2$, and $A(\bar{t}_{nc}(3)) \neq A(\bar{t}_c(3)$, then m_c can have any value.
If $A(\bar{t}_{nc}(3)) = A(\bar{t}_c(3))$, and $m_{nc} > 3$, and $A(\bar{t}_{nc}(4)) \neq A(\bar{t}_c(4)$, then m_c can have any value.

If $A(\bar{t}_{nc}(4)) = A(\bar{t}_c(4))$, and $m_{nc} > 4$, and $A(\bar{t}_{nc}(5)) \neq A(\bar{t}_c(5))$, then m_c can have any value.
...
If $A(\bar{t}_{nc}(m_{nc})) = A(\bar{t}_c(m_{nc}))$, then m_c must be $> m_{nc}$.
If $A(\bar{t}_{nc}(m_{nc} + 1)) = A(\bar{t}_c(m_{nc} + 1))$, then m_c must be $> m_{nc} + 1$.
...

A corresponding set of statements holds if we fix $\bar{t}_c$ and then consider all pairs $< \bar{t}_c, \bar{t}_{nc} >$, where where $\bar{t}_{nc}$ is any counterexample infinite tuple.

Does this give us the basis for a contradiction?

Another approach based on (1) and (2) is the following: raise the level i, beginning at $i = 2$, through successive levels 2, 3, 4, ... and consider the properties of $\{\bar{t}_c(i)\}$ and of $\{\bar{t}_{nc}(i)\}$, keeping in mind:

(3) that each of these sets is an (infinite) set of prefixes of anchor tuples. In other words, each of these sets is an (infinite) set of prefixes of first tuples in tuple-sets, and

(4) that each of these sets is complete, and

(5) that each i-level tuple-set has exactly one first i-level tuple (the anchor tuple), and

(6) that beyond a minimum level i_0 there are both non-counterexample and counterexample anchor tuples, and that the set of each of these is incomplete (otherwise, there would be two anchor tuples in some tuple-set).

One conclusion we can draw from these facts can be expressed informally as: each non-counterexample infinite tuple $\bar{t}_{nc}$ is eventually an element of an incomplete set, and similarly for each counterexample infinite tuple $\bar{t}_c$. Formally: for each pair $< \bar{t}_{nc}, \bar{t}_c >$ there exists a level which is equal to the maximum of the marks of $\bar{t}_{nc}$, $\bar{t}_c$ such that, for all greater levels i, $A(\bar{t}_{nc}(i)) \neq A(\bar{t}_c(i))$.

However, this fact does not contradict (1) or (2), because as i increases, there is always, by "Lemma 5.0" on page 16, a residue of complete counterexample prefixes and a residue of complete non-counterexample prefixes.

It is worth investigating where "Lemma 18.0: Statement and Proof" on page 93 can give us a contradiction despite this fact. That lemma states that, for each range element y (for example, the range element 1), and for each exponent sequence A, there exists an x that maps to y via A possibly followed by a single buffer exponent.

Show That Lemma 5.0 Makes Counterexample Tuples Impossible

(*Note*: we now believe this implementation will not work. The reason is given at the end of this sub-section. We describe the implementation just in case a reader might see a way to overcome the flaw in our argument.).

1. "Lemma 5.0" on page 16 states that if counterexamples exist, then each tuple-set contains an infinity of counterexample tuples and an infinity of non-counterexample tuples.

2. Assume counterexamples exist. Then by "Lemma 8.7" on page 23, there exists a minimum level i_0 such that for all greater levels i, the set of i-level non-counterexample anchor tuples is incomplete (because some i-level exponent sequences are associated with counterexample anchor tuples and therefore not with non-counterexample tuples).

3. Each domain element x of the $3x + 1$ function is eventually the first element of an anchor tuple, because for each such x, there exists a minimum i such that x is less than the distance between first elements of i-level tuples successive at level i in some i-level tuple-set. That is, there exists a minimum x such that

$$x < 2 \cdot (2^{a_2})(2^{a_3})\dots(2^{a_i})$$

(See "Lemma 1.0" on page 12.)

If t_c is an i-level counterexample anchor tuple in a tuple-set T_A, where $A = \{a_2, a_3, a_4, ..., a_i\}$, let $t_c(1, j)$, $1 \leq j \leq i$, and $i \geq i_0$, denote the prefix of t_c consisting of the elements at levels 1 through j. Let $A(t_c(1, j))$ denote the exponent sequence associated with that prefix.

Since the set of non-counterexample anchor tuples is incomplete at level $i \geq i_0$, there must be a prefix $A(t_c(1, j))$ of the i-level exponent sequence associated with at least one i-level counterexample anchor tuple t_c that differs from the corresponding prefix of *all* i-level exponent sequences associated with i-level non-counterexample tuples. That is, there must be a prefix $t_c(1, j)$ such that, for *all* i-level non-counterexample anchor tuples t_{nc}, $A(t_c(1, j)) \neq A(t_{nc}(1, j))$.

However, by Lemma 5.0, for some larger $i = i'$, there will be at least one non-counterexample anchor tuple t_{nc}' such that $A(t_{nc}'(1, j)) = A(t_c(1, j))$. Therefore, to maintain the necessary incompleteness property of non-counterexample anchor tuples at all levels greater than i_0, there must be an i'-level counterexample anchor tuple t_c' such that for *all* i'-level non-counterexample anchor tuples t_{nc}', $A(t_c'(1, j')) \neq A(t_{nc}'(1, j'))$, where $j' > j$.

This argument is repeated for prefixes at each level i, where i increases without limit. Thus, the differing counterexample prefixes continue to grow longer and longer.

4. .But if the exponent sequence associated with an infinite counterexample tuple differs from the exponent sequences associated with all non-counterexample infinite tuples, as must be the case for the incompleteness property to hold at all levels $i \geq i_0$, then the first level at which this difference occurs is fixed, and holds for all larger levels. It is nonsensical to speak of the difference as somehow increasing for a fixed counterexample infinite tuple. This contradiction gives us a proof of the $3x + 1$ Conjecture.

Now to the flaw in this argument. That there *is* a flaw is made clear immediately upon considering the $3x - 1$ function. A counterexample, namely, 5, appears already at level 2. And so the set of i-level non-counterexample anchor tuples is incomplete for all levels i, where $i \geq 2$. And yet according to our argument, this cannot happen!

The flaw can be described via the following example. Consider the 3-level counterexample anchor tuple <5, 7, 5>, which is associated with the exponent sequence $A = \{1, 2\}$. Although there can be no 3-level non-counterexample anchor tuple that is associated with A (because that would imply there are two anchor tuples in the same tuple-set, which is impossible), Lemma 5.0 states that at *some* level i', there will be at least one i'-level non-counterexample anchor tuple t_{nc} such that $A(t_{nc}(1, 3)) = \{1, 2\}$. Lemma 5.0 specifies nothing about the exponent sequence associated with the rest of t_{nc}, that is, associated with the suffix exponent sequence $A(t_{nc}(4, i'))$. And so it is entirely possible that there exists an i'-level counterexample anchor tuple t_c such that $A(t_c(1, 3)) = A(t_{nc}(1, 3)) = \{1, 2\}$.

A strategy that is closely related to the one we have described applies the same reasoning to *suffixes* of non-counterexample anchor tuples. By “Lemma 18.0: Statement and Proof” on page 93, if y is a range element (non-counterexample or counterexample) then for each exponent sequence A, there exists an x that maps to y via A, possibly followed by one “buffer” exponent. Thus, for each non-counterexample range element y, all exponent sequences A map to y via A, with the possibility of the additional buffer exponent. Thus either the exponent sequences of counterexample and non-counterexample anchor tuples differ only in their last exponent, or the exponent sequences of *prefixes* of counterexample anchor tuples are “pushed down” without limit, since these prefixes offer the only possibility of exponent sequences of counterexample anchor tuples differing from the exponent sequences of non-counterexample anchor tuples.

The flaw is similar to that for the previous strategy in this sub-section, and is exemplified by the existence of counterexamples to the $3x - 1$ Conjecture.

Show That No Counterexample Anchor Tuple Exists (Flawed Strategy)

The following reasoning is not correct, but we offer it to show the strategy implementation we have in mind. Perhaps the reader can find a way to correct the error.

Clearly the odd, positive integer 1 is an anchor (hence a range element) in each tuple-set T_A such that 1 is mapped to by any exponent A. By “Lemma 18.0: Statement and Proof” on page 93, we know that, for each range element y (whether non-counterexample or counterexample) and for each exponent sequence A, there exists an x that maps to y via A possibly followed by a “buffer” exponent. Now 1 is mapped to by all even exponents. Therefore, for all $i \geq 2$, 1 is mapped to by all $(i+1)$-level exponent sequences $A^*\{a_{i+1}\}$, where A is any i-level exponent sequence, and a_{i+1} is an even exponent.

This means that no counterexample anchor can be mapped to by one of these exponent sequences. (Otherwise, a tuple-set would have a non-counterexample anchor tuple and a counterexample anchor tuple, which is impossible.) The error in our reasoning occurs in the next sentence.

But then, since Lemma 18.0 applies to counterexamples as well as non-counterexamples, this means that for all counterexample anchors y_c, y_c must be mapped to by an odd exponent only.

This statement is erroneous because it is possible that, for each exponent sequence A, if $A^*\{a_{i+1}\}$maps to 1, where a_{i+1} is even, then $A^*\{a_{i+1}'\}$maps to y_c, where y_c is mapped to by even exponents, a_{i+1}' is therefore even *but* $a_{i+1}' \neq a_{i+1}$. (If y_c is mapped to by odd exponents, then $A^*\{a_{i+1}\}$affords no potential contradiction, because a_{i+1} is even and a_{i+1}' is odd.)

Show That No Counterexample Anchor Tuple Exists (Second Flawed Strategy)

1. We begin by reviewing the following facts:

- Each odd, positive integer, whether non-counterexample or counterexample, is the first element of an infinite tuple.
- Each infinite tuple has a mark m. By definition, this means that for each infinite tuple, there is a least level (namely, m) at which a prefix of the infinite tuple is an anchor tuple (first m-level tuple in an m-level tuple-set). All greater-level prefixes are likewise anchor tuples.
- A finite prefix of an infinite tuple we call simply a *tuple*.

2. For each level i, where $i \geq 2$, and for each i-level tuple-set, form the set S_i = the set of all non-counterexample infinite tuples having mark $\leq i$. Each set S_i is non-empty because the set S_2 is non-empty (all 2-level anchor tuples are non-counterexample tuples).

Assume counterexamples exist. Then for each S_i , form the set U_i consisting of the set of all pairs $<t_{nc}, t_c>$, where t_{nc} is an i-level non-counterexample tuple and t_c is an i-level counterexample tuple such that the exponent sequences associated with the two tuples are the same. We know that this pairing is always possible by "Lemma 5.0" on page 16.

3. We now ask about the mark of each infinite counterexample tuple whose i-level prefix is the i-level counterexample tuple in a pair $<t_{nc}, t_c>$. Clearly, the mark must be greater than i, because if it were less than or equal to i, that would mean there are two anchor tuples in the same tuple-set, which is impossible, by definition of *anchor tuple*.

4. But clearly step 3 implies that no counterexample infinite tuple has a finite mark, which means no counterexample infinite tuple exists, hence neither do any counterexamples.

We must now check our argument by interchanging the terms "non-counterexample" and "counterexample" in step 2. We know that, based on computer tests, all S_i for $2 \leq i \leq 35$ are empty. But there must be an infinity of consecutive levels $i > 35$ such that S_i is not empty. Our conclusion must be that no *non*-counterexample infinite tuple has a mark, and hence that non-counterexamples do not exist, which we know is false.

We conclude that counterexamples do not exist.

The error in this reasoning is that it is entirely possible for one or more exponent sequences to be missing from the exponent sequences associated with the non-counterexample tuples in some S_i (step 2). The only possible reason is that such exponent sequences are associated with counterexample anchor tuples. And so the proof fails at that point.

Show That No Counterexample Anchor Tuple Exists (Third Flawed Strategy)

1. If a counterexample exists, then for all levels i greater than some minimum i, the set S of all first i-level tuples in all i-level tuple-sets consists of counterexample and non-counterexample tuples.

2. No two tuples in S can be associated with the same exponent sequence because that would mean that two i-level tuple-sets were defined by the same exponent sequence, which is impossible by definition of tuple-set.

3. By "Lemma 18.0: Statement and Proof" on page 93, we know that, for each range element y (whether non-counterexample or counterexample) and for each exponent sequence A, there exists an x that maps to y via A possibly followed by a "buffer" exponent.

4. But then by step 3 there exist exponent sequences that do not map to 1, namely, the exponent sequences that map to i-level elements of counterexample tuples in S. This contradiction to Lemma 18.0 gives us our proof.

The Flaw In The No-Counterexample-Anchor-Tuple-Exists Strategy

The error in our reasoning is as follows: just because there does not exist a non-counterexample i-level first tuple associated with a certain i-level exponent sequence A, in the set of all i-level tuple-sets, does not mean that there does not exist a non-counterexample i-level tuple associated with A. In fact, "Lemma 5.0" on page 16 tells us that there is an infinity of such i-level tuples in each i-level tuple-set!

So there is no violation of Lemma 18.0.

Show That the Set of all Tuple-sets Is the Same Whether or Not Counterexamples Exist

This is the strategy we use in our proof of the $3x + 1$ Conjecture. See our paper, "A Solution to the 3x + 1 Conjecture", on occampress.com.

Earlier remarks on this strategy are the following.

Objections to the *comparison* of the two cases, counterexamples exist, and counterexamples do not exist, have fallen into several categories. These are listed in our paper, "Is It Legitimate to Begin a Sentence With 'If Counterexamples Exist, Then...' " on occampress.com, along with our replies to the objections.

Perhaps the strategy will be more convincing if the reader considers a version of the $3x + 1$ function that initially acts *simultaneously* on the entire set of odd, positive integers. Then, if the exponent is 1, the result is the *set* of range elements congruent to 5 mod $2 \cdot 3^{(2-1)} = 5$ mod 6. If the exponent is 2, the result is the *set* of range elements congruent to 1 mod $2 \cdot 3^{(2-1)} = 1$ mod 6..

We can designate this initial behavior of the $3x + 1$ function as $\mathbf{C}_{\{1\}}(\mathbf{x}) = \mathbf{y}$ in the first case, and as $\mathbf{C}_{\{2\}}(\mathbf{x}) = \mathbf{y}'$ in the second case.

We then apply $\mathbf{C}$, the set-argument version of the$3x + 1$ function, to the set $\mathbf{y}$ or the set $\mathbf{y}'$, for any exponent a_3, and again arrive at a set of range elements, in this case, a set whose elements are congruent mod $2 \cdot 3^{(3-1)} =$ mod 18. And so on.

It should be clear that this process always yields the same results (the same sets of range elements) regardless if counterexamples exist or not.

Show That If All *i*-Level Anchors are Non-Counterexamples, Then So Are All (*i*+1)-Level Anchors

If the following Conjecture is true, it will directly imply the truth of the $3x + 1$ Conjecture, since computer tests show that the set of all odd, positive integers < $2 \cdot 3^{35-1}$ are non-counterexamples, so that Conjecture R1 would allow a simple inductive proof that all odd, positive integers map to 1.

Conjecture R1.

Let S_i denote the set of odd, positive integers that are less than $2 \cdot 3^{i-1}$. This set consists of all i level anchors (these contain no multiples of 3), plus all multiples of 3 in the range given.

Let S'_i denote the set of all x that map to elements of S_i . Then S'_i includes the set of all odd, positive integers y such that $2 \cdot 3^{i-1} < y < 2 \cdot 3^i$.

Discussion of possible proof: It seems that the proof would be laborious but not conceptually difficult. Furthermore, we have examples to guide us. Thus, consider S_3 = the set of all odd, positive integers that are less than $2 \cdot 3^{3-1} = 18$, that is, the set $S_3 = \{1, 5, 7, 9, 11, 13, 15, 17\}$. Now since if any element of a "spiral" maps to a non-counterexample, then all elements of the "spiral" do, we obtain immediately from S_3 the odd, positive integers 4•5 + 1 = 21, 4•7 + 1 = 29, 4•9 + 1 = 37, and 4•11 + 1 = 45. These integers lie between $2 \cdot 3^{3-1} = 18$ and $2 \cdot 3^{4-1} = 54$ and hence are in S'_3.

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Possible Strategies Based on the Minimum Counterexample

If a counterexample exists, then there is a minimum counterexample. For details on the possibility of proving the $3x + 1$ Conjecture from this fact, see "Strategy of Proving There Is No Minimum Counterexample" in the first file of our paper, "The Structure of the $3x + 1$ Function: An Introduction" on occampress.com.

Possible Strategies Using Induction on Anchors

As a result of computer tests of the $3x + 1$ function[1], we can state that, for all i, where $2 \le i \le 35$, all i-level anchors are non-counterexamples. This fact suggests the possibility of an inductive proof of the $3x + 1$ Conjecture. Such a proof would first define a function $t(i)$ that would yield, for each i, (1) the number of levels down ("depth") in the infinite set of recursive "spiral"s[2] with base element 1 and (2) the maximum number of branches to the right ("width") for each node, that would yield all the anchors for level i.

Clearly, the computer would be a virtual necessity in arriving at a formula for the value of $t(i)$ for each i. As a start, we might try a depth and width of $2 \cdot 3^{i-1}$ — as long as the value of the depth and width is finite, the value doesn't matter.

One of the longest tuples that begins with a small odd, positive integer is the tuple whose first element is 27. The tuple <27, 41, 31, ..., 1> has 42 elements. Since 27 is a level 4 anchor, our tentative depth formula works for it, since the depth for level 4 is $2 \cdot 3^{4-1} = 54$, which is greater than 42.

In general, assuming, for the moment, no multiples of 3, the number of nodes (odd, positive integers) for a depth of k and a width of w is $1 + w + w^2 + w^3 + ... w^k = (w^{k+1} - 1)/(w - 1)$.

The inductive proof would then show that for each i, the set of anchors for all i-level tuple-sets, consists solely of non-counterexamples. This would then imply that no counterexamples exist.

Possible Strategies Based on Recursive "Spiral"s

Note: A strategy that we believe yields a proof of the $3x + 1$ Conjecture, is the basis of "Third Proof of $3x + 1$ Conjecture" in our paper, "A Solution to the $3x + 1$ Problem", on occampress.com.

We repeat what we said at the start of the section "Possible Strategies Based on Tuple-sets" on page 52: "Tuple-sets have one important advantage over recursive 'spiral's, namely, that if a counterexample exists, then by "Lemma 5.0" on page 16, each tuple-set contains a countable infinity of counterexample tuples, as well as a countable infinity of non-counterexample tuples. On the other hand, an infinite set of recursive 'spiral's with base element 1 cannot contain *any*

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1. See, e.g., Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html.
2. See "Section 2. Recursive 'Spiral's" in the paper, "The Structure of the $3x + 1$ Function: An Introduction" on the web site www.occampress.com

counterexample tuples, and an infinite set of recursive 'spiral's with base element a counterexample, cannot contain *any* non-counterexample tuples."

It is important to keep in mind that the structure of the infinite set of recursive "spiral"s for the $3x - 1$ function is the same as that for the $3x + 1$ function. But the distance in the former case between successive elements x, x' of a given "spiral" is $x' = 4x - 1$, whereas for the $3x + 1$ function it is $x' = 4x + 1$. Thus, for example, in the $3x - 1$ function, the "spiral" of elements mapping to 1 is 1, 3, 11, 43, ..., and $3 = 4(1) - 1$, $11 = 4(3) - 1$, etc.

Yet this minor difference is sufficient to allow counterexamples in the case of the $3x - 1$ function, and furthermore, counterxamples that appear early (5, 7 are two counterexamples). But no counterexample to the 3x + 1 function is known in all odd, positive integers up to well above 10^{15}.

It is also important to keep in mind that if counterexamples to the $3x + 1$ function exist, then for each range element y in the countable infinity of counterexample range elements, there exists an infinite set of recursive "spiral"s with base element y. Furthermore, this set has the same structure, and the same distance between successive elements x, x' of each "spiral", namely, $x' = 4x + 1$.

A problem that occurs in the contemplation of the infinite resursive "spiral"s with base element 1 is that of determining the set of first elements of all "spiral"s. For example, 1 is such a first element (the elements of the "spiral" are 1, 5, 21, 85, ...), as is 3 (the elements of the "spiral" are 3, 13, 53, 213, ...), as is 7 (the elements are 7, 29, 117, ...), ...

It turns out that tuple-sets provide us with a valuable first step toward determining the set of first elements of all "spiral"s. The reason is as follows. The first element of a "spiral" maps to a range element (in one iteration of the $3x + 1$ function) via the exponent 1 or via the exponent 2. But the set of all 2-level tuple-sets in the tuple-set T_A, where $A = \{1\}$ is the set of all 2-tuples that map to a range element via the exponent 1, and similarly for the set of all 2-level tuple-sets in the tuple-set $T_{A'}$, where $A' = \{2\}$. So the set of first elements of all "spiral"s in the infinite set of recursive "spiral"s whose base element is 1 is a subset of the set of all first elements of 2-tuples in $T_A \cup T_{A'}$. The subset is a proper subset only if counterexamples exist.

Of course, also to be kept in mind is that the set of odd, positive integers that directly or indirectly map to 1 is the same regardless if counterexamples exist or not (the laws of arithmetic are not subject to the truth or falsity of the $3x + 1$ Conjecture). See "Lemma 8.8" on page 32. T

Obviously, we would have our proof of the $3x + 1$ Conjecture if there existed a closed-form function that, for any infinitary tree generated by a single rule, e.g., the definition of the $3x + 1$ function or of the $3x - 1$ function or of the $3x + C$ function, where C is an integer, would return a finite representation of the set of all elements at all nodes of the tree. But so far as we know, no such function has yet been discovered.

A Proof of the 3*x* + 1 Conjecture Using Recursive "Spiral"s

See our paper, "A Solution to the $3x + 1$ Problem".

The rest of this section was written before the above proof was discovered.

The Simplest Strategy Using Recursive "Spiral"s

The simplest strategy is based on the following facts:

• Every non-counterexample is an element of the 1-tree;

• For each finite exponent sequence (possibly followed by a buffer exponent — see "Lemma 18.0: Statement and Proof" on page 93), there exists a path upward — toward the root of the 1-tree — that is associated with this sequence. (The path is, of course, a tuple.) In other words, by abuse of language, we can say that "every exponent sequence maps to 1", or, in other words, "every exponent sequence is associated with a non-counterexample anchor tuple". But this we already knew, from "Lemma 5.0" on page 16.

• Each tuple is the prefix of an infinite tuple, and each infinite tuple has a minimum prefix that is an anchor tuple. (By definition, the level of the last element of that minimum prefix is the *mark* of the infinite tuple.) All extensions of that anchor tuple are also anchor tuples. So, by abuse of languge, we say that "every tuple is eventually an anchor tuple, and remains one thereafter (that is, for all higher levels)".

Then our task is simply to show that all this implies that every anchor tuple is a non-counterexample anchor tuple. One obstacle that must be overcome is the presence of possible buffer exponents. In brief, we must eliminate the possibility that the exponent sequence associated with each counterexample anchor tuple, always differs in at least the last exponent, from the exponent sequence associated with a non-counterexample anchor tuple at the same level. If that were so, then counterexample anchor tuples could co-exist with non-counterexample anchor tuples at any level i, and that would deprive us of a contradiction that would prove the Conjecture.

Whatever we come up with, we must remember that it must pass the 3x — 1 Test (see "Definition of the "3x – 1 Test"" on page 104). This would seem to be especially challenging, since it appears that "every exponent sequence maps to 1" in the case of the $3x – 1$ function, as in the case of the $3x + 1$ function.

Strategy Based on a Set Equation

See "Apppendix B — Analysis of a Failed Strategy" on page 96.

Strategy of "Filling-in" of Intervals in the Base Sequence Relative to 1

At present, we believe there is at least one mathematician (and probably several) in the world who could, from the material in this section, either construct a proof of the $3x + 1$ Conjecture, or make a major, publishable, advance toward such a proof.

This strategy was first discussed in the section, "Strategy of "Filling-in" of Intervals in the Base Sequence Relative to 1", in the first file of our paper, "The Structure of the $3x + 1$ Function: An Introduction" on occampress.com.

Definition of "Filling-in" Strategy

The Filling-in Strategy is described by the following conjecture, which is clearly equivalent to the $3x + 1$ Conjecture:

Conjecture 4.0

Each interval in the base sequence relative to 1, that is, in the sequence $S_1 = \{1, 5, 21, 85, 341, ..., \}$, is eventually filled by elements that map to 1.

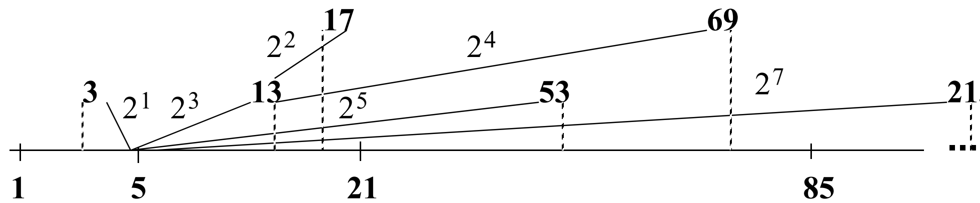


Fig. 5. Illustration of part of the "filling-in" process.

The reason we are motivated to attempt a proof of Conjecture 4.0 is the following fact. Suppose y is non-counterexample range element. Then a "spiral" maps to y in one iteration of the $3x + 1$ function. *Each successive element of the "spiral" is an element of successive intervals in S_1* (Lemma 14.0 in the section, "Three Important Lemmas", in the first file of the paper, "The Structure of the $3x + 1$ Function: An Introduction" on occampress.com). This is true for all non-counterexample range elements. (The same fact holds for each counterexample range element, if counterexamples exist.)

Since we know, by "Lemma 13.0: Statement and Proof" on page 89, that a countable infinity of range elements map to 1, we might wonder (naively) why that does not give us a proof of the $3x + 1$ Conjecture. The answer is the "forward-movement" problem, which we now explain.

The "Forward-Movement" Problem

Suppose we have a succession of intervals, each four times larger than the previous. In particular, I_{1+} contains 2 elements, I_{2+} contains 8 elements, I_{3+} contains 32 elements, etc. It is easy to see that interval I_{i+} contains 2^{2i-1} elements. (The reason for the "+" is that each of these intervals contains the element of $S_1 = \{1, 5, 21, 85, 341, ..., \}$ that immediately precedes it. This convention will be of use to us, as the reader will learn below.)

Suppose we have an infinite sequence of "spiral"s $s_1, s_2, s_3, ...$ as follows:

The "spiral" s_1 places one element in each of $I_{1+}, I_{2+}, I_{3+}, ...$ Thus each interval from I_{1+} on contains one element.

The "spiral" s_2 places one element in each of $I_{2+}, I_{3+}, I_{4+}, ...$ Thus each interval from I_{2+} on contains two elements.

The "spiral" s_3 places one element in each of $I_{3+}, I_{4+}, I_{5+}, ...$ Thus each interval from I_{3+} on contains three elements.

Etc.

It is clear that, because of the exponential growth in the size of intervals, no interval will ever be filled with “spiral” elements. The reason is that the “spiral”s “move forward” too rapidly to fill in any interval.

On the other hand, suppose we have an infinite series of “spiral”s s_1', s_2', s_3', ... as follows:

The “spiral” s_1' places one element in each of I_{1+}, I_{2+}, I_{3+}, ... Thus each interval from I_{1+} on contains one element.
The “spiral” s_2' places one element in each of I_{1+}, I_{2+}, I_{3+}, ... Thus each interval from I_{1+} on contains two elements.
The “spiral” s_3' places one element in each of I_{1+}, I_{2+}, I_{3+}, ... Thus each interval from I_{1+} on contains three elements.
Etc.

Clearly, all intervals will eventually be filled with “spiral” elements. There is no forward movement problem. However, this process does not apply to the $3x + 1$ function.
What we would like is for the number of “spiral” elements in each interval to increase like the left-hand side of the following equations:

In interval I_{1+}: $1 + 1 = 2^{2(1) - 1} = 2$;
In interval I_{2+}: $1 + 1 + 2^1 + 2^2 = 2^{2(2) - 1} = 8$;
In interval I_{3+}: $1 + 1 + + 2^1 + 2^2 + 2^3 + 2^4 = 2^{2(3) - 1} = 32$;
Etc.

Here, there is no foreward-movement problem. Clearly, each interval is eventually filled in with “spiral” elements.

Our goal, in the Filling-in Strategy, is to show that in fact this is the case for the $3x + 1$ Problem. But if we are to achieve this goal, we need to have before us some facts about “spiral”s, intervals, and levels (the last term to be defined below). We now provide these facts.

Here are the initial facts that we must deal with in connection with the forward-movement problem:
Let y be a range element in an interval of the sequence $S_1 = \{1, 5, 21, 85, 341, ... \}$ of elements that map to 1 in one iteration of the $3x + 1$ function. (The next sub-section has details on intervals.) Then y is mapped to either by all odd exponents or by all even exponents. For each case, there are three possibilities for the first three elements of the “spiral” that maps to y:
3, e, o;
o, 3, e;
e, o, 3.

where “3” means that the “spiral” element is a multiple of 3, and hence not a range element; “e” means the “spiral” element is mapped to by all even exponents; “o” means the “spiral” element is mapped to by all odd exponents.
The first two possibilities are the worst cases, because, in the case (3, e, o), it means that for the third “spiral” element x we have $(3x + 1)/2^5 = y$, or $x \approx (32/3)y$, or x lies between $10y$ and $11y$.

Now $4y + 1$ is in the next interval, $4(4y + 1) + 1 = 16y + 4 + 1$ is in the interval after that. So x is in the second interval forward from that of y. But since x is mapped to by all odd exponents, it is mapped to by the exponent 1, so the x' that maps to x in this case yields a smaller number than x, which is in our favor.

In the even exponent case (o, 3, e), we have, for the third "spiral" element, $(3x + 1)/2^6 = y$, or $x \approx (64/3)y$, or x lies between $21y$ and $22y$. Now $4y + 1$ is in the next interval, $4(4y + 1) + 1 = 16y + 4 + 1$ is in the interval after that, and $4(4(4y + 1) + 1) + 1 = 64y + 16 + 4 + 1$. So x is in the second or third interval forward from that of y. But since x is mapped to by all even exponents, it is mapped to by the exponent 2, so the x' that maps to x in this case yields a larger number than x, which is not in our favor.

In the worst case, the forward-movement could be three intervals *per descent in level* from y. But the forward-movement for the other cases will be less, and we must always keep in mind that if we subtract a finite number of intervals from a countable infinity of successive intervals, we are left with a countable infinity of successive intervals. However, if the size of the intervals is growing exponentially, then we may be leaving ample space for counterexample elements in these intervals. Which brings us to the following fact:

Fundamental Fact of the Moving-Forward Problem

If the filling-in rate is greater than or equal to the moving-forward rate, then there is no moving-forward problem.

By this we mean the following: since each interval contains four times the number of elements in the previous interval, if the filling-in rate is greater than four times per interval, then there is no moving-forward problem. See (2) under "The Most Promising Implementations of the Filling-in Strategy", below.

"Spiral"s, Intervals, and Levels

We begin by repeating the definition of "spiral" (see "Section 2. Recursive 'Spiral's" in the first file of the paper, "The Structure of the $3x + 1$ Function: An Introduction", on the website occampress.com): if y is a range element, then the set of x that map to y in one iteration of the $3x + 1$ function is a "spiral".

There are an infinite number of x in each spiral. These x map to y either by all odd exponents or by all even exponents (Lemma 13.0). Thus the first element of the spiral maps to y by either the exponent 1 or by the exponent 2.

If y is a non-counterexample, then all x in the "spiral" mapping to y are non-counterexamples. (And similarly for counterexamples.)

The set of first elements of all non-counterexample "spiral"s is a subset of the set of first elements of all the 2-tuples in all the 2-level tuple-sets $T_{\{1\}}$ and $T_{\{2\}}$. For further details, see "Strategy Based on the Application of "Spiral"s to 2-level Tuple-sets" in the first file of the paper, "The Structure of the 3x + 1 Function: An Introduction" on occampress.com)

The distance function on "spiral"s is as follows: if x, x' are successive elements of a "spiral", then $x' = 4x + 1$ (Lemma 11.0 in the first file of our paper, "The Structure of the $3x + 1$ Function: An Introduction" on occampress.com).

Let $S_1 = \{1, 5, 21, 85, 341, ... \}$. This is the "spiral" that maps to 1 in one iteration of the $3x + 1$ function.

Each "spiral" — including the first element of each "spiral" — that maps, directly or indirectly, to 1 is a descendant of exactly one element of $S_1 = \{1, 5, 21, 85, ... \}$.

We say that an odd, positive integer that maps to 1 in k iterations of the function is *at level k*. The level of a "spiral" is the level of its first element. Call the set of "spiral"s that map directly or indirectly to 1, the 1-*tree*. Since the 1-tree is oriented vertically, we will speak of a level that is "lower" than a given level, or a certain number of levels "down" from a given level, even though the level number is higher (larger).

The set of all k-level non-counterexamples is the set of all odd, positive integers that are the first elements of all anchor tuples in all $(k + 1)$-level tuple-sets such that the anchor ($(k + 1)$-level element) of the anchor tuple is 1. Thus, the entire 1-tree consists of all tuples having 1 in some extension.

As we descend through successive levels in the 1-tree, the thought might occur to us: How is this descent possible, given that we cannot go into the odd, negative integers, since they are not elements of the domain or range of the $3x + 1$ function? The answer is that in general, the descent yields larger and larger numbers: all exponents greater than 1 have that effect. For example, the level-2 range element 13 is mapped to by the level-3 range element 277 via the exponent 6. We must also keep in mind that the set of all integers at each level is simply a subset of the odd, positive integers. The $3x + 1$ function, both in the "upward" or "forward" direction and in the "downward" or "inverse" direction, can be thought of as merely re-arranging, at each level, a subset of the odd, positive integers (Lemma 4.75 in "A Solution to the 3x + 1 Problem" on occampress.com).

Let I_i, where $i \geq 1$, denote the ith interval in S_1. Thus $I_1 = \{3\}$, $I_2 = \{7, 9, 11, 13, 15, 17, 19\}$.

Let I_{i+}, the "expanded interval", where $i \geq 1$, denote I_i preceded by the ith element of S_1. Thus $I_{2+} = \{5, 7, 9, 11, 13, 15, 17, 19\}$.

Let $|I_i|$ denote the number of elements in I_i. Then $|I_i| = 2^{2i-1} - 1$.

Let $|I_{i+}|$ denote the number of elements in I_{i+}. Then $|I_{i+}| = 2^{2i-1}$ and $|I_{(i+1)+}| = 4|I_{i+}|$.

A total of $|I_{i+}|$ "spiral"s are represented in I_{i+}. Each "spiral" has exactly one element in I_{i+}.

If all intervals from $|I_{1+}|$ through $|I_{k+}|$ are filled with non-counterexamples then:

- There is one element in $|I_{k+}|$ for each "spiral" that started in *any* interval $|I_{1+}|$ through $|I_{k+}|$. This means that, for example, for each "spiral" s that started in interval I_{2+}, there exists a countable infinity of successive intervals each of which contains an element of s.
- Thus, for each j, the $|I_{j+}|$ elements in I_{j+} consist of one element from each of the $|I_{j-1+}|$ "spiral"s having elements in I_{j-1+}, plus one element from each of the "spiral"s that start in I_{j+}. There must be $3\,|I_{j-1+}|$ of these latter, new "spiral"s, since $|I_{j+}| = 4\,|I_{j-1+}| = |I_{j-1+}| + 3\,|I_{j-1+}|$. But some of these might be "spiral"s from deep descendants of much earlier "spiral"s.

By computer test, we know that at least the first 26 intervals are filled with non-counterexmples. There are thus elements of $2^{2 \cdot 26-1}$ "spiral"s in I_{26+}. Not all of these "spiral"s are at the same level, however! Thus, e.g., the "spiral" {3, 13, 53, ... }, which has an element in I_{26+}, is at level 2, because 3 maps to 1 in two iterations of the $3x + 1$ function. But the "spiral" {7, 29, 117, ... }, which also has an element in I_{26+}, is at level 5, because 7 maps to 1 in five iterations of the $3x + 1$ function. There exists a maximum level "spiral" in each interval, hence in I_{26+}. Since it is known, from computer tests, that the anchors of all 35-level tuple-sets are non-counterexamples, this suggests that the maximum level "spiral" in I_{26+} is at about level 35.

Successive elements of a "spiral" are in successive intervals I_{i+} (Lemma 14.0 in the section "Three Important Lemmas" in the first file of the paper, "The Structure of the 3x + 1 Function: An Introduction" on occampress.com).

The elements of a "spiral" follow the pattern ...3, *e*, *o*, 3, ..., where "3" means: "multiple-of-3, hence not mapped to by any odd, positive integer"; "*e*" means "mapped to by all and only exponents of even parity"; "*o*" means "mapped to by all and only exponents of odd parity". (See proof of Lemma 18.0 in "A Solution to the $3x + 1$ Problem" on occampress.com.)

As we move forward through the elements of a "spiral", each occurrence of an "e" element means that we have added one element to *each* of a countable infinity of successive intervals, and similarly for each occurrence of an "o" element. Thus for each triple of "spiral" elements we move through, we have added two elements to *each* of a countable infinity of successive intervals.

In reckoning the "spiral" elements in a given expanded interval I_{i+}, we must include the "spiral" elements generated by the first element of I_{i+}, that is, by the *i*th element of S_1, and also, possibly, the "spiral" elements generated by the first element of $I_{(i+1)+}$.

The most important properties of a "spiral" are: (1) the first element; (2) the level of the first element, hence of the "spiral"; (3) the element of the "spiral" $S_1 = \{1, 5, 21, 85, ... \}$ from which the first element, hence all elements, are descended; (4) the parity of the exponents mapping to the base element of the "spiral".

Downward Extensions of "Spiral"s in Triples of Intervals and of "Spiral" Elements

In each consecutive triple of successive intervals, we know that each "spiral" yields one range element that is mapped to by even exponents only, and one range element that is mapped to by odd exponents only. The reason we know this is that the pattern of successive elements in any "spiral" is ...3, e, o, 3..., as we explained in the previous sub-section.

From this fact, we can construct, for each "spiral" having elements in each interval of a "triple", a binary tree of unlimited depth. Here is how the construction works.

Let I_{k+} denote the largest interval that contains solely non-counterexamples. By computer tests, we know that $k > 26$. Let a equal the number $|I_{k+}|$ of elements in I_{k+}. We know that $a = 2^{2k+-1}$.

We also know that the infinity of successive intervals following I_{k+} each contains an element of each of the a spirals having elements in I_{k+}.

Let *trip(k, n)* denote the *n*th triple of successive intervals following the interval I_{k+}.

Let s be a "spiral" having an element in each of the successive intervals in *trip*(k, 1). Two of these elements are range elements. One is mapped to by all even exponents (thus establishing a new "spiral" s_1) and the other is mapped to by all odd exponents (thus establishing a second new "spiral" s_1'). Each of these two new "spiral"s consists of an infinity of successive triples of elements. In the first triple of each "spiral", s_1 and s_1', there are two range elements. One is mapped to by all even exponents (thus establishing a new "spiral" s_2) and the other is mapped to by all odd exponents (thus establishing a second new "spiral" s_2'). Each of these two "spiral"s consists of an infinity of successive triples of elements. Etc.

So if we descend *j* levels, we establish $2^1 + 2^2 + 2^3 + ... + 2^j = 2^{j+1} - 2$ new "spiral"s.

Therefore, each of the three intervals in the next triple, that is, in *trip*(*k*, 2), and all subsequent triples, contains $a + a(2^{j+1} - 2)$ non-counterexample elements, each an element of a separate "spiral". However, by the reasoning we have just demonstrated, there are in addition $(a + a(2^{j+1} - 2))(2^{j+1} - 2)$ additional non-counterexample elements in *trip*(*k*, 2), each an element of a separate "spiral". Each of these elements is an element of a new "spiral" which sends an element into each of countable infinity of successive intervals.

We can proceed like this without limit. Each of the three intervals in each new triple contains the number of elements *m* in the previous triple, plus $(2^{j+1} - 2)m$ new elements, each of which is an element of a separate "spiral".

Since the number of elements in the third interval of each triple is 4^3 times the number in the third interval of the previous triple, and since $4^3 = 2^6$, it seems we might have some hope of proving that an interval beyond I_{k+} is completely filled with non-counterexamples, thus proving the 3*x* + 1 Conjecture. The reason this would constitute a proof is that the successive elements of at least one counterexample "spiral" would have to "skip over" the filled-in interval, and that is prohibited by Lemma 14.0 in the section, "Three Important Lemmas", in the first file of the paper, "The Structure of the 3*x* + 1 Function: An Introduction" on occampress.com.

It is important to keep in mind that *arbitrarily deep* downward extensions of *each* extended interval I_{1+}, I_{2+}, ... send forth "spiral"s to an infinity of successive intervals. We have not considered these "spiral"s in our discussion up to this point.

The Most Promising Implementations of the Filling-in Strategy

These implementations make clear that we have other options than simply proving that each interval I_{i+} is eventually filled in with non-counterexamples! The implementations are:

(1) Prove that *just one* interval following the first interval in which counterexamples appear, is filled with non-counterexamples. Just one. This contradicts Lemma 14.0 (in the section, "Three Important Lemmas" in the first file of the paper, "The Structure of the 3x + 1 Function: An Introduction" on occampress.com), which implies that no interval can be "skipped over" by successive elements of a (non-counterexample) "spiral".

(2) Prove that the number of non-counterexamples in a countable infinity of successive intervals is always increasing (as a result of the always-increasing number of "spiral"s) as we move through successive triples of intervals. This implies that there cannot be a fixed number of non-counterexamples in these intervals. A fixed number would allow room for counterexamples. The previous section, "Downward Extensions of "Spiral"s in Triples of Intervals and of "Spiral" Elements" on page 70, offers grounds for cautious optimism about this implementation.

(3) Prove that there is "not enough room" for counterexamples in the intervals in the "spiral" $S_1 = \{1, 5, 21, 85, 341, ... \}$ in addition to non-counterexamples. (Clearly, this implementation is

closely related to the previous one.) Specifically, if counterexamples do not exist, then all elements of all intervals are filled with non-counterexamples. If counterexamples do exist, then some elements of some of these intervals are filled with counterexamples. Yet if counterexamples exist, each counterexample y_c is an element of a "spiral". Now if the counterexample y_c is not an element of an infinite cycle, then each iteration of y_c yields an element of another "spiral". And similarly in the downward direction if y_c is a range element (in which case it yields infinities of "spiral"s). If y_c is an element of an infinite cycle, then it appears that further difficulties arise in "finding room" for the "spiral"s that are produced by y_c.

In any case, it seems difficult to believe that all these "spiral"s would be occupied by non-counterexamples if counterxamples did not exist.

Some readers might reply that only one of the two cases "counterexamples do not exist" and "counterexamples exist" holds. They might assert that it is illegitimate and indeed meaningless to speak of counterexamples somehow occupying locations that *would be* occupied by non-counterexamples if counterexamples do not exist. It is entirely possible that counterexamples "make all the room they need" if in fact they exist.

The trouble with this counterargument is that (1) it ignores the fact that for all elements that are known, by computer test, to map to 1 (at least the first 26 intervals of S_1 = {1, 5, 21, 85, 341, ... }) the 1-tree containing these elements is *the same* regardless if counterexamples exist or not. Informally, the two cases are not "disjoint"; (2) it ignores the fact that the intervals in S_1 are the same regardless if counterexamples exist or not. Counterexamples are not some "additional type" of odd, positive integer lying outside these intervals.

(4) Prove that the *density* of odd, positive integers in the 1-tree implies that all intervals in S_1 = {1, 5, 21, 85, ... } are eventually filled in. Informally, the density is the number of odd, positive integers in an "area" of the 1-tree defined by a number of levels and a number of successive "spiral" elements. If the density is sufficiently large, then a range of one or more intervals in S_1, must be filled in.

(5) Another implementation is one based on elementary facts about tuple-sets and recursive "spiral"s. Consider a 2-level element *y* of a 2-level tuple $<x, y>$ in a 2-level tuple-set T_A. This element, being a range element, is mapped to by all exponents of one and only one parity. Furthermore, we know that all *x* in the tuples $<x, y>$ are elements of a "spiral", and are separated by the distance $4x + 1$. Thus, in *each* tuple-set defined by an exponent of that parity, *y* is the second element of a tuple in that tuple-set. If *y* is a counterexample range element, then we immediately have that a countable infinity of 2-tuples are counterexample tuples.

If we can show that if a 2-level tuple *t* in a 2-level tuple-set T_A is a non-counterexample tuple, then the next 2-level tuple t' in T_A is a non-counterexample tuple, we will have our proof of the $3x + 1$ Conjecture.

The 2-level element *y* of a 2-level tuple $<x, y>$ in a 2-level tuple-set is itself a 1-level element of a 2-level tuple $<y, z>$ in some 2-level tuple-set. Thus, for example, 7 is the 2-level element of the 2-level tuple <9, 7> in the 2-level tuple-set T_A, where $A = \{2\}$, and 7 is the 1-level element of the 2-level tuple <7, 11> in the 2-level tuple-set $T_{A'}$, where $A' = \{1\}$.

This strategy is developed under "Strategy Based on the Application of 'Spiral's to 2-level Tuple-sets" in the first file of the paper, "The Structure of the $3x + 1$ Function: An Introduction" on occampress.com.

A Challenge to the Reader

We pose the following challenge to the reader. Let I_{k+} be the largest extended interval that is entirely filled with non-counterexamples. We know, from what we have established in this sub-section, that I_{k+} and each successive interval beyond I_{k+} contains 2^{2k-1} non-counterexamples, each an element of a "spiral". Furthermore, we know that, by the recursive process we have described, each non-multiple-of-3 in the successive intervals beyond I_{k+} gives rise to an infinity of "spiral"s. The challenge is to explain how the interval $I_{(k+1)+}$ could *not* be also filled with non-counterexamples, considering the following facts:

• The set of "spiral"s whose elements fill all extended intervals up to I_{k+} is the same whether or not counterexamples exist;

• The elements of these "spiral"s in all intervals beyond I_{k+} are likewise the same whether or not counterexamples exist;

• The descendants of each range element in each of these "spiral"s are likewise the same whether or not counterexamples exist;

• The number of elements in each expanded interval from the first (I_{1+}) on, is the same whether or not counterexamples exist. Intervals do not somehow "exapand" to accomodate counterexamples.

Other Strategies Based on Recursive "Spiral"s

Other strategies based on recursive "spiral"s are discussed in "Section 2. Recursive 'Spiral's" in the first file of the paper, "The Structure of the 3*x* + 1 Function: An Introduction" and in the second file of "The Structure of the 3*x* + 1 Function", both on the web site occampress.com.

Possible Strategies Based On Topology

These are described in the paper, "The Structure of the 3x + 1 Function: An Introduction", www.occampress.com, in the section "Strategy of Using a Topology Defined on Tuples or Tuple-sets".

Open Questions

At present, there are several Open Questions that we feel are of fundamental importance:

(1) A question that we are sure has been asked ever since the $3x + 1$ Problem was given to the world in the early 1930s, is, informally: "What makes certain odd, positive integers yield 1 — "go to 1" — after repeated iterations of the $3x + 1$ function?" Or, in other words, "Why are there non-counterexamples?" In particular, "Why does the number of iterations differ so much between different odd, positive integers? For example, 3 yields 1 in 2 iterations of the function; 11 yields 1 in 4 iterations; 21 yields 1 in 1 iteration; 27 yields 1 in 41 iterations."

As we recently discovered (and so we now regard Open Question (1) as no longer Open), the answer is remarkably simple: "Because all non-counterexamples are elements of an infinitary tree with 1 as the root (see the section, "Graphical Representation of the Set J as Recursive 'Spiral's" , p. 25 in our paper, "Are We Near a Solution to the $3x + 1$ Problem?" on occampress.com). The set J is the set of all non-counterexamples; a recursive "spiral" is the set of odd, positive integers that map to a non-counterexample range element of the function.

Each (finite) non-counterexample tuple is an upward path in the 1-tree.

In our opinion, the efforts of mathematicians to answer the Question have been hampered by their adhering to the original definition of the $3x + 1$ function, in which each division by 2 is a separate node in the tree representing computations by the function. In our definition, all successive divisions by 2 are collapsed into a single exponent of 2, thus making possible tuple-sets, which reveal the underlying, and, we feel, beautiful, structure of the $3x + 1$function

(2) Why do counterexamples to the $3x – 1$ Conjecture appear already at level 2, whereas no counterexample to the $3x + 1$ Conjecture has been discovered at levels 2 through at least level 35? One answer to this question — though not one that is of the depth that we seek — is given in "Why Are There Counterexamples to the 3x – 1 Conjecture?" on page 110. Essentially, the reason is that at least the counterexamples 5 and 7 simply "fall out of the arithmetic" of the Lemma 1.0 distance functions.

This Open Question also applies to "3x + 1-like" functions (see "Appendix C — "3x + 1 -like" Functions" on page 99). The functions of this type that we have investigated, and that have counterexamples to the corresponding conjecture, all have a counterexample among the small odd, positive integers.

(3) How is it possible that the following two facts hold for the tuple-sets over the odd, negative integers, yet seem not to hold for the tuple-sets over the odd, positive integers?

(I) for each $i \geq 2$, the set of i-level counterexample and non-counterexample negative anchor tuples is complete;

(II) for each $j \geq 2$, there exists an $i > j$ such that the set of j-level prefixes of all i-level negative counterexample anchor tuples is complete, and similarly for the set of j-level prefixes of all i-level negative non-counterexample anchor tuples.

See "How Is the Interleaving of Counterexample and Non-Counterexample Anchor Tuples Possible?" on page 109.

(4) Is it possible for arbitrarily long extensions of a tuple $<y_c>$ to contain no element less than y_c? If it is not, then we have ***a proof of the 3x + 1 Conjecture***, because such a sequence of extensions would define a counterexample tuple, namely, the counterexample tuple generated by the minimum counterexample y_c. (See "Strategy of Proving There Is No Minimum Counterexample" in the first part of our paper, "The Structure of the 3x + 1 Function: An Introduction" on occampress.com.) The attempt to prove no such counterexample tuple exists might be helped by referring to Table 1, "Distances between elements of tuples consecutive at level i" on page 13, and by considering tuple-sets over the negative integers. The $3x - 1$ function, where counterexample tuples are abundant, is the negative of the $3x + 1$ function over the negative integers. It follows easily from the Distance functions ("Lemma 1.0" on page 12), that no infinite counterexample tuple in the $3x + 1$ function, can be associated with the same exponent sequence as an infinite counterexample tuple in the $3x - 1$ function.

(5) What is the relationship between the two structures underlying the $3x + 1$ function, namely, tuple-sets and recursive "spiral"s? By "the relationship" we mean, ideally, a closed form function that takes an *i*-level non-counterexample tuple as input, and shows where this tuple is located in (a) its *i*-level tuple-set and where it is located in (b) the infinite set of recursive "spiral"s with base element 1. A first step toward an answer is given in "Mechanism of the Relationship Finally Discovered" on page 40.

(6) Is it in fact the case that, in the $3x - 1$ function, for all $i \geq 2$, there is exactly one anchor tuple for each *i*-level exponent sequence, hence exactly one *i*-level tuple-set for each *i*-level exponent sequence? (This is the case in the $3x + 1$ function.

(7) Is it ever legitimate, in a proof, to make use of the fact that a certain fact is not known, when an analogous fact is known in a similar problem? For example, we do not know, at the time of this writing, if counterexamples to the $3x + 1$ Conjecture exist. Therefore, we believe it is legitimate to write, "if counterexamples exist..." and "if counterexamples do not exist..."

However, in the case of the $3x - 1$ function, we know that counterexamples exist — 5 is the smallest one. So whereas we can legitimately write, in the $3x - 1$ case, "if counterexamples exist ..." we cannot legitimately write, "if counterexamples do not exist ..." because here the antecedent is false, and, as is often stated informally, "false implies anything". In other words, if the consequent is true, then the implication is true, and if the consequent is false, then the implication is likewise true. This ambiguity, we claim, shows that a proposed proof of the $3x + 1$ Conjecture that uses the phrases, "if counterexamples exist..." and "if counterexamples do not exist..." does not apply to the $3x - 1$ Conjecture, as some critics claim.

At present, therefore, we consider this Open Question to be no longer Open.

(8) Is it possible to prove the $3x + 1$ Conjecture using only the 1-tree (see "Graphical Representation of the Set J as Recursive "Spiral"s" on page 28) and basic results in this paper? The goal here is to find a proof that does *not* use the Comparison Strategy. A first attempt is described under "Strategy of "Filling-in" of Intervals in the Base Sequence Relative to 1" on page 65.

References

[1] Jeff Lagarias, The $3x + 1$ Problem and Its Generalizations, *American Mathematical Monthly*, **93** (1985), 3-23.

[2] Günther J. Wirsching, *The Dynamical System Generated by the 3n + 1 Function*, Springer-Verlag, Berlin, Germany, 1998.

Appendix A — Statement and Proof of Each Lemma

Lemma 1.0: Statement and Proof

(a) Let $A = \{a_2, a_3, ..., a_i\}$, *where* $i \geq 2$, *be a sequence of exponents, and let* t_k, t_n *be tuples consecutive at level i in* T_A. *Then d(i, i), the distance between* t_k *and* t_n *at level i, is defined to be the absolute value of the difference between the level i elements of* t_k *and* t_n, *that is, is defined to be* $|t_{k(i)} - t_{n(i)}|$, *and is given by*:

$$d(i, i) = 2 \cdot 3^{(i-1)}$$

(b) Let t_k, t_n *be tuples consecutive at level i in* T_A. *Then d(1, i), the distance between* t_k *and* t_n *at level* 1, *is defined to be the absolute value of the difference between the level 1 elements of* t_k *and* t_n, *that is, is defined to be* $|t_{k(1)} - t_{n(1)}|$, *and is given by:*

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3})\ldots(2^{a_i})$$

Thus, in Fig. 1 in the section "Tuple-set" on page 8, the distance $d(3, 3)$ between $t_{8(3)} = 35$ and $t_{4(3)} = 17$ is $2 \cdot 3^{(3-1)} = 18$. The distance $d(1, 2)$ between $t_{12(1)} = 23$ and $t_{10(1)} = 19$ is $2 \cdot 2^1 = 4$.

Proof:

The proof is by induction.

Proof of Basis Step for Parts (a) and (b) of Lemma 1.0:

Let t_r and t_s be the first and second 2-level tuples, in the standard linear ordering of tuples based on their first elements, that are consecutive at level $i = 2$ in T_A. (See Fig. 2 (1).)

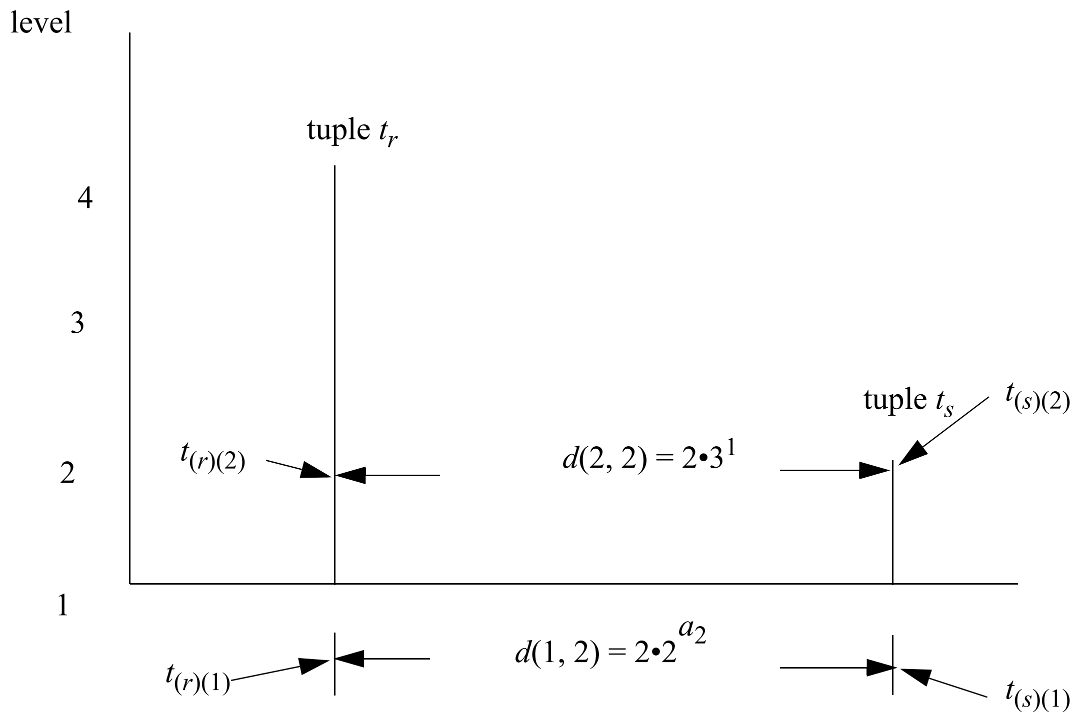


Fig. 2 (1). Illustration for proof of Basis Step of Lemma 1.0.

Then we have:

$$\frac{3t_{(r)(1)} + 1}{2^{a_2}} = t_{(r)(2)} \tag{1.1}$$

and since, by definition of $d(1, 2)$,

$$t_{(s)(1)} = t_{(r)(1)} + d(1, 2)$$

we have:

$$\frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}} = t_{(s)(2)} \tag{1.2}$$

Therefore, since, by definition of $d(i, i)$,

$$t_{(r)(2)} + d(2, 2) = t_{(s)(2)}$$

we can write, from (1.1) and (1.2):

$$\frac{3t_{(r)(1)} + 1}{2^{a_2}} + d(2, 2) = \frac{3(t_{(r)(1)} + d(1, 2)) + 1}{2^{a_2}}$$

By elementary algebra, this yields:

$$2^{a_2} d(2, 2) = 3 \cdot d(1, 2)$$

Now $d(2, 2)$ must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level i, it must be the smallest such even number, whence it follows that $d(2, 2)$ must = $3 \cdot 2$, and necessarily

$$d(1, 2) = 2 \bullet 2^{a_2}$$

A similar argument establishes that $d(2, 2)$ and $d(1,2)$ have the above values for every other pair of tuples consecutive at level 2.

Thus we have our proof of the Basis Step for parts (a) and (b) of Lemma 1.0.

Proof of Induction Step for Parts (a) and (b) of Lemma 1.0

Assume the Lemma is true for all levels j, $2 \le j \le i$.

Let t_r, t_s be tuples consecutive at level i, and let t_r, t_f be tuples consecutive at level $i + 1$. (See Fig. 2 (2).)

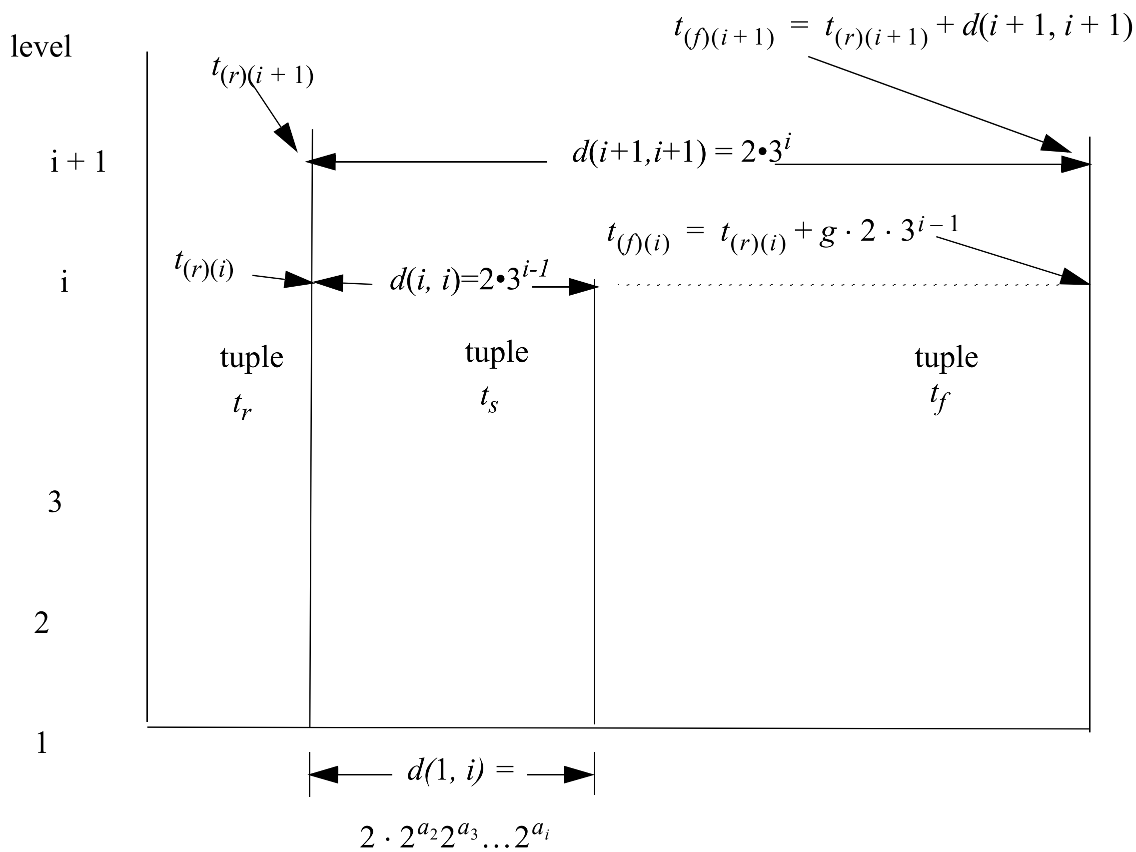


Fig. 2 (2). Illustration for proof of Induction Step of Lemma 1.0.

Then we have:

$$\frac{3t_{(r)(i)} + 1}{2^{a_{i+1}}} = t_{(r)(i+1)}$$

and since, by definition of $d(i, i)$,

$$t_{(f)(i)} = t_{(r)(i)} + g \cdot d(i, i)$$

for some $g \geq 1$, we have:

$$\frac{3(t_{(r)(i)} + g \cdot d(i, i)) + 1}{2^{a_{i+1}}} = t_{(f)(i+1)}$$

Thus, since

$$t_{(r)(i+1)} + d(i+1, i+1) = t_{(f)(i+1)}$$

we can write:

$$\frac{3t_{t_{(r)(i)}} + 1}{2^{a_{i+1}}} + d(i+1, i+1) = \frac{3(t_{t_{(r)(i)}} + gd(i, i)) + 1}{2^{a_{i+1}}}$$

This yields, by elementary algebra:

$$2^{a_{(i+1)}} d(i+1, i+1) = 3 \cdot gd(i, i)$$

As in the proof of the Basis Step, $d(i+1, i+1)$ must be even, since it is the difference of two odd, positive integers, and furthermore, by definition of tuples consecutive at level $i+1$, it must be the smallest such even number. Thus $d(i+1, i+1) = 3 \cdot d(i, i)$, and

$$g \cdot d(i, i) = 2^{a_{i+1}} d(i, i) \quad .$$

Hence

$$g = 2^{a_{i+1}}$$

Now g is the number of tuples consecutive at level i that must be "traversed" to get from $t_{(r)}$ to $t_{(f)}$. By inductive hypothesis, d(1, i) for *each* pair of these tuples is:

$$d(1, i) = 2 \cdot 2^{a_2} \cdot 2^{a_3} \cdot \ \ldots \cdot 2^{a_i}$$

hence, since

$$g = 2^{a_{i+1}}$$

we have

$$d(1, i+1) = d(1, i) \cdot 2^{a_{i+1}}.$$

A similar argument establishes that $d(i+1, i+1)$ and $d(1, i+1)$ have the above values for every pair of tuples consecutive at level $i+1$.

Thus we have our proof of the Induction Step for parts (a) and (b) of Lemma 1.0. The proof of Lemma 1.0 is completed. □

Lemma 2.0: Statement and Proof

For each exponent a_2, a tuple-set T_A, where $A = \{a_2\}$, exists.

Proof:

By Lemma 13.0 (see “Lemma 13.0: Statement and Proof” on page 89) we know that each range element is mapped to by all exponents of one parity only. Then since 5 is mapped to by 3 via the exponent 1, we know that 5 is mapped to by all odd exponents. Since 1 is mapped to by 1 via the exponent 2, we know that 1 is mapped to by all even exponents. Both 1 and 5 are level-2 anchors, since each is less than $2 \cdot 3^{2-1} = 6$. Therefore each tuple <x, 5>, where x maps to 5 via the odd exponent a_2 is the anchor tuple of a tuple-set, and each tuple <x', 1>, where x maps to 1 via the even exponent a_2', is the anchor tuple of a tuple-set. The result follows by Lemma 1.0 (a) and (b) (see “Lemma 1.0: Statement and Proof” on page 77), which assures us of an infinite number of tuples in each 2-level tuple-set. □

Lemma 3.0: Statement and Proof

Each i-level tuple-set, where $i \geq 2$, can be extended by each even or odd exponent a_{i+1}.

Proof:

By Lemma 2.0 (see “Lemma 2.0: Statement and Proof” on page 82), for each exponent a_2, a tuple-set $T_{A'}$, where $A' = \{a_2\}$, exists. So we show that for each exponent $a_2 = a_{i+1}$, the sequence of first elements of all tuples in $T_{A'}$ has at least one element in common with the sequence of i-level elements in T_A.

The sequence of i-level elements in the i-level tuple-set T_A is given by

$$2 \cdot 3^{i-1}k + y \qquad (3.1)$$

where $k \geq 0$ and y is an i-level anchor, that is, y is an odd, positive integer that is less than or equal to, and relatively prime to, $2 \cdot 3^{(i-1)}$.

The sequence of 1-level elements of $T_{A'}$ is given by

$$\frac{2^{a_2}y' - 1}{3} + j2 \cdot 2^{a_2} \qquad (3.2)$$

where $y' = 1$ or 5 is a 2-level anchor and $j \geq 0$ (see "Lemma 1.0: Statement and Proof" on page 77). Specifically, y' is 1 if $a_2 = a_{i+1}$ is even, and y' is 5 if $a_2 = a_{i+1}$ is odd. The left-hand term of (3.2) gives the value of the first element x of the level-1 sequence of $T_{A'}$ because

$$\frac{3x+1}{2^{a_2}} = y'$$

and an anchor, namely, y', is the smallest i-level element (in this case 2-level element) of an i-level tuple-set. The right-hand term of (3.2) is j times the difference between successive first elements of $T_{A'}$ (see "Lemma 1.0: Statement and Proof" on page 77).

Setting (3.1) equal to (3.2), we must prove that a solution j, k exists to the equation

$$2 \cdot 3^{i-1}k + y = \frac{2^{a_2}y' - 1}{3} + j2 \cdot 2^{a_2}$$

Multiplying through by 3, then dividing through by 2, which we can do since $3y + 1$ is even, we get

$$3^i k + \frac{3y+1}{2} = 2^{a_2 - 1}y' + 3j2^{a_2}$$

Rearranging terms, we have

$$3^i k - 3j2^{a_2} = -\frac{3y+1}{2} + 2^{a_2 - 1}y' \tag{3.3}$$

or

$$3(3^{i-1}k - j2^{a_2}) = -\frac{3y+1}{2} + 2^{a_2 - 1}y'$$

The right-hand side of the equation must be a multiple of 3, and so we can divide both sides by 3 and write:

$$3^{i-1}k - 2^{a_2}j = U$$

This is an equation of the form

$$au + bv = c$$

and a basic fact of Diophantine Equations states that such an equation has a solution u, v if and only if (a, b) divides c. In our case,

$$(3^{i-1}, 2^{a_2}) = 1$$

and so (3.3) has a solution j, k.

Lemma 1.0 (see "Lemma 1.0: Statement and Proof" on page 77) then assures us of an infinity of i-level elements in T_A that have extensions via the exponent $a_2 = a_{i+1}$, thus creating the tuple-set $T_{A''}$, where A'' = $\{a_2, a_3, ..., a_i, a_{i+1}\}$. □

Lemma 4.0: Statement and Proof

For each exponent sequence $A = \{a_2, a_3, ..., a_i\}$, where $i \geq 2$, there exists a tuple-set T_A generated by A.

Proof:

The proof is by induction.

Basis Step:

By Lemma 2.0 (see "Lemma 2.0: Statement and Proof" on page 82) we know that there is a 2-level tuple-set for each exponent a_2.

Inductive Step:

Assume the Lemma is true for all j-level exponent sequences $2 \leq j \leq i$. But then by Lemma 3.0 (see "Lemma 3.0: Statement and Proof" on page 82) it is true for all tuple-sets generated by $(i + 1)$-level exponent sequences. .□

Lemma 4.5: Statement and Proof

For each $i \geq 2$, the number of i-level tuple-sets is countably infinite.

Proof:

Each i-level exponent sequence is a string of one or more of the symbols 1, 2, 3, ..., 8, 9, "," . (Strings involving ",,...,", however, that is, involving two or more commas in succession, do not occur. Nor do strings that begin with ",".) There is a countable infinity of such strings. □

Lemma 4.75: Statement and Proof

For each $i \geq 2$, the set of all i-level elements of all i-level tuples in all i-level tuple-sets is the set of all range elements of the 3x + 1 function.

Proof:

We use an inductive proof.

Basis step

The Lemma is certainly true for all 2-level tuple-sets, since the set of all first elements of all 2-level tuples in all 2-level tuple-sets is the domain of the $3x + 1$ function, and the set of all second elements in all 2-level tuples in all 2-level tuple-sets is therefore the range of the $3x + 1$ function.

Inductive step

Assume the Lemma is true for all levels i, where $2 \leq i \leq k$. Assume now that at least one range element is absent from the set of all $(k + 1)$-level elements of all $(k + 1)$-level tuples in all $(k + 1)$-level tuple-sets.

But it is easily shown (see proof in "Lemma 18.0: Statement and Proof" on page 93) that each range element is mapped to, in one iteration of the $3x + 1$ function, by an infinity of range elements. Therefore an infinity of range elements must be absent from the set of all k-level elements of all k-level tuples in all k-level tuple-sets, contrary to the first assumption in our inductive step.
□

Lemma 5.0: Statement and Proof

Assume a counterexample exists. Then for all i ≥ 2, each i-level tuple-set contains an infinity of i-level counterexample tuples and an infinity of i-level non-counterexample tuples.

Proof:

1. Assume counterexamples exist. Then:

There is a countable infinity of non-counterexample range elements.

Proof: Each non-counterexample maps to a range element, by definition of *range element*.

Each range element is mapped to by an infinity of elements ("Lemma 13.0: Statement and Proof" on page 89). A countable infinity of these are range elements (proof of "Lemma 18.0: Statement and Proof" on page 93).

There is a countable infinity of counterexample range elements.

Proof: same as for non-counterexample case.

2. For each finite exponent sequence A, and for each range element y, non-counterexample or counterexample, there is an x that maps to y via A possibly followed by a buffer exponent ("Lemma 18.0: Statement and Proof" on page 93). The presence of the buffer exponent does not change the fact that x is the first element of a tuple associated with the exponent A.. □

Lemma 5.5: Statement and Proof

Let a be a finite exponent sequence such that if x maps to y via a, then y > x. Then there does not exist a counterexample x such that the infinite tuple <x, ... > is associated with the exponent sequence {a, a, a, ... }.

Proof:

Assume the contrary. Then there exists a counterexample x such that x is the first element of the infinite tuple <*x*, ... > that is associated with the exponent sequence $\{a, a, a, ...\}$. But x maps to y via a, and by hypothesis $y > x$, so y is the first element of the infinite tuple <*y*, ... > and this infinite tuple is likewise associated with the exponent sequence $\{a, a, a, ...\}$. Therefore, in the infinite sequence of tuple-set extensions associated with the infinite sequence $\{a\}$, $\{a, a\}$, $\{a, a, a\}$, ... of exponent sequences, there must occur an i-level tuple-set T_A in which <*x*, ... > and <*y*, ... > have i-level prefixes that are tuples consecutive at level i. The infinite tuples <*x*, ... > and <*y*, ... > have $(i + j)$-level prefixes in all j-level extensions of T_A. But since x and y are the same for all these prefixes, the level 1 distance function defined by part (b) of "Lemma 1.0" on page 12 is violated, and this contradiction gives us our proof. □

Lemma 6.0: Statement and Proof

Let t be the i-level anchor tuple in an i-level tuple-set, where $i \geq 2$. *Then the last element y of t, that is, the i-level element of t (which is the anchor), is a number* less than $2 \cdot 3^{(i-1)}$.

Proof:

By definition of *i-level anchor tuple*, t is the first i-level tuple in an i-level tuple-set. Hence there are no i-level tuples to the left of t under our convention for ordering tuples from left to right in a tuple-set. By the distance function defined in part (a) of "Lemma 1.0" on page 12, the distance between the last elements of consecutive i-level tuples in an i-level tuple-set is $2 \bullet 3^{(i-1)}$. An argument similar to that used in the proof of part (a) of Lemma 1.0 (see "Lemma 1.0: Statement and Proof" on page 77), but in the "leftward" direction, shows that, if the value of the i-level element of an i-level tuple t in an i-level tuple-set is greater than $2 \bullet 3^{(i-1)}$, then there exists an i-level element of an i-level tuple t' to the left of t. But if there is no i-level tuple to the left of t, it follows that the last element y of t must be less than $2 \bullet 3^{(i-1)}$. □

Lemma 7.0: Statement and Proof

(a) For each i-level tuple-set T_A*, where* $A = \{a_2, a_3, ..., a_i\}$*, the set of all i-level elements of all i-level tuples is a reduced residue class mod* $2 \bullet 3^{(i-1)}$.

(b) The set of all such reduced residue classes, over all i-level tuple-sets T_A*, is a complete set of reduced residue classes mod* $2 \bullet 3^{(i-1)}$.

Proof:

Part (a): Let T_A be an i-level tuple-set. Since the first i-level tuple t in T_A is an anchor tuple, the last element y of t is an anchor. By Lemma 6.0 (see "Lemma 6.0: Statement and Proof" on page 86), y is an odd, positive integer not divisible by 3 that is less than $2 \bullet 3^{i-1}$ — in other words, y is the minimum element of a reduced residue class mod $2 \bullet 3^{i-1}$.

Part (b): The set of all i-level elements of all i-level tuples in all i-level tuple-sets is the set of range elements of the $3x + 1$ function ("Lemma 4.75" on page 15). This set includes the set U of range elements that are less than $2 \bullet 3^{i-1}$. Since a range element is an odd, positive integer that is not a multiple of 3, the set U consists of all minimum reduced residues mod $2 \bullet 3^{i-1}$ — that is, the complete set of minimum reduced residues. The result follows from the fact that the distance between i-level elements of successive i-level tuples in an i-level tuple-set is $2 \cdot 3^{i-1}$ ("Lemma 1.0: Statement and Proof" on page 77). □

Lemma 8.0: Statement and Proof

For each odd, positive integer x there exists a minimum $i = i_0$ such that for each $i \geq i_0$, x is the first element of the first i-level tuple in some i-level tuple-set, that is, x is the first element of an i-level anchor tuple in some i-level tuple-set. In terms of infinite tuples, this lemma states: if x is an odd, positive integer, then in the infinite tuple $\bar{t} = <x, y, y', ... >$, there exists a minimum level i_0 such that:

$\bar{t}(i_0)$ is the i_0-level anchor tuple in an i_0-level tuple-set;
$\bar{t}(i_0 + 1)$ is the $(i_0 + 1)$-level anchor tuple in an $(i_0 + 1)$-level tuple-set;
$\bar{t}(i_0 + 2)$ is the $(i_0 + 2)$-level anchor tuple in an $(i_0 + 2)$-level tuple-set;
etc.

(Of course, the $(i_0 + k + 1)$-level tuple-set, where $k \geq 0$, must be an extension of the $(i_0 + k)$-level tuple-set by the same exponent by which the anchor tuple is extended.)

Proof:

Let x be an odd, positive integer. Then x is the first element of an infinite tuple $\bar{t} = <x, y, ... >$. With each increment of i, $i \geq 2$, the element of $\bar{t}$ at level i increases by at most a factor of 2, since for all exponents except 1, $C(y) < y$, and for exponent 1, $C(y) < 2y$. However, with each increment of i, $2 \cdot 3^{(i-1)}$ increases by a factor of 3. Therefore, a level $i = i_0$ must eventually be reached such that the element y' of $\bar{t}$ at level i is less than $2 \cdot 3^{(i-1)}$. But then by definition y' is an anchor, and hence the prefix $<x, y, ..., y'>$ is an anchor tuple. By our rule, "once an anchor tuple, always an anchor tuple" (see under "Mark" on page 20), the final part of our result follows. □

Lemma 10.0: Statement and Proof

No multiple of 3 is a range element.

Proof :

If

$$\frac{3x+1}{2^a} = 3m$$

then $1 \equiv 0 \bmod 3$, which is false. □

Lemma 11.0: Statement and Proof

Each odd, positive integer (except a multiple of 3) is mapped to by a multiple of 3 in one iteration of the 3x + 1 function.

Proof:

Since the domain of the $3x + 1$ function is the odd, positive integers, the only relevant generators are $3(2k + 1)$, $k \geq 0$. We show that, for each odd, positive integer y not a multiple of 3, there exists a k and an a such that

$$y = \frac{(3(3(2k+1))+1)}{2^a}, \quad (11.1)$$

where a is necessarily the largest such a, since y is assumed odd.

Rewriting (11.1), we have:

$$y2^{a-1} - 5 = 9k . \quad (11.2)$$

Without loss of generality, we can let $y \equiv r \bmod 18$, where r is one of 1, 5, 7, 11, 13, or 17 (since y is odd and not a multiple of 3, these values of r cover all possibilities mod 18). Or, in other words, for some q, r, $y = 18q + r$. Then, from (11.2) we can write:

$$18(2^{a-1})q + (2^{a-1})r - 5 = 9k . \quad (11.3)$$

Since the first term on the lefthand side is a multiple of 9, $(2^{a-1})r - 5$ must also be if the equation is to hold. We can thus construct the following table. (Certain larger a also serve equally well, but those given suffice for purposes of this proof.)

Table 3: Values of *r*, *a*, for Proof of Lemma

r	a	$(2^{a-1})r - 5$
1	6	27
5	1	0
7	2	9
11	5	171
13	4	99
17	3	63

Given q and r (hence y), we can use r to look up a in the table, and then solve (11.3) for integral k, thus producing the multiple of 3 that maps to y in one iteration of the $3x + 1$ function. □

Lemma 12.0: Statement and Proof

For each range element y there exists an infinity of x that map directly to y. Specifically,
If

$$\frac{3x+1}{2^a} = y$$

Then, for all n ≥ 1,

$$\frac{3(x + (2^{a+2(0)} + 2^{a+2(1)} + \ldots + 2^{a+2(n-1)})y) + 1}{2^{a+2(n)}} = y$$

Proof:

The proof is a matter of straightforward algebra.

From the antecedent, we have:

$$x = \frac{2^a y - 1}{3}$$

Substituting into the left-hand side of the consequent, multiplying the term in parentheses by 3, cancelling two 1's, and factoring out $(2^a)(y)$ yields:

$$\frac{2^a y(1 + 3(2^0 + 2^2 + 2^4 + \ldots + 2^{2(n-1)}))}{2^{a+2(n)}}$$

The 2^as cancel, the term (1 + 3(...)) is easily shown to equal $2^{2(n)}$, the $2^{2(n)}$ in numerator and denominator cancel, and we are left with *y*, which gives us our result. ☐

Remark

Lemma 12.0 and Lemma 11.0 (see "Lemma 11.0: Statement and Proof" on page 87) imply that if a counterexample exists, then there is an infinity of counterexamples.

Lemma 13.0: Statement and Proof

Each range element y is mapped to, in one iteration of the 3x + 1 function, by all exponents of one parity only.

The following proof is an edited version of a proof by Sanjai Gupta. Any errors it contains are entirely our own.

Proof:

Fix a range element y, and suppose that x maps to y via the exponent a. Now a is either even or odd, hence $a = 2n + h$, where h is either 0 or 1. Since $y = (3x + 1)/2^a$, it follows that $(2^a)y = 3x+1$. Reduce the equation mod 3, and we get $(2^h)y \equiv 1 \bmod 3$, by the following reasoning: $(2^a)y \equiv 1 \bmod 3$ implies $(2^{2n+h})y \equiv 1 \bmod 3$ implies $2^{2n} 2^h y \equiv 1 \bmod 3$ implies $2^h y \equiv 1 \pmod 3$ because $2^{2n} = 4^n \equiv 1 \bmod 3$.

Since y is fixed, either $y \equiv 1$ or $y \equiv 2 \bmod 3$. (We know that y, a range element, is not a multiple of 3 by Lemma 10.0 (see "Lemma 10.0: Statement and Proof" on page 87)). If $y \equiv 1 \bmod 3$, then we have $2^h(1) \equiv 1 \bmod 3$, which implies that h must be 0. If $y \equiv 2 \bmod 3$, then we have $(2^h)(2) \equiv 1 \bmod 3$, implying that h must be 1, which proves the Lemma. □

Lemma 14.0: Statement and Proof

There exists an explicit construction of the tuple-set whose exponent sequence is associated with a given tuple.

Proof:

Let x be the first element of a tuple and let $\{a_2, a_3, ..., a_{n+1}\}$ be the sequence of exponents associated with the first n extensions of the tuple $\langle x \rangle$. The last element of the tuple is given by:

$$\frac{3^n x + r}{2^a}$$

where

$$a = \sum_{i=2}^{n} a_i$$

The term r is most easily calculated by iterating from $x = 0$, then multiplying by the appropriate power of 2, as shown in the table at the end of this proof. We want the integral x that produce odd outputs:

$$\frac{3^n x + r}{2^a} = 2k + 1$$

which gives

$$3^n x - 2^{a+1} k = 2^a - r$$

This is a standard linear Diophantine equation. Since $(3^n, 2^{a+1}) = 1$, and 1 divides the righthand side of the equation, the equation has a solution. One solution is:

$$x_0 = (-(2^a - r))\left(\frac{2^{2 \cdot 3^{n-1} \cdot (a+1)} - 1}{3^n}\right)$$

$$k_0 = (-(2^a - r))(2^{(2 \cdot 3^{n-1} - 1)(a+1)})$$

Note that the ratio in the expression for x_0 is an integer because

$$2^{2 \cdot 3^{n-1}} \equiv 1 \quad mod \quad 3^n$$

The general solution is:

$$x = x_0 + t \cdot (-2^{a+1})$$

$$k = k_0 - t \cdot 3^n$$

where t ranges over the integers. Thus, the x's are the inputs that iterate with the specified exponents and

$$2k + 1 = 2k_0 - t \cdot 2 \cdot 3^n + 1$$

are the outputs.

Table 4: Successive values of *n*, the *x* term, and *r* in proof of Lemma 14.0

n	***x* term**	***r***	**level of tuple element yielded, i.e., *i* in a_i**
1	3^1x	1	2
2	3^2x	$3^1 + 2^{a_2}$	3

Table 4: Successive values of *n*, the *x* term, and *r* in proof of Lemma 14.0

n	***x* term**	***r***	**level of tuple element yielded, i.e., *i* in a_i**
3	3^3x	$3^2 + 3^1 2^{a_2} + 2^{a_2} 2^{a_3}$	4
4	3^4x	$3^3 + 3^2 2^{a_2} + 3^1 2^{a_2} 2^{a_3} + 2^{a_2} 2^{a_3} 2^{a_4}$	5
. . .	. . .	. . .	. . .

□

Lemma 15.0: Statement and Proof

For each range element y, and for each finite sum a of exponents, a domain element x exists that maps to y via a sum a′ that contains a.

Proof:

We are looking for an x such that the sequence of iterations represented by

$$\frac{3^n x + r}{2^a}$$

where n, a, and r are defined as in Lemma 14.0 (see “Lemma 14.0: Statement and Proof” on page 90), lead to a computation that ends with y. n, a, and r are determined by the exponent sequence we want. There also has to be an optional buffer iteration between the above and y, for example, to allow for parity constraints on the exponent leading to y (see “Lemma 12.0: Statement and Proof” on page 89). Thus, for example, if y is mapped to by even exponents, and our exponent sequence a ends with an odd exponent, then there must be a buffer exponent following the sequence a. So, we want

$$\frac{3\left(\frac{3^n x + r}{2^a}\right) + 1}{2^j} = y$$

or

$$\frac{3^{n+1}x + 3r + 2^a}{2^{a+j}} = y$$

which gives

$$3^{n+1}x = (2^a y)2^j - 3r - 2^a \tag{15.1}$$

or

$$(2^a y)2^j \equiv 3r + 2^a \quad mod \quad 3^{n+1}$$

We are looking for x and j. Since y is a range element, it cannot be a multiple of 3 (see "Lemma 10.0: Statement and Proof" on page 87). Therefore $2^a y$ is relatively prime to 3^{n+1}, as is $3r + 2^a$. Since 2^j, where $j \geq 0$, is an element of a reduced residue class mod 3^{n+1}, the congruence is solvable. Hence we can find j, and then, from (15.1), x. □

Remarks

The result would hold for each finite number of buffer exponents following the exponent sum a, since they do not change the fact that a tuple generating each exponent sequence whose sum is a is guaranteed by the proof.

A recursive proof of the Lemma is possible because the set of odd, positive integers mapping to a range element y in one iteration of the $3x + 1$ function C includes an infinite subset each element of which is mapped to by an infinity of even exponents, and an infinite subset each element of which is mapped to by an infinity of odd exponents. (See "Lemma 13.0: Statement and Proof" on page 89, and Lemma 15.0, p. 57, in our paper, "The Structure of the $3x + 1$ Function: An Introduction" on the web site www.occampress.com).

Lemma 18.0: Statement and Proof

Let y be a range element of the 3x + 1 function. Then for each finite exponent sequence A, there exists an x that maps to y via A possibly followed by a "buffer" exponent. (If y is mapped to by even exponents, and our exponent sequence A ends with an odd exponent, then there must be a "buffer" exponent following A, and similarly if y is mapped to by odd exponents and A ends with an even exponent.)

Proof:

1. Each range element y is mapped to by all exponents of one parity ("Lemma 13.0: Statement and Proof" on page 89).

2. Each range element y is mapped to by a multiple of 3 ("Lemma 11.0: Statement and Proof" on page 87).

Each range element is mapped to by an infinity of range elements (“Lemma 11.0: Statement and Proof” on page 87).

3. Let y be a range element and let $S = \{s_1, s_2, s_3, \ldots\}$ be the set of all odd, positive integers that map to y in one iteration of the $3x + 1$ function. In other words, S is the set of all elements in a “spiral”. Furthermore, let the s_i be in increasing order of magnitude. It is easily shown that $s_{i+1} = 4s_i + 1$.

(In Fig. 18, $y = 13$, $S = \{17, 69, 277, 1109, \ldots\}$

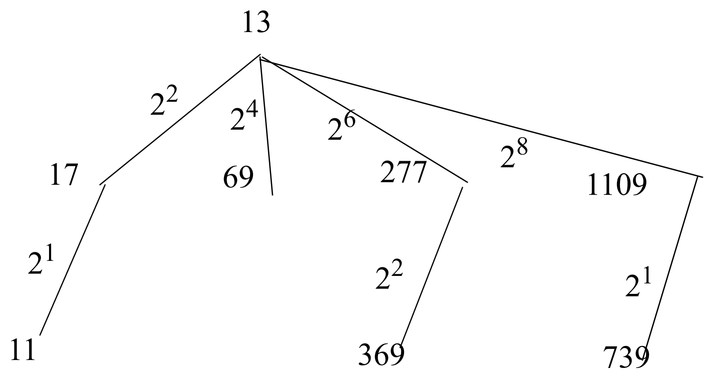


Fig. 18

4. If s_i is a multiple of 3, then $4s_i + 1$ is mapped to, in one iteration of the $3x + 1$ function, by all exponents of even parity.

To prove this, we need only show that x is an integer in the equation

$$4(3u) + 1 = \frac{3x + 1}{2^2}$$

Multiplying through by 2^2 and collecting terms we get

$$(48u) + 4 = 3x + 1$$

and clearly x is an integer.

5. If s_j is mapped to by all even exponents, then $4s_j + 1$ is mapped to, in one iteration of the $3x + 1$ function, by all exponents of odd parity.
(The proof is by an algebraic argument similar to that in step 4.)

6. If s_k is mapped to by all odd exponents, then $4s_k + 1$ is a multiple of 3.
(The proof is by an algebraic argument similar to that in step 4..)

7. The Lemma follows by an inductive argument that we now describe.

Let *y* be a range element. It is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. As a consequence of steps 1 through 6, we can represent an infinite sub-sequence of the sequence by

...3, 2, 1, 3, 2, 1, ...

where
"3" means "this odd, positive integer is a multiple of 3 and therefore is not mapped to by any odd, positive integer";
"2" means "this odd, positive integer is mapped to by all even exponents";
"1" means "this odd, positive integers is mapped to by all odd exponents".

Each type "2" and type "1" odd, positive integer is mapped to by all exponents of one parity. Thus it is mapped to by an infinite sequence of odd, positive integers. We can represent an infinite sub-sequence of the sequence by

...3, 2, 1, 3, 2, 1, ...

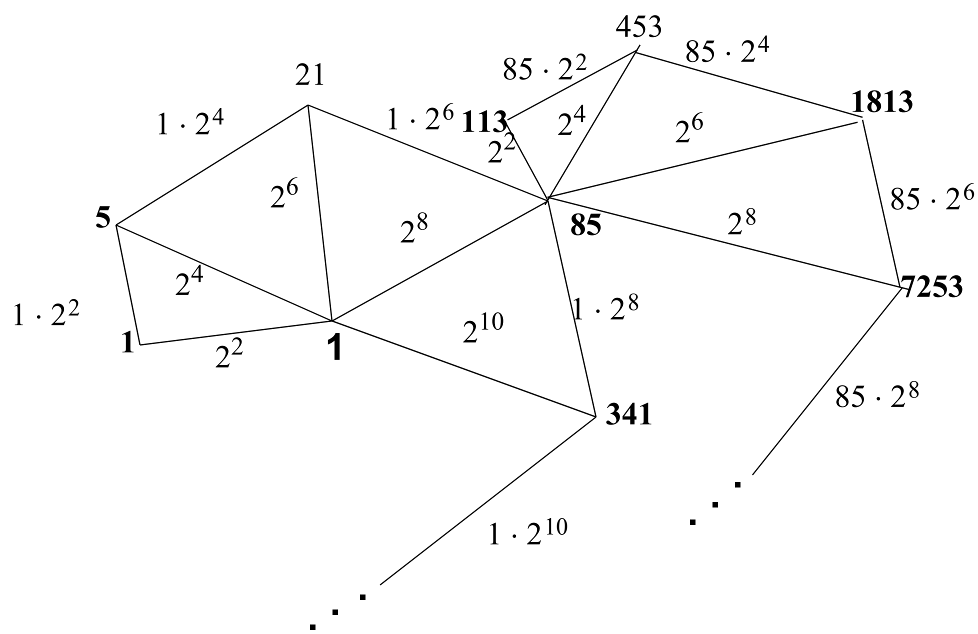
where each integer has the same meaning as above. (See proof under "Properties of *y*-Trees" in our paper, "A Solution to the 3x + 1 Problem", on occampress.com.)

Temporarily ignoring the case in which a buffer exponent is needed, it should now be clear that, for each range element *y*, and for each finite sequence of exponents *B*, we can find a finite path down through the infinitary tree we have just established, starting at the root *y*. The path will end in an odd, positive integer *x*. Let *A* denote the path *B* taken in reverse order. Then we have our result for the non-buffer-exponent case. The buffer-exponent case follows from the fact that the buffer exponent is one among an infinity of exponents of one parity. Thus *y* is mapped to by an infinite sequence of odd, positive integers. We then simply apply the above argument.. □

Apppendix B — Analysis of a Failed Strategy

In early 2009 we attempted to prove the $3x + 1$ Conjecture using the inverse of the $3x + 1$ function — specifically, the inverse of 1. Our motivation was as follows:

It dawned on us that all odd, positive integers that are known to map to 1 — namely, 1, 3, 5, 7, 9, 11, ..., up to about $5.76 \cdot 10^{18}$, by computer test[1] — map to 1 regardless if counterexamples exist or not. We then thought of the structure of the set of all odd, positive integers that are inverses of 1, a structure we have elsewhere called the "infinite set of recursive 'spiral's whose base element is 1." (See "Section 2. Recursive 'Spiral's" in the first file of the paper "The Structure of the $3x + 1$ Function: An Introduction" on the web site www.occampress.com.) Following is a diagram of part of this structure.



Recursive "spirals" structure of computations produced by the $3x + 1$ function.

Bold-faced numbers are range elements (21 and 453 are multiples of 3, hence not range elements). Partial "spirals" surrounding the base elements 1 and 85 are shown. The line connecting 1813 to 85 is marked with a 2^6 because $(3 \cdot 1813 + 1)/ 2^6 = 85$. The line connecting 453 to 1813 is marked $85 \cdot 2^4$ because $453 + 85 \cdot 2^4 = 1813$. The exponents of 2 are not always even, of course. The "spiral" of numbers (not shown) mapping to 341 has odd exponents.

It is easily shown that {all odd, positive integers that map to 1 in *one* iteration of the $3x + 1$ function} = {1, 5, 21, 85, 341, ... }. This set is a recursive "spiral". Lemma 11.0 in the above-referenced "...Introduction" paper, states that if y is an element of a "spiral", the next element is $4y + 1$.

It is also easily shown that each recursive "spiral" contains an infinity of range elements and an infinity of multiples-of-3.

1. See results of tests performed by Tomás Oliveira e Silva, www.ieeta.pt/~tos/3x+1/html. All odd, positive integers to at least $20 \cdot 2^{58} \approx 5.76 \cdot 10^{18}$ have been tested and found to be non-counterexamples.

We then defined the set J from this diagram as follows:

Let J denote

{all odd, positive integers that map to 1 in *one* iteration of the $3x + 1$ function} $\cup$
{all odd, positive integers that map to 1 in *two* iterations of the $3x + 1$ function} $\cup$
{all odd, positive integers that map to 1 in *three* iterations of the $3x + 1$ function} $\cup$
...

We stated that

(1) Each element of J maps to 1 regardless whether counterexamples exist,

or, in other words,

(2) J is the same set regardless whether counterexamples exist.

Our justification was that the contrary would imply that the laws of arithmetic — in particular, those governing the elements of each "spiral" in the above structure — were sensitive to the truth or falsity of the $3x + 1$ Conjecture, which is absurd.

However, statements (1) and (2) drew strong criticism from virtually all readers. Many declared the statements were meaningless. But we persisted, and eventually arrived at the following argument:

Let V = the set of odd, positive integers, and let C = the set of counterexamples. Then (1) implies:

$J \cup C = V = J$, and therefore C is empty. Hence we have a proof of the $3x + 1$ Conjecture.

We received many objections to this argument, most of which we didn't understand. Then Jonathan Kilgallin sent us the following counterargument, which we consider irrefutable. It is that our argument can be applied equally to the $3x - 1$ function. But there we know that counterexamples exist (5 and 7 form an infinite cycle, and thus are counterexamples). Therefore our argument is invalid.

We feel it is important to understand the fault in our reasoning even apart from the $3x - 1$ counterargument. The fault rests in our confusing of domains of discourse.

Case I. Let our domain of discourse = W = the set of odd, positive integers that map to 1 under the $3x + 1$ function. Then (1) and (2) hold, and we can legitimately write:

$J \cup C = W = J$,

because, in fact, there are no counterexamples in W, hence $C = \varnothing$.

Case II. Now let our domain of discourse = V = the set of odd, positive integers. Then, although (1) and (2) hold, and we can write

$J \cup C = V,$

it is not necessarily true that

$V = J.$

We will not know until we have a proof or disproof of the $3x + 1$ Conjecture.

However, we emphasize that (1) and (2) hold in both cases. In Case I, J has only one value in the domain of discourse, in Case II, J has two possible values in the domain of discourse. Just as we might know that an equation has one solution, x, but we do not know, until we solve for x, if x is real or complex.

Perhaps a better way to understand the counterintuitive fact that J is a single, fixed set in both cases is as follows. Let S_1 denote the set containing the singleton set that is the set of odd, positive integers. Let S_2 denote the set of all proper subsets of the odd, positive integers. Then the $3x + 1$ Problem asks if J is an element of S_1 or of S_2.

Appendix C — "3x + 1 - like" Functions

Generalizations of the 3x + 1 Function

During the course of our attempts to find a proof of the $3x + 1$ Conjecture, we were occasionally encouraged to check if our proposed proof also constituted a proof of the $3x - 1$ Conjecture. If the answer was Yes, then our proof must be wrong, since the $3x - 1$ Conjecture is false (5 and 7 are counterexamples). We began referring to this check as the *3x – 1 Test.*

But the existence of the $3x - 1$ function encouraged us to investigate what we called 3x + C functions, where C is an odd, positive integer We have been told that the $3x + 3^k$ function, where k is a positive integer, was first defined and investigated by Barry Brent in 1993. We have so far been unable to find anything about $3x + C$ functions in the literature.

One class of Some of these $3x + C$ functions we now call *3x + 1-like functions* (see definition below).

Another generalization of the $3x + 1$ function is $3x + C$ functions whose domain includes the negative integers. The negative of the $3x - 1$ function over the negative integers is the same as the $3x + 1$ function over the negative integers, a fact that provides some insight into the nature of the $3x + 1$ function. See, for example, "Why Are There Counterexamples to the 3x – 1 Conjecture?" on page 110.

A further generalization would be $Ax + B$ functions, where A and B are integers.

Finally, for all the above functions, we can generalize the denominators.

Definition of "3x + C Function" and the "3x + C Problem"

We define a $3x + C$ function F_C as

$$F_C(x) = \frac{3x + C}{2^{ord_2(3x + C)}}$$

where C and x are odd, positive integers, with $C = 2^k - 3$, where $k \geq 2$. However, we also include the $3x - 1$ function in the $3x + C$ functions, as explained below. We have not yet investigated other negative C. We call such functions *3x + 1-like functions*.

Since for each k in the definition, $F_C(2^k - 3) = 2^k - 3$, we have a counterexample to the $3x + (2^k - 3)$ Conjecture, namely, the infinite cycle of $2^k - 3$ terms.

In the case of the $3x - 1$ function, the smallest counterexample begins with 5, yielding the infinite cyclic tuple <5, 7, 5, ... >. (In the $3x + 1$ function, 5 is the first element of the non-counterexample 2-level anchor tuple <5, 1>.) Thus 5 and 7 are counterexamples to the $3x - 1$ Conjecture.

In the case of the $3x + 5$ function, 5 is a counterexample because it yields the infinite cyclic tuple <5, 5, ...>. Another counterexample is 19, yielding the infinite cyclic tuple <19, 31, 49, 19, ... >. (In the $3x + 1$ function, 19 is the first element of the non-counterexample 4-level anchor tuple <19, 29, 11, 17>.) Since any odd, positive integer that maps to a counterexample, is itself a counterexample, it turns out that, as the reader can verify, all odd, positive integers less than $2 \bullet 3^{3-1} = 18$ except 1 and 9 are counterexamples!

For all other 3x + C functions, C is a counterexample, because $(3C + C)/2^2 = C$, giving rise to the infinite cycle, <C, C, C, ... >.

A Relationship Between 3x + C Tuples and 3x + 1 Tuples

We are indebted to a computer scientist for the statement and proof of the following Lemma. We have edited the proof slightly, so any errors are entirely our fault.

Lemma 14.8

For each 3x + C function that is a 3x + 1-like function other than the 3x – 1 function, the tuple <Cx, Cy, Cy′, ..., Cz> is a 3x + C tuple iff the tuple <x, y, y′, ..., z> is a 3x + 1 tuple.

Proof (only if part):

Assume Cu is an element of a $3x$ + C tuple. Then

$$\frac{3(Cu)+C}{2^{ord_2(3(Cu)+C)}} = \frac{C(3u+1)}{2^{ord_2(C(3u+1))}} = \frac{C(3u+1)}{2^{ord_2(3u+1)}}$$

The denominator of the middle term equals the denominator of the right-hand term because for each $3x$ + 1-like function except the $3x$ – 1 function, C is an odd, positive integer (see “Lemma 15.0” on page 101). Thus C does not contain 2 as a factor and therefore has no effect on the value of the ord_2 function.

The right-hand term gives us our desired result. □

Proof (if part):

Let x be an element of a $3x$ + 1 tuple. Then:

$$\frac{3x+1}{2^{ord_2(3x+1)}} = y \rightarrow \frac{C(3x+1)}{2^{ord_2(3x+1)}} = Cy \rightarrow \frac{3Cx+C}{2^{ord_2(3x+1)}} = Cy \rightarrow \frac{3Cx+C}{2^{ord_2(3Cx+C)}} = Cy$$

The right-most equation gives us our result.

The denominators in the last two fractions are equal — that is, $\mathrm{ord}_2(3Cx + C) = ord_2 3(3x + 1) = ord_2(3x+1)$ — because 3 does not contain 2 as a factor, and therefore has no effect on the value of the ord_2 function. □

Remark

Lemma 14.8 shows that, informally, the $3x$ + 1 function is embedded in each $3x$ + 1-like function.

All Positive *C* That Give Rise to 3*x* + 1-like Functions

The following Lemma shows that the 3*x* + 1 function is embedded in each 3*x* + 1-like function. Thus, if a counterexample to the 3x + 1 Conjecture exists, then a counterexample to each 3*x* + 1-like function Conjecture exists.

Lemma 15.0

Let C define a 3x + C function F_C. Then F_C gives rise to a 3x + 1-like Problem iff $C = -1$ or $C = -1 + 2^1 + 2^2 + 2^3 + ... + 2^k$.

Proof (if part):
Let $C = -1$. Then by direct calculation we confirm that $F_{-1}(1) = 1$.

Let $C = -1 + 2^1 + 2^2 + 2^3 + ... + 2^k$. Then
(1)

$$\frac{3(1) - 1 + 2^1 + 2^2 + \dots + 2^k}{2^{k+1}} = \frac{1 + 2^0 + 2^1 + 2^2 + \dots + 2^k}{2^{k+1}} = \frac{2^{k+1}}{2^{k+1}} = 1$$

□

Proof (only if part):
If F_C gives rise to a 3*x* + 1-like Problem, then by definition there must exist a $k + 1$ such that

$$\frac{3(1) + C}{2^{k+1}} = 1$$

We find that solutions to this equation are $C = -1$ and $C = -1 + 2^1 + 2^2 + 2^3 + ... + 2^k$. □

The First Few 3x + 1-like Functions

The first few 3*x* + 1-like functions are the 3*x* – 1 function, the 3*x* + 1 function, the 3*x* + 5 function, the 3*x* + 13 function, and the 3*x* + 29 function.

On Trivial Infinite Cycles in 3*x* + 1-like Functions

The definition of 3*x* + 1-like functions, along with "Lemma 15.0" on page 101 and its Corollary, make clear that there are at least two trivial infinite cycles in each 3*x* + 1-like function: <1, 1, 1, ... > and <*C*, *C*, *C*, ... >. In the 3*x* + 1 case, and only in this case, these cycles are the same. A naive question arises: if we are going to identify, for each 3*x* + 1-like function, one of these infinite cycles as "the fundamental (trivial) cycle", which one should it be? So far, researchers have regarded <1, 1, 1,....> in the 3*x* + 1 case as being "fundamental", not least because it is part of the definition of the 3*x* + 1 Problem. If we do the same for all 3*x* + 1-like functions, then the <*C*, *C*, *C*, ... > cycles are counterexamples to the 3*x* + *C* Conjecture. On the other hand, if we make the <*C*, *C*, *C*, ... > cycles fundamental, then each <1, 1, 1, ... > is a counterexample! In the $3x + 1$ case, this wrecks the definition of the $3x + 1$ Problem.

A computer scientist has suggested that we define, for each $3x + 1$-like functions except the $3x + 1$ function, *both* <1, 1, 1, ... > *and* <*C*, *C*, *C*, ... > as fundamental (trivial) cycles, call each odd, positive integer that maps to *either* 1 or *C*, a non-counterexample to the $3x + C$ Conjecture, and call all other odd, positive integers, counterexamples. This convention has the advantage that each of the two fundamental (trivial) cycles has an obvious relationship to <1, 1, 1, ... > in the $3x + 1$ function.

The fact that there are the two trivial infinite cycles, <1, 1, 1, ... > and <*C*, *C*, *C*, ... >, for each $3x + C$ function that is a $3x + 1$-like function — two such cycles *except* when C = 1 (our familiar $3x + 1$ function) suggests a possible "convergence" strategy for proving the $3x + 1$ Conjecture: show that, as *C* decreases, a certain crucial property converges to a value such that counterexamples cannot exist for the $3x + 1$ function.

Conjectures Concerning 3*x* + 1-like Functions

We will regard it as remarkable if the following conjectures are true, because it will mean that the countable infinity of $3x + 1$-like functions all have the same structure.

Conjecture *C*1

The tuple-set structure holds for all these functions. In particular, the distance functions established by parts (a) and (b) of Lemma 1.0 are the same for all these functions. Furthermore, for each 3x + 1-like function, and for each i ≥ 2, the set of i-level anchors is the same as the corresponding set of i-level anchors in the 3x + 1 function.

The Conjecture holds in tests of the $3x – 1$, $3x + 5$, and $3x + 13$ functions. In particular, for each $3x + 1$-like function, the set of anchors for each 2-level tuple-set is {1, 5}, which is the same as for the $3x + 1$ function. The exponents for successive $3x + 1$-like functions beginning with the $3x + 1$ function, are successive, as shown in the following calculations:

$(3 \cdot 1 + 1)/2^2 = 1$; $(3 \cdot 1 + 5)/2^3 = 1$; $(3 \cdot 1 + 13)/2^4 = 1$; $(3 \cdot 1 + 29)/2^5 = 1$; ...
$(3 \cdot 3 + 1)/2^1 = 5$; $(3 \cdot 5 + 5)/2^2 = 5$; $(3 \cdot 9 + 13)/2^3 = 5$; $(3 \cdot 17 + 29)/2^4 = 5$; ...

In addition, each argument that yields 5 appears to be the previous argument plus an increasing power of 2.

Conjecture *C*2

The recursive "spiral"s structure holds for all these functions. In particular, the distance between successive elements x, x′ in a "spiral" is given by x′ = 4x + C.

The Conjecture holds in tests of the $3x – 1$, $3x + 5$, and $3x + 13$ functions.

Conjecture C3

For each 3x + 1 - like function F_C*, where C is positive, the infinite cycle <C, C, C, ... > exists.*

The Conjecture is true for *C* = 1, 5, 13, and 29. However, for the case *C* = 1, and only for that case, the infinite cycle <*C*, *C*, *C*, ...> = <1, 1, 1, ... > , the trivial cycle. For all other positive *C* (if the conjecture is true), the trivial cycle and the cycle <*C*, *C*, *C*, ... > are different.

If the Conjecture is true, then we have a proof that all 3*x* + *C* Conjectures, where *C* is positive, are false.

An Obvious Strategy For Using the 3*x* + 1-like Functions to Prove the 3*x* + 1 Conjecture

If conjectures *C*1, *C*2 and *C*3 are true, then an obvious strategy for proving the 3*x* + 1 Conjecture would be to assume a counterexample to the 3*x* + 1 Conjecture and then show that it implies a contradiction in at least one of the 3*x* + 1-like functions. Or, we could proceed in the opposite direction, and show that the 3*x* + 1-like functions prevent a counterexample to the 3*x* + 1 Conjecture. Obviously, "Lemma 15.0" on page 101 will be of use in such a strategy.

The 3x + 5 Function: A Few Tuples

Observe that 3, 7, 11 and 13 map to a non-trivial infinite cycle 19, ..., 19, and thus are counterexamples to the 3x + 5 Conjecture.

<1, 1, ...>
<3, 7, 13, 11, 19, 31,49,19, ... >
<5, 5, ...>
<7, 13, 11, 19, 31, 49, 19, ... >
<9, 1, 1, ...>
<11, 19, 31, 49, 19, ... >
<13, 11, 19, 31, 49, 19, ... >
...
<19, 31, 49, 19, ... >

The 3x + 13 Function: A Few Tuples

<1, 1, ...>
<3, 11, 23, 41, 17, 1, 1, ...>
<5, 7, 17, 1, 1, ...>
<7, 17, 1, 1, ...>
<9, 5, 7, 17, 1, 1, ...>
<13, 13, ...>
...
<65, 13, 13, ...>

The 3x + 29 Function: A Few Tuples

Observe that 3, 5, 7, 11, 13 and 19 map to a non-trivial infinite cycle 11, ..., 11, and thus are counterexamples to the 3x + 29 Conjecture.

Since the tuples <3, 19, 43> and <11, 31, 61> are both associated with the exponent sequence {1, 1}, they are in the same 3-level tuple-set, and, in fact are consecutive at level 3. In accordance with part (a) of Lemma 1.0 the distance between their third elements is 18.

<1, 1, ...>
<3, 19, 43, 79, 133, 107, 175, 277, 215, 337, 65, 7, 25, 13, 17, 5, 11, 31, 61, 53, 47, 85, 71, 121, 49, 11, ... >
<5, 11, 31, 61, 53, 47, 85, 71, 121, 49, 11, ... >
<7, 25, 13, 17, 5, 11, 31, 61, 53, 47, 85, 71, 121, 49, 11, ...>
<11, 31, 61, 53, 47, 85, 71, 121, 49, 11, ... >
<13, 17, 5, 11, 31, 61, 53, 47, 85, 71, 121, 49, 11, ... >
<19, 43>

The 3*x* – 1 Function

Definition of 3*x* – 1 Function

The definition of the $3x - 1$ function is similar to that of the $3x + 1$ function. That is, for x an odd, positive integer, the $3x - 1$ function C' is defined as:

$$C'(x) = \frac{3x-1}{2^{ord_2(3x-1)}}$$

where $ord_2(3x - 1)$ is the largest exponent of 2 such that the denominator divides the numerator. The importance of the $3x - 1$ function for our purposes is that it is the negative of the $3x + 1$ function on the odd, negative integers. Thus, for example,

$$-\left(\frac{3(7)-1}{2^2}\right) = -5 = \left(\frac{3(-7)+1}{2^2}\right)$$

The $3x - 1$ Conjecture asserts that for each odd, positive integer x (or for each odd, negative integer x' for the $3x + 1$ function over the odd, negative integers) repeated iterations of the $3x - 1$ eventually terminate in 1 (or –1 in the negative case).

However, the $3x - 1$ Conjecture is known to be false. Among the counterexamples are 5 and 17 (–5 and –17 in the negative case).

A useful test for the correctness of a proof of the $3x + 1$ Conjecture is to see if it also proves that the $3x - 1$ Conjecture is true. If it does, then we know that the proof is invalid.

Definition of the "3x – 1 Test"

The Test consists simply of seeing if a proposed proof of the $3x + 1$ Conjecture also applies to the $3x - 1$ Conjecture.

Argument for the Test

If a proposed proof also proves the $3x – 1$ Conjecture, the proposed proof must contain an error, because the Conjecture is false, since 5, 7, and 17 are known counterexamples. So the Test *can* be used to find an error in a proof. But by no means is it guaranteed to find an error.

Arguments Against the Test

In our experience, the Test is used by readers with no time (or inclination) to examine a proposed proof of the $3x + 1$ Conjecture in detail. So these readers simply assert, "You must convince me that your proof does not also apply to the $3x – 1$ Conjecture." Our arguments against this use of the Test are as follows.

What Does It Mean for a Proof To "Pass the Test"?

There is no question but that the Test can reveal errors in a proposed proof of the 3x + 1 Conjecture. In fact, it has done so twice for us. But if a proof does *not* pass the Test, the matter does not end there. One must check that the reason is an error in the proof, and not in the fact that one or more $3x + 1$ lemmas simply do not apply to the $3x – 1$ function (see below under "What Does It Mean for a Proof to "Fail the Test"?" on page 106).

On the other hand, the best that the skeptical reader, or the author, can say, is, "The proposed $3x + 1$ proof does not seem to also prove the $3x – 1$ Conjecture." But "does not seem to also prove" is not at all the same thing as "does not prove". Reader and/or author may have overlooked something. Furthermore there may be errors that the Test could never reveal.

3*x* – 1 Test Requirement Implies 3*x* +1-like Function Test Requirement

A requirement that a proposed proof of the $3x + 1$ Conjecture be subjected to, and pass, the $3x – 1$ Test carries with it an implication that the proof should also pass a similar test for *each* of the countable infinity of $3x + 1$-like functions. The reason for this implication is that we have absolutely no reason to believe that the $3x – 1$ Test suffices. But there is a countable infinity of $3x + 1$-like functions, and at present we have no reason to believe that we could even determine which ones have counterexamples to their respective conjectures, much less determine how a test like the $3x – 1$ Test could be applied to all the functions that do have counterexamples.

Furthermore it almost goes without saying that the implication extends to *all* proposed proofs of the $3x + 1$ Conjecture, whether or not they are based on tuple-set and recursive "spiral" strategies or on completely different strategies, e.g., partial differential equation strategies. We strongly suspect that the $3x + 1$ research community will have a few things to say about this requirement.

But why should the rest of the mathematics community be spared? The requirement of the $3x – 1$ Test is ultimately a requirement that throughout mathematics, all proofs of conjectures must be accompanied by a proof that the proof does not also prove a false conjecture. We strongly suspect that the *mathematics community* will have a few things to say about this requirement.

General Insistence on Similar Tests Would Bring Mathematics to a Stop

If the rule were established that a proof of a conjecture must carry with it a proof that it does not also prove any other conjecture for which counterexamples are known, mathematics would come to a stop. In addition, a fundamental theorem in the foundations of mathematics would be contradicted. This theorem states that if a proof is correct, then its correctness can be verified by machine (computer program). However, if it is decided that a proof of a conjecture is not correct unless one can show that it does not also prove the correctness of known-false conjectures, then this theorem is contradicted, since it may not even be possible for the machine (program) to determine what all the relevant known-false conjectures *are*, much less actually confirm that the proof in question does not apply to each of them.

A Proof Is Correct or Incorrect Within Its Own Context

A proof must stand on its own, not the least reason being the theorem in foundations of mathematics that says that a correct proof can be verified by machine (program). If a proof contains an error, then there must be a way to determine that error within the proof itself.

What Does It Mean for a Proof to "Fail the Test"?

If someone asserts that a proposed proof of the 3x + 1 Conjecture also "applies" to the $3x - 1$ Conjecture, does that mean that each statement in the proof holds for the $3x - 1$ function as well? Or does it mean that "if appropriate changes are made", then the proof holds? But if *any* changes are made in a proof, it should not be surprising if the proof turns out to be invalid. Furthermore, what are "appropriate changes"? No reader who has claimed that a proposed proof of the $3x + 1$ Conjecture also applies to the $3x - 1$ Conjecture, has demonstrated to us that in fact all the lemmas used in our proof, also apply to the $3x - 1$ function.

In fact, we know that at least one $3x + 1$ lemma does *not* hold for the $3x - 1$ function, and that is the lemma that defines the distance function on the inverse of the $3x + 1$ function (Lemma 11.0 in our paper, "The Structure of the $3x + 1$ Function: An Introduction" on occampress.com). This distance function is as follows. If y is a range element, then in both the $3x + 1$ and the $3x - 1$ functions, y is mapped to by all exponents of one parity only. For example, in the $3x + 1$ function, 5 is mapped to via odd exponents (each element of the set {3, 13, 53, ... } maps to 5 via an odd exponent). If x, x' are successive elements of such a set, then $x' = 4x + 1$. In the case of the $3x - 1$ function, however, $x' = 4x - 1$. For example, in the $3x - 1$ function, 5 is mapped to via even exponents (each element of the set {7, 27, 107 ... } maps to 5 via an even exponent).

However, it does seem that the distance function in the "forward" direction of the $3x + 1$ function, that is, the distance function for tuple-sets ("Lemma 1.0" on page 12) does in fact also hold for the $3x - 1$ function.

We now prove several elementary facts about the $3x - 1$ function. From here on we will use its negative version, denoting it as C', although we will refer to this negative version as the $3x - 1$ function.

Elementary Facts About the 3*x* – 1 Function

Lemma 9.0

For no odd, negative integer $-u$ *is it the case that* $C'(-u)$ *is positive.*

Proof:

$(3(-u) + 1)/(\ ord_2(3(-u) - 1))$ is negative because $(3(-u) + 1)$ is negative and $ord_2(3(-u) + 1)$ is positive. □

Lemma 9.05

The negative of the 3x – 1 function over the odd, positive integers = the 3x + 1 function over the odd, negative integers. That is, for all odd, non-zero integers u,

$$-\left(\frac{3(u)-1}{2^2}\right) = -w = \left(\frac{3(-u)+1}{2^2}\right)$$

Proof:
Follows directly from algebra on the equation in the statement of part (*a*).□

Lemma 9.1
If y is an anchor for the 3x + 1 function at level i , then $y - 2 \cdot 3^{(i-1)}$ *is an i-level anchor for the (negative of the) 3x – 1 function.*
Proof:
It is reasonable to define an anchor for the $3x - 1$ function analogously to an anchor for the $3x + 1$ function, that is, to define an anchor for the $3x - 1$ function as an odd, negative integer y' that is relatively prime to $2 \cdot 3^{(i-1)}$ and greater than $-2 \cdot 3^{(i-1)}$. It is clear that $y - 2 \cdot 3^{(i-1)}$ is such an integer. In fact, since y is a minimum positive residue of the integers mod $2 \cdot 3^{(i-1)}$ that is relatively prime to $2 \cdot 3^{(i-1)}$, it follows that $y - 2 \cdot 3^{(i-1)}$ is a maximum negative residue of the integers mod $2 \cdot 3^{(i-1)}$ that is relatively prime to $2 \cdot 3^{(i-1)}$. □

Thus, for example, consider the level-3 anchors. Since $2 \cdot 3^{(3-1)} = 18$; the level-3 anchors for the $3x + 1$ function are 17, 13, 11, 7, 5, 1. For the negative of the $3x - 1$ function, we get, for the level-3 anchors:

17 – 18 = –1;
13 – 18 = –5;
11 – 18 = –7;
7 – 18 = –11;
5 – 18 = –13;
1 – 18 = –17.

Lemma 9.15, the "Mirroring" Lemma
If $C(x) = y$, and $x < y$, then $C(-x) = -y$, and $-x > -y$; If $C(x) = y$ and $x > y$, then $C(-x) = -y$ and $-x < -y$.

Proof:
Follows from a basic fact of arithmetic: if $a < b$, then $-a > -b$; if a > b, then $-a < -b$. □

Remark:
The reason for the name "Mirroring Lemma" is that the property it describes is the same as that which holds for a point in front of a vertical, say, six-foot, mirror, and the image of the point in the mirror. If we place a measuring tape on the floor, perpendicular to the mirror, then we can imagine minus signs in front of each number on the tape in the image in the mirror. And as we

move the point toward the mirror, that is, as the point passes downward through positive integers, its image moves upward through negative integers.

The mirroring effect does not apply to the $3x - 1$ function. Thus $(3(5) - 1)/2^1 = 7$ in the $3x - 1$ function, and $(3(3) + 1)/2^1 = 5$ in the $3x + 1$ function. In both cases, the exponent 1 results in an increase in the value of $C(x)$.

Lemma 9.2

For each $i \geq 2$, the set of all i-level anchor tuples for the $3x - 1$ function is complete.

Proof:

Follows directly from (1) the fact that the set of all i-level anchor tuples for the $3x + 1$ function is complete, and (2) "Lemma 9.1" on page 107. □

Lemma 9.3

"Lemma 1.0" on page 12 and "Lemma 5.0" on page 16 apply to the $3x - 1$ function.

Proof:

Each lemma applies to the $3x - 1$ function because in our negative definition, the $3x - 1$ function is simply the $3x + 1$ function extended into the odd, negative integers, and this does not affect the proof of either Lemma, or of referenced lemmas.. □

Lemma 9.4

Let u be an odd, negative integer, and let $\bar{t}_u = <u, u', ... >$ be the infinite tuple it generates. Let $A(\bar{t}_u)$ be the infinite exponent sequence associated with $\bar{t}_u$. Let x be an odd, positive integer, and let $\bar{t}_x = <x, x', ... >$ be the infinite tuple it generates. Let $A(\bar{t}_x)$ be the infinite exponent sequence associated with $\bar{t}_x$.

Then $A(\bar{t}_u) \neq A(\bar{t}_x)$.

Proof:

Assume the contrary. Then u, x are at a distance $d = x - u$ apart. But as the length of their respective tuples increases, they nevertheless remain in the same succession of tuple-set extensions, by our hypothesis. However, by "Lemma 1.0" on page 12, a level i must eventually be reached such that d is less than the minimum distance $d(1, i)$ between first elements of successive i-level tuples, where

$$d(1, i) = 2 \cdot (2^{a_2})(2^{a_3})\ldots(2^{a_i})$$

and the exponent sequence A for the i-level tuple-set T_A is $\{a_2, a_3, ..., a_i\}$. This impossibility gives us our proof. □

Remark: Lemma 9.4 implies that there does not exist a counterexample x to the $3x + 1$ Conjecture such that $A(\bar{t}_x) = A(\bar{t}_u)$ for any counterexample u to the $3x - 1$ Conjecture. In passing we point out that by "Lemma 5.0" on page 16 applied to the $3x - 1$ function, we know that for each i, the set $\{\bar{t}_u(i)\}$ of all i-level prefixes of all $3x - 1$ counterexample infinite tuples $\bar{t}_u$ is complete. The next lemma is another way of expressing this fact. It shows how a certain class of $3x + 1$ counterexample tuples are "pushed away" to infinity, and hence to non-existence.

Lemma 9.5

Let u be a counterexample to the 3x – 1 Conjecture, and let $\bar{t}_u$ = *<u, u′, ... > be the infinite tuple it generates. Let* $A(\bar{t}_u(j))$ *be the exponent sequence associated with the prefix* $\bar{t}_u(j)$. *And similarly for counterxamples x to the 3x + 1 Conjecture. Then for all counterexamples x to the 3x + 1 Conjecture:*

If $A(\bar{t}_x(2)) = A(\bar{t}_u(2))$ *then* $x - u$ *must be* $\geq 2 \bullet 2^{a_2}$*; and*

If $A(\bar{t}_x(3)) = A(\bar{t}_u(3))$ *then* $x - u$ *must be* $\geq 2 \bullet 2^{a_2} 2^{a_3}$ *; and*

If $A(\bar{t}_x(4)) = A(\bar{t}_u(4))$ *then* $x - u$ *must be* $\geq 2 \bullet 2^{a_2} 2^{a_3} 2^{a_4}$ *; and*

...

Proof:

Same argument as in the proof of “Lemma 9.4” on page 108, plus part (b) of the distance function lemma, namely,“Lemma 1.0” on page 12□

How Is the Interleaving of Counterexample and Non-Counterexample Anchor Tuples Possible?

“Lemma 5.0” on page 16, which also applies to the 3*x* – 1 function, states that if counterexamples exist, then each *i*-level tuple-set, where *i* ≥ 2, contains an infinity of counterexample tuples and an infinity of non-counterexample tuples. Since for each level *i* ≥ 2, there are both non-counterexample and counterexample anchor tuples in the case of the 3*x* – 1 function, this would seem to imply that, for each level *i* > 2, the set of non-counterexample anchor tuples is complete, and similarly for the set of counterexample anchor tuples. (At level 2, this does not hold.) This in turn would imply that each *i*-level tuple-set has two anchor tuples, one non-counterexample and the other counterexample, which is impossible.

But in fact the existence of non-counterexample and counterexample anchor tuples is possible if the following is always the case, namely, that for each level *i* > 2, there is a maximum *j* < *i* such that the set of *j*-level non-counterexample anchor tuple *prefixes* is complete, and similarly for the set of *j*-level counterexample anchor tuple *prefixes*. For longer prefixes, the corresponding sets are not complete, although the set of both non-counterexample and counterexample *i*-level anchor tuples is always complete.

We now believe that we have made our attempts to answer the question of interleaving unnecessarily difficult. For, regardless if the anchor tuple of an *i*-level tuple-set is non-counterexample or counterexample, the rest of the tuples in the tuple-set, both non-counterexample and, if counterexamples exist, counterexample, are all associated with the same exponent sequence *A* that defines the tuple-set. However, the mark of each infinite tuple having a prefix (tuple) in the tuple-set must be greater than *i*: otherwise, the tuple-set would have two anchor tuples, which is impossible.

And so successive extensions of the anchor tuple define successive extensions of the original tuple-set, with the marks of all tuples in each tuple-set other than the anchor tuple being greater than the level of the tuple-set. However, the tuples in these extensions grow farther apart (by part (a) of Lemma 1.0) because some tuples drop out when their exponent sequences are no longer the same as those of the extensions of the original anchor tuple.

Thus we believe that the phrase "interleaving of counterexample and non-counterexample anchor tuples" is misleading. Each tuple-set has exactly one anchor tuple, non-counterexample or counterexample. The interleaving refers to the set of all anchor tuples at a given level *i*. In general, we can say only that this set is always complete (is associated with the set of all *i*-level exponent sequences).

Note: later reflection based on consideration of recursive "spiral"s for the 3x – 1 function, incline us to assert that the following holds for this function:

Every finite exponent sequence is eventually the prefix of a non-counterexample anchor tuple, *and* of a counterexample anchor tuple. However, unlike the case of the 3*x* + 1 function, the *suffixes* of these two tuples will differ.

3x – 1 Anchor Tuples and a Failed Proof of the 3x + 1 Conjecture

The reader will naturally wonder how "Lemma 9.5" on page 109 can hold, given the fact that for each level *i* there is a complete set of positive anchor tuples (non-counterexample and counterexample). How is it possible that the distance function is not violated by the facts that (1) each negative counterexample *u* is eventually (that is, at some level) a negative anchor tuple, and remains so for an infinity of successive levels, and that (2) for each level *i*, there must be a positive anchor tuple $\bar{t}_x(i)$ such that $A(\bar{t}_u(i)) = A(\bar{t}_x(i))$?

The answer is clear from the lemma statement itself. It is *different* positive anchor tuples that fulfill the role of providing successive matching exponent sequences for the prefixes of $\bar{t}_u$. A single positive *x* does not give rise to all these positive anchor tuples.

If the reader asks what happens to the prefixes of infinite tuples $\bar{t}_x$ once their exponent sequences no longer match those of prefixes of $\bar{t}_u$, the answer is that they are associated with the exponent sequences of different negative anchor tuples.

We see, therefore, that the 3*x* – 1 anchor tuples limit the possible exponent sequences for positive counterexample tuples. But not sufficiently for a proof of the 3*x* + 1 Conjecture, because, for example, it is possible that all negative counterexamples give rise to infinite cycles. If so, then it is still possible that positive counterexamples exist that do not give rise to infinite cycles.

Why Are There Counterexamples to the 3x – 1 Conjecture?

Two known counterexamples to the 3*x* – 1 Conjecture are 5 and 17, or, in our alternate version of the function, –5 and –17. Both counterexamples give rise to infinite loops. For –5 we have the infinite tuple <–5, –7, –5, ...> and for –17 we have the infinite tuple <–17, –25, –37, –55, –41, –61, –91, –17, ...>. As in the case of the 3*x* + 1 function, –5 (or 5) is the base element of an infinite set of recursive "spiral"s in the 3*x* – 1 function, and similarly for –17 (or 17). We conjecture that these infinite sets are disjoint, and in fact that the three infinite sets of recursive "spiral"s with base elements –5 (or 5), – 17 (or 17), and –1 (or 1) are disjoint.

If we sharpen our question to, "Why does the infinite cycle <–5, –7, –5, ...> (or <5, 7, 5, ...>), which consists of the counterexamples –5 (or 5) , and –7 (or 7), exist?", then the answer is simple: the cycle is a consequence of the distance functions *d*(1, 3), *d*(2, 3) and *d*(3, 3) in "Lemma 1.0" on page 12 operating on the 3-level tuple <11, 17, 13> in the 3-level tuple-set T_A , where $A = \{1, 2\}$.

Specifically, subtracting $d(1, 3) = 2 \cdot 2^1 \cdot 2^2 = 16$ from 11 gives us –5. Subtracting *d*(2, 3) =

$2 \bullet 3 \bullet 2^2$ (see "Distances between elements of tuples consecutive at level i" on page 13) = 24 from 17 gives us –7. Finally, subtracting $d(3, 3) = 2 \bullet 3^{3-1} = 18$ from 13 gives us –5.

Thus the distance functions give us the tuple <–5, –7, –5> in the odd, negative integers, which is the negative of the tuple <5, 7, 5> for the 3*x* – 1 function.

The 5*x* + 1 Function, and *Nx* + *C* Functions in General

This Appendix has been devoted to 3*x* + *C* funct.ions. However, we must not overlook the fact that there are *Nx* + *C* functions where *N* is any integer. I do not know to what extent, if any, these functions have been studied. The question of the denominators that should be considered for each *N*, *C* is no doubt important in itself.

The 5*x* + 1 function, with the same denominator as for the 3*x* + 1 and other functions considered above, is interesting because it exhibits the same property we observed in these other functions, namely, that if counterexamples exist, then there are counterexamples already for small numbers.

In the case of the 5*x* + 1 function, there is a tuple, <5, 13, 33, 83, 13, ...>, which contains a non-trivial infinite cycle, hence a sequence of counterexamples. The first two elements of this tuple are the first two elements of the second 2-tuple in a 5*x* + 1 tuple-set (assuming, of course, that the tuple-set structure holds for this function).

A study of all the *Nx* + *C* functions might begin with an attempt to somehow "line up" the functions, so that corresponding elements of corresponding tuples for each function could easily be compared. It might then be possible to answer such questions as, Why is it that if counterexamples exist, at least one of them is a smal, odd, positive integer? (See "Why Are There Counterexamples to the 3x – 1 Conjecture?" on page 110.)