.Section 2. Recursive "Spiral"s

Definitions

Recursive "Spiral"

A *recursive "spiral"* is the infinity of domain elements that map to a given range element in one iteration of the $3x + 1$ function, as established in the proof of Lemma 5.0. (See Fig. 4.) Each range element in the infinity of elements in turn sets up a recursive "spiral", etc. Thus the infinite set of all "spiral"s relative to a given range element are a *self-similar* structure ([4], p. 34).

The recursive "spiral" structure has been independently discovered by at least two researchers besides the author, although the author is not aware of anything in the literature that explicitly deals with this structure.

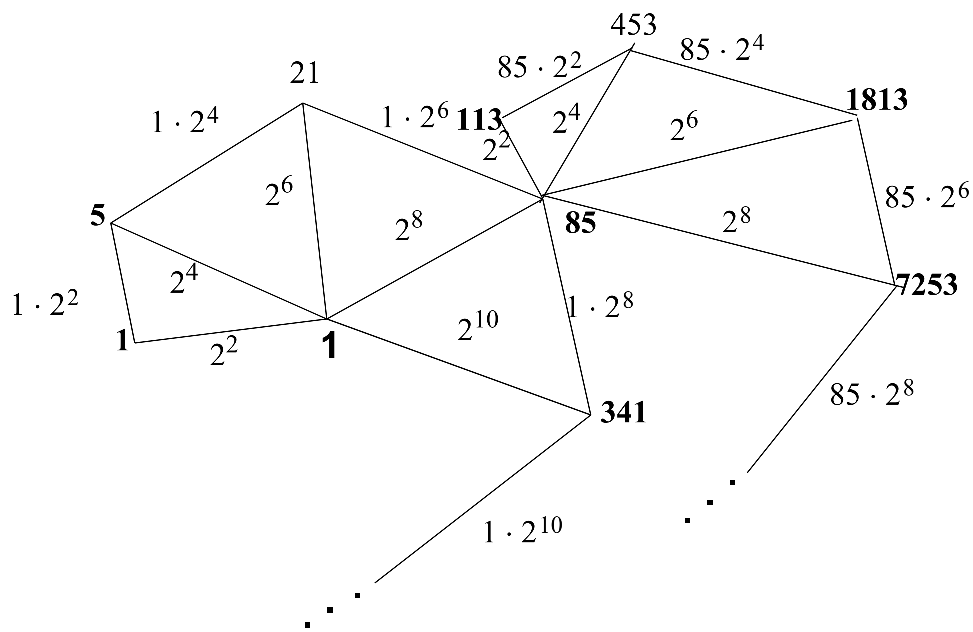

Fig. 4. Recursive "spirals" structure of computations produced by the $3x + 1$ function.

Bold-faced numbers are range elements (21 and 453 are multiples of 3, hence not range elements). Partial "spirals" surrounding the base elements 1 and 85 are shown. The line connecting 1813 to 85 is marked with a 2^6 because $((3 \cdot 1813) + 1)/ 2^6 = 85$. The line connecting 453 to 1813 is marked $(85 \cdot 2^4)$ because $453 + (85 \cdot 2^4) = 1813$. The exponents of 2 are not always even, of course. The "spiral" of numbers (not shown) mapping to 341 has odd exponents.

Level *i* and Base Sequence of "Spiral"

Let *y* be a range element, e.g., 1. We define *y* to be at *level 0 relative to y*. We now define all *x* that map to *y* in a single iteration to be at *level 1 relative to y*. (*Warning*: no suggestion is intended that the term *level* as defined here is the same as the term *level* as defined above for tuple-sets, although, as we shall see, there is a strong relationship between the two terms.) We define all domain elements of level 1 iterations that map to y to be level 1 *elements*. These elements constitute a sequence (a "spiral" in Figure 4). Specifically, they constitute a unique *base sequence* (relative to y). Thus, for example, the base sequence relative to 1 is the sequence {1, 5, 21, 85, 341, ... }. We define all x that map to a level 1 element in a single iteration to be at level 2 relative to y, and similarly for level 2 elements. And so on for all levels i. When y is understood, we will sometimes eliminate the phrase, "relative to y". The expressions *level i sequence* and *level i "spiral"* thus mean the same thing.

We define the range element mapped to, in a single iteration, by each element of a "spiral", to be a *center element*, because it is the center of a "spiral", as shown in Figure 4. Often, we will call a center element a *base element*. The infinite set of elements that map to a given base element corresponds, in [3] (p. 21), to a *predecessor set*, although unlike a predecessor set, the infinite set of elements that map to a given base element contains no even numbers.

We say that elements of the base 1 sequence map *directly* to the base element, and that elements of level i, $i > 1$, map *indirectly* to the base element.

We define a *path*, relative to any base element y, to be a finite sequence of elements of "spirals" at levels $i, i - 1, i - 2, ... 0, i \geq 0$, such that the "spiral" element at level j, $1 \leq j \leq i$ maps directly to the element at level $j - 1$ in a single iteration. Thus, e.g., <13, 5, 1> is a path. A path is thus the equivalent of a tuple. Each path defines an exponent sequence, e.g., in the case of our example, the sequence {3, 4}.

Some examples of elements at different levels: If $y = 1$, then 1, 5, 21, 85, 341, ... are level 1 elements relative to 1. They also constitute the base sequence relative to 1, the center, or base element. For the center element (or base element) 5 in this base sequence, the level 1 elements are 3, 13, 53, 213, 853, ... These are level 2 elements relative to 1.

We define the set of odd, positive integers lying *between* any two successive elements of a level i sequence to be *interval*s of that sequence. Thus, for example, 7, 9, 11, 13, 15, 17, 19, are the elements of the second interval in the level 1 sequence, 1, 5, 21, 85, 341, ... When necessary, we number the intervals in a given sequence starting with 0.

Distance Functions on "Spiral"s

The proof of Lemma 5.0 implicitly defines two distance functions on "spirals": one, between any "spiral" element and the base element of the "spiral", and the other between successive elements of a "spiral". We will refer to these simply as *"spiral" distance functions*, specifying which one we mean as required. We give these functions in the next lemma.

Lemma 11.0. *(a) The distance between the jth element, $j \geq 1$, of a "spiral", and the base element y of the "spiral", is given by $|(2^k y - 1)/3 - y|$, where k is the jth element in the sequence <1, 3, 5, ..., > or the sequence <2, 4, 6, ..., > as established by y.*

(b) The distance between successive elements x, x' of a "spiral" is given by 3x + 1.

Proof:

(a) Follows directly from Lemma 5.0.

(b)(a) By Lemma 5.0 we have

$$\frac{3x+1}{2^j} = y$$

and

$$\frac{3x'+1}{2^{j+2}} = y$$

so that

$$\frac{3x+1}{2^j} = \frac{3x'+1}{2^{j+2}}$$

and hence

$$2^2x+1 = x'$$

and thus

$$x'-x = (2^2x+1)-x = 3x+1$$

□

The following pair of lemmas is of interest because of the presence of the base sequence relative to 1.

Lemma 12.1:

$$2^2+2^4+2^6+\ldots+2^{2j} = s(j+1)-1 = 2\cdot(2\cdot s(j))$$

where s(j) is the jth element of the base sequence relative to 1.

Thus, for example:

$$2^2+2^4 = 20 = 21-1 = 2\cdot(2\cdot 5)$$

$$2^2+2^4+2^6 = 84 = 85-1 = 2\cdot(2\cdot 21)$$

$$2^2+2^4+2^6+2^8 = 340 = 341-1 = 2\cdot(2\cdot 85)$$

The proof is given along with the proof of Lemma 12.2.

Lemma 12.2:

$$2^1 + 2^3 + 2^5 + \ldots + 2^{2j-1} = (2 \cdot s(j))$$

where s(j) is the jth element of the base sequence relative to 1.

Thus, for example:

$$2^1 + 2^3 = 10 = (2 \cdot 5)$$

$$2^1 + 2^3 + 2^5 = 42 = (2 \cdot 21)$$

$$2^1 + 2^3 + 2^5 + 2^7 = 170 = (2 \cdot 85)$$

Proof:
The following proof is by Michael O'Neill.

$$s(j) = \frac{2^{2j} - 1}{3}$$

$$= \frac{2^{2j-1} + 2^{2j-2} + \ldots + 1}{3}$$

$$= \frac{3 \cdot 2^{2j-2} + 3 \cdot 2^{2j-4} + \ldots + 3}{3}$$

EQ1

$$= 2^{2j-2} + 2^{2j-4} + \ldots + 1$$

The proof of Lemma 12.1 follows by multiplying through EQ1 by 4 and the proof of Lemma 12.2 follows by multiplying through EQ1 by 2. □

Summary of Properties of Recursive "Spiral"s

For readers with limited time, we now provide a table that summarizes our results — both those above and those to follow — on recursive "spiral"s.

Note: some table-rows may have the same content as other rows, though under different properties. This redundancy is deliberate, the purpose being to aid understanding and to make the looking up of properties easier.

Table 1: Some important properties of recursive "spiral"s

Property	Value of property	Reference
Self-similarity	Each element of a "spiral" is the base element of a "spiral" each element of which is the base element of... Also, for all base points *y*, the infinite set of "spiral"s relative to *y* is path-similar to every other such infinite set, i.e., all paths, as defined by finite exponent sequences, exist in each such infinite set.	[4], p. 34
Set of elements in a "spiral"	$\{x \mid x = (2^k y - 1)/3\}$, where *y* is the base element of the "spiral" and all *k* are either even or odd, depending on y. Thus, the number of elements in a "spiral" is infinite.	Lemma 5.0
Distance between *j*th element of a "spiral" and its base element *y*	$\lvert(2^k y - 1)/3 - y\rvert$, where *k* is the *j*th element of {1, 3, 5, ... } or {2, 4, 6, ...}, depending on *y*.	Lemma 11.0
Distance between successive elements x, x', of a "spiral"	$3x + 1$	Lemma 11.0

Table 1: Some important properties of recursive "spiral"s

Property	Value of property	Reference
Number of levels in the infinite set of "spiral"s relative to any given base element	Infinite	Lemma 5.0
In the infinite set of "spiral"s relative to any given base element, number of paths defined by any given exponent sequence $A = \{a_2, a_3, ..., a_i\}$.	Infinite, i.e., there are an infinite number of paths for *each* exponent sequence, as in tuple-sets.	Lemma 7.0
Congruence classes to which base element and "spiral" elements belong	For all $i \geq 2$, and for each base element y: (1) y is an element of a reduced residue class mod $2 \cdot 3^{(i+1)-1}$; (2) the elements of the base sequence (i.e., "spiral" having y as base element) are elements of a sequence s of all reduced residue classes mod $2 \cdot 3^{(i-1)}$, with s being repeated endlessly over all elements of the"spiral".	Lemma 15.85.

Possible Strategies for Proving Conjecture 1 Using "Spiral"s

Strategy of Proving Existence of a Certain Map Between Tuples and Paths in "Spirals"

Probably the most direct approach to a proof of Conjecture 1 using "spiral"s would be by proving there exists a one-one onto map between tuples in tuple-sets and finite paths in the infinite set of "spiral"s whose base element is 1. Such a proof would prove Conjecture 1 because the set of all tuple-sets represents the set of all finite computations by the $3x + 1$ function.

A major step in the direction of such a proof is outlined in the third section of this paper, which shows how tuple-sets and recursive "spirals" can be merged into a single structure. But at this point we will merely set forth several lemmas which may prove of value.

Lemma 13.0. *A bijection exists between tuples and "spiral" elements that eventually map to 1.*

The following proof is by Michael O'Neill.

Proof:

We can construct the set of "spiral" elements that map to 1 by specializing the result of Lemma 5.0 to $y = 1$. By Lemma 3.24, each finite sequence of exponents gives an infinite set of exponents, $j = j_0 + (2 \cdot 3^{i-1})t$, where j_0 is the least positive value and t is a parameter. The one fine point is that we must disallow j_0 from being 2, since that would allow loops of 1s. We can use the next higher j for j_0. If we prevent loops, then the set of j will represent unique odd, positive integers that map to 1, and these odd, positive integers will be all such odd, positive integers, since all possible exponent sequences are used. This set of odd, positive integers can be indexed by the exponent sequence and the parameter t.

Likewise, each tuple-set is generated by a given finite sequence of exponents, and contains an infinite set of tuples which can be indexed by a parameter t, as in Lemma 6.0. So, the whole set of tuples can be indexed by exponent sequence and the parameter t, just as for the "spiral" elements.

Clearly, a one-one and onto map is formed by equating the corresponding indices. □

The reason the bijection described in Lemma 13.0 is not the desired mapping that would prove Conjecture 1 is that it does not prove that the "spiral" elements that eventually map to 1 are, in fact, all the odd, positive integers.

Strategy of "Filling-in" of Intervals

1. Consider the range element 1.

2. Consider the level 1 "spiral" elements that map to 1, i.e., consider the base sequence {1, 5, 21, 85, 341, ...} and its intervals. Our goal will be to show that, as the number of levels approaches infinity, all these intervals are eventually "filled in" by odd, positive integers that eventually map to 1. To do this, we need to show the following:

3. No element of a higher level sequence is "wasted" by being mapped "on top of" one of the base sequence elements. This is proved in Lemma 13.5.

4. No interval of the base sequence is "skipped over" by successive elements of a higher level sequence once that sequence gets started. This is proved in Lemma 14.

5. The first elements of higher level sequences don't increase so rapidly that "holes" are left behind them. This is Conjecture 4.

Lemma 13.5. *For all base sequences, and for all levels $i \geq 2$ relative to a base sequence, no element of a level i sequence is an element of the base sequence. Thus, in particular, no level i sequence, $i \geq 2$, is an element of the base sequence relative to 1, i.e., of the base sequence {1, 5, 21, 85, 341, ... }.*

This Lemma is necessary to ensure that no higher level elements are "wasted" in the filling-in process as a result of being mapped "on top of" elements of the base sequence relative to 1.

Proof:

Assume to the contrary that an element of a higher level sequence is an element of a base level sequence, and hence maps to the base element via some exponent j.

Then, by Lemma 5.0, so does every other element of that higher level sequence map to the base element via an exponent of the same parity as j. Hence the higher level sequence is identical with the base sequence, contrary to hypothesis. □

Lemma 14.0. *For any "spiral" at any level i ≥ 1, the sequence of elements of the "spiral" map to successive intervals of the base sequence.*

The following proof is by Michael O'Neill.

Proof:
The base sequence relative to $n = 1$ is:

$$\frac{2^k - 1}{3}$$

where k is even.
The base sequence relative to $n > 1$ is:

$$\frac{2^k n - 1}{3}$$

where k is even or odd depending on n.
The initial element of the base sequence relative to n must fall between some pair of elements of the base sequence relative to 1. Assume that the exponent for the n-sequence is odd (even works the same way). Assume that the 1-sequence interval bracketing its start has exponents k and $k + 2$. Then:

$$\frac{2^{k+2} - 1}{3} > \frac{2n - 1}{3} > \frac{2^k - 1}{3}$$

or

$$2^{k+2} > 2n > 2^k$$

Then multiply by
2^{2j}:

$$2^{k+2j+2} > 2^{2j+1}n > 2^{k+2j}$$

which shows that nesting is preserved. □

Of course, this Lemma in no way states what the first interval of the base sequence mapped to is. This Lemma also in no way requires that only one element map to a given interval, but that does us no harm as far as a proof is concerned.

The following calculations illustrate the truth of Lemma 14.0:

The base sequence relative to 1 is {1, 5, 21, 85, 341, 1365, 5461, 21845, ...}

The level 1 sequence relative to 5 (level 2 relative to 1) is {3, 13, 53, 213, 853, 3413, 13653, ...}. At least the first 7 elements of this sequence are elements of successive intervals of the base sequence relative to 1.

The level 1 sequence relative to 13 (level 3 relative to 1) is {17, 69, 277, 1109, 4437, 17749, ...}. At least the first 6 elements of this sequence are elements of successive intervals of the base sequence relative to 1.

Lemmas 13. 5 and 14.0 show that any sequence at any level “does the right thing”, namely, it doesn’t waste elements, and it adds at least one element to each successive interval of the base sequence relative to 1 once the higher level sequence gets started. The most difficult task, however, is to prove that first elements of higher level equences increase “sufficiently slowly”, so that no element in the base sequence is “skipped over”. This would have to be proved (Conjecture 4), which would, of course, then prove Conjecture 1.

Conjecture 4. On the Filling-in of Intervals in the Base Sequence
Every interval in the base sequence relative to 1 is eventually filled by elements that map to 1.

(The author will pay $100 for the first correct proof or disproof of this Conjecture provided it is not trivially implied by another result.)

We must prove that first elements of higher level sequences increase “sufficiently slowly”. Lemma 13.5 relieves us of having to worry about higher level sequences providing “too many” elements because Lemma 13.5 assures us that higher level sequence elements cannot map on top of other sequence elements. We begin with an observation:

If the exponent $a_j = 1$, the x producing one iteration of the $3x + 1$ function will be *less than* the range element y it produces — the only exponent when this occurs. This is in our favor, of course. Otherwise, x will be greater than the base element, except in the case when $y = 1$. We need to prove, in Conjecture 4, that x will never be “too large”. Let us consider the “spiral” whose base element is 13, namely, the sequence {17, 69, 277, 1109, 4437, 17749, ...}, which produces 13 via the exponents 2, 4, 6, 8, 10, 12, ... respectively. 17 is larger than 13, but not a great deal larger.

An approach to a proof might be to show that the rate of increase of the first elements in each level $i + 1$ sequence is less than the rate of increase of the number of interval elements in successive intervals in the base sequence.

We should keep in mind the following:

The author has heard, from a source he considers reliable, that as of Nov., 1998, Conjecture 1 had been verified for all odd, positive integers up to about 56 quadrillion, i.e., $56 \bullet 10^{15}$.

Since the number of elements in successive intervals of the base sequence relative to 1 are 2^2, 2^4, 2^6, ... this means that, **at least the first 26 intervals in the base sequence are known to be filled with odd, positive integers that map to 1**:

$$((2^2)^1) + ((2^2)^2) + ((2^2)^3) + \ldots + ((2^2)^{26}) = \frac{(2^2)^{27} - 1}{2^2 - 1} - 1 < 2^{54} < 56 \cdot 10^{15}$$

Now since, by Lemma 5.0, the structure of computations mapping to any range element counterexample is similar to that of computations mapping to 1, it seems plausible that a similarly long sequence of intervals must be completely filled in by odd, positive integers that map to the base odd, positive integer for the smallest of the counterexamples. Therefore the odd, positive integers that map to 1 must somehow manage to “leap over” this long sequence of filled-in counterexample intervals before the odd, positive integers that map to 1 can “find a place” again. We know this leap must occur because the number of levels is infinite, and the number of odd, positive integers in a given point of the “spiral” at any level is infinite. The bounds on the distances between successive elements of a sequence, plus Lemma 14.0, seem to argue against such a thing being possible. Perhaps a proof can be constructed from this observation alone.

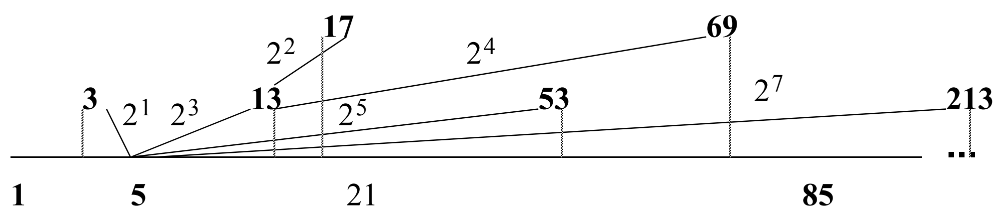

Fig. 5. Illustration of part of the “filling-in” process.

A conjecture that would imply the truth of Conjecture 4 is the following.

Conjecture 5. Let y be any base element, and let x, x' be successive elements of the base sequence (“spiral”) relative to y. Then all elements of the interval between x and x' are filled by “spiral” elements at no higher level than $3x + 2$ relative to y.

The author will pay $100 for the first correct proof of this Conjecture.

Fig. 5.3. illustrates portions of levels 0, 1, and 2 relative to the base element 1.

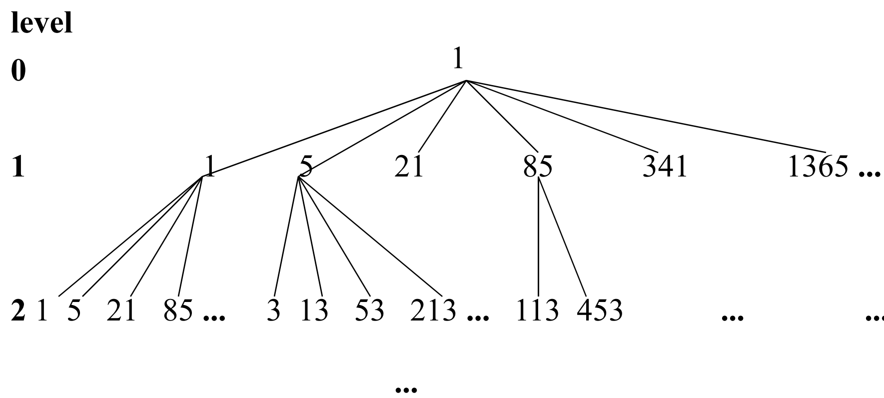

Fig. 5.3. Illustration for Conjecture 5.

Strategy of Proving There Is No Minimum Counterexample

We now prove several lemmas which might enable us to prove, using arguments based on the recursive "spiral"s structure, that there is no minimum counterexample.

Lemma 15.0. *The parity of exponents mapping from successive range elements of any "spiral" alternates. The alternation sequence is not affected by the presence of multiples-of-3 in the "spiral".*

Proof:

The following proof is by Michael O'Neill.

The difference between two successive elements of the "spiral" with base element b is:

$$\frac{b \cdot 2^{j+2} - 1}{3} - \frac{b \cdot 2^{j} - 1}{3} = \frac{b \cdot 2^{j}(4-1)}{3} = b \cdot 2^{j}$$

where, for a given "spiral", j is either always even or always odd. Since even (odd) powers of two are congruent to 1 (2) mod 3, an even (odd) "spiral" will have the residue sequence 1, 2, 0, 1, 2, 0, ... (2, 1, 0, 2, 1, 0, ...). The starting point of the sequence will depend on b.

Nothing maps to an element with residue 0 mod 3. Here are the elements mapping to the residues 1 and 2:

$$\frac{2^{j}(3n+1) - 1}{3} = \frac{3 \cdot 2^{j}n + 2^{j} - 1}{3}$$

$$\frac{2^{j}(3n+2) - 1}{3} = \frac{3 \cdot 2^{j}n + 2^{j+1} - 1}{3}$$

For the right-hand sides to be integral, the first requires j be even and the second requires it odd. So, the exponent sequence for an even (odd) "spiral" will be even, odd, skip, even, odd, skip, ... (odd, even, skip, odd, even, skip, ...). This proves the Lemma. □

The next lemma shows that, for any given range element y at any level $i > 2$, the smallest exponent produced by a multiple-of-3, is completely predictable.

Lemma 15.5. *For each element y of the two top rows {5, 11, 17, 23, ... } and {1, 7, 13, 19, ...} of all 2-level tuple-sets, and hence for each y in the range of the 3x + 1 function, the smallest exponent mapping to y by a multiple-of-3 is given by the following table:*

Table 2:

y	**Smallest exponent a_2 such that** $\frac{3(3x)+1}{2^{a_2}} = y$ **x odd**
$y \equiv 5 \bmod 3 \bullet 2 \bullet 3^1$	1
$y \equiv 11 \bmod 3 \bullet 2 \bullet 3^1$	5
$y \equiv 17 \bmod 3 \bullet 2 \bullet 3^1$	3
$y \equiv 1 \bmod 3 \bullet 2 \bullet 3^1$	6
$y \equiv 7 \bmod 3 \bullet 2 \bullet 3^1$	2
$y \equiv 13 \bmod 3 \bullet 2 \bullet 3^1$	4

Proof:

The proof for each case is similar. We give the proofs for only two cases.

Case $y \equiv 5 \bmod 3 \bullet 2 \bullet 3^1$

We begin by observing that 3 maps to 5 via the exponent 1. Then if y is the kth element from 5 such that the distance between successive elements is $3 \bullet 2 \bullet 3^1$, then the element that maps to y is, by Lemma 1.0 (b), $3 + (3 \bullet 2 \bullet 2^1)k$, so we write:

$$\frac{3(3 + (3 \cdot 2 \cdot 2^1)k) + 1}{2^1} = 5 + (3 \cdot 2 \cdot 3^1)k$$

hence

$$9 + 3^2 \cdot 4k + 1 = 10 + 3^2 \cdot 4k$$

This is an equality, and the proof is obtained by simply reversing the sequence of steps.

Case $y \equiv 7 \bmod 3 \cdot 2 \cdot 3^1$

Since 9 maps to 7 via the exponent 2, we write, again utilizing Lemma 1.0 (b):

$$\frac{3(9 + (3 \cdot 2 \cdot 2^2)k) + 1}{2^2} = 7 + (3 \cdot 2 \cdot 3^1)k$$

hence

$$27 + 3^2 \cdot 8k + 1 = 28 + 3^2 \cdot 8k$$

This is an equality, and the proof is obtained by simply reversing the sequence of steps.□

Lemma 15.75. *In the following table, let the values of y be the minimum residues of the reduced residue classes mod 18 to which a base element of a recursive "spiral" must belong (see Lemma 15.5). Let 1, 2, 3, ..., 20 in the top horizontal column denote exponents. Then:*

(a) the value in the cell defined by any y and any exponent is the minimum residue of the reduced residue class mod 18 of the "spiral" element that maps to y in one iteration of the 3x + 1 function. Furthermore,

(b) the table holds for all exponents congruent mod 18. Thus, e.g., if y is a base element that is congruent to 7 mod 18, then all "spiral" elements x mapping to y via any exponent congruent to 10 mod 18, are congruent to 13 mod 18.

Table 3: Description of all "spiral" elements mapping to any base element *y* in one iteration of 3*x* + 1 function (see notes immediately following table

y	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1		1		5		*		13		17		*		7		11		*		1
5	*		13		17		*		7		11		*		1		5		*	
7		*		1		5		*		13		17		*		7		11		*
11	7		11		*		1		5		*		13		17		*		7	

Table 3: Description of all “spiral” elements mapping to any base element *y* in one iteration of 3*x* + 1 function (see notes immediately following table

y	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
13		17		*		7		11		*		1		5		*		13		17
17	11		*		1		5		*		13		17		*		7		11	

Notes:

(1) Columns 19 and 20 are included simply to show the repetition of values in columns 1 and 2, respectively, as required by the Lemma.

(2) “*” denotes “spiral” elements that are multiples of 3. These cannot by mapped *to* (Lemma 0.2), although they map to *y*.

Proof:

(a) Follows directly from calculation.

(b) Follows from the fact that $2^k \equiv 2^{k + (\phi(18) = 6)}$ mod 18, for all $k \geq 1$. □

Remarks

(1) The above table describes the infinite set of recursive “spiral”s relative to any base element, including base elements which are counterexamples. Every path in such a set can be represented by a sequence of cells, as can the “inverse” of any path (in which we begin with the last element in the path).

(2) The above Lemma makes clear that not only does the existence of a counterexample imply the existence of an infinity of counterexamples (Lemma 5.0), it also implies the existence of an infinity of counterexamples in each of the reduced residue classes mod 18.

(3) The above Lemma also makes clear that, at least up to congruence classes, there is no difference between the behavior of a counterexample and of a non-counterexample. Any path through a set of infinite “spiral”s whose base element is a counterexample, has a duplicate in a path through a set of infinite “spiral”s whose base element is a non-counterexample. (This fact is also implied by the definition of tuple-set.)

(4) The similarity of the above table to the tables summarizing extensions of top rows in 3-level tuple-sets, in the sub-section “Generating Level-(i + 1) Top Rows from Level-i Top Rows” on page 37, suggests that for each $2 \cdot 3^{i-1}$, $i \geq 2$, there exists a pair of such tables, one for tuple-sets and one for recursive “spiral”s. These tables may make it possible to prove that every tuple in every tuple-set is contained in the infinite set of “spiral”s whose base element is 1. This would imply the truth of Conjecture 1.

Lemmas 15.0 and 15.75 suggest a generalization to all $i \geq 2$, a generalization which turns out to be true, as the next lemma shows.

Lemma 15.85. *For each $i \geq 2$, and for each base element y, the successive elements (ignoring multiples of 3) in any “spiral” whose base element is y consist of elements from a fixed sequence of all reduced residue classes mod $2 \cdot 3^{i-1}$, the same sequence of classes being repeated endlessly in any given “spiral”.*

Proof:
Follows directly from Lemmas 7.1 and 7.3. □

Lemma 15.87. *For each base element y, and for each exponent sequence A, A maps to an infinite number of elements of the base sequence relative to y.*

Proof: Follows directly from Lemma 15.85, which says, in effect, that, for any $i \geq 2$, there exists an infinity of elements in the base sequence relative to *y* that are the last elements of *i*-level tuples in the tuple-set T_A defined by *A*. □

Conjecture 12. On the Existence of Paths With the "Less-to-greater" Property

Let *S* be the infinite set of recursive "spiral"s relative to a base element *y*. For each possible base element (i.e., for each range element) there exists such an *S*. Let {*S*} denote the set of all such *S*. Then, by Lemma 15.0, for each "spiral" in each such *S*, one of the following sequences describes the successive "spiral" elements:

3, o, e, 3, o, e, 3, o, e, ...;
o, e, 3, o, e, 3, o, e, 3, ...;
e, 3, o, e, 3, o, e, 3, o,...;
3, e, o, 3, e, o, 3, e, o, ...;
e, o, 3, e, o, 3, e, o, 3,...;
o, 3, e, o, 3, e, o, 3, e,...;

where "e" means that even parity exponents map to the "spiral" element, "o" means that odd parity exponents map to the "spiral" element, and "3" means that the "spiral" element is a multiple of 3, and hence, by Lemma 0.2, no exponents map to it.

Now in each "spiral" in each *S* in {*S*}, let the value $2^n/3$ be assigned to each branch which represents division by 2^n. Thus, e.g., 2/3 will be assigned to each branch that represents division by 2, and 4/3 will be assigned to each branch that represents division by $2^2 = 4$. Let the value 1 be assigned to each 3 in each "spiral". The rationale lying behind these values is that, e.g., if $(3x + 1)/2^1 = y$, then *x* is about (2/3)*y* (the 1 becomes negligible for large *x*), and if $(3x + 1)/2^2 = y$, then *x* is about (4/3)*y* (the 1 again neglible for large *x*). And so on for all exponents.

Then, for each *S* in {*S*} except the one whose base element is 1, there exists a path (succession of branches) from the base element such that, if the branch values are multiplied together, the result will be less than 1.

The author will pay $100 to the first person who proves or disproves this conjecture.

A proof of this Conjecture would constitute a proof of Conjecture 1 because it would mean that there is no minimum counterexample. Since each node (except a multiple of 3) is *always* mapped to by all even or all odd exponents (Lemma 5.0), then the existence, in each *S* except the *S* having 1 as base element, of a path the product of whose element values is less than 1 would mean that, for any assumed minimum counterexample, there is another, smaller counterexample.

It is conceivable that a proof might be obtained by showing that the attempt to construct a counterexample to what the Conjecture asserts, will never terminate. This proof might be accomplished in conjunction with a computer program that would generate assignments of the above sequences to "spirals", then test path products.

The answer to the following question might prove useful in proving there is no minimum counterexample.

Question 2. On the Number of Sequences, For Each *i*, Having the "Less-to-greater Property
Let $P(i) = \{p = (t_2)(t_3)...(t_i) \mid t_j = 2^h/3, 2 \le j \le i, h \ge 1, p < 1\}$. Let $|P(i)|$ denote the cardinality of $P(i)$. What is $|P(i)|$ for each $i \ge 2$?

The author will pay $75 for the first correct answer to this question.

Question 2 asks for the number of exponent sequences of length 1, 2, 3, ... such that, for each tuple defined by such a sequence, *x*, the first element of the tuple, is less than *y*, the last element of the tuple. The *p* are the path products, as described in Conjecture 12. It is important to keep in mind that different sequences may produce the same path product. But although each of these different sequences may produce different *x*, *y*, for all the sequences it will be the case that $x < y$.

It is tempting to hope that for some *i*, $|P(i)| \ge 2 \cdot 3^{i-1}$, because then (we may hope to argue), the number of exponent sequences of length $i - 1$ that yield path products < 1, is ≥ the number of equivalence classes of buffer exponents (see Lemma 7.1) and thus (we may hope to argue) we can be assured that (1) for each *y*, there exists an exponent sequence such that *x*, under this sequence, yields *y*, and that (2) *x* is < *y*, thus showing that there can be no minimum counterexample. Unfortunately, we have no reason to believe that if $|P(k)| \ge 2 \cdot 3^{i-1}$, then there must be at least one exponent sequence whose path product is < 1 for each equivalence class of buffer exponents. All such exponent sequences could be in the equivalence class whose minimum buffer exponent is $2 \cdot 3^{i-1-1}$.

A Way to Reduce Computation Time in Computer Testing of Conjecture 1

The concept of a path product, along with Lemma 1.0, leads to the following technique for computer testing of Conjecture 1, a technique that would appear to drastically reduce the amount of computation time required. The reasoning is as follows.

If a counterexample exists, then there is a minimum counterexample. Consider any sequence *A* of exponents such that (a) the path product of the exponents is < 1 and such that (b) the first exponent is 1. These two conditions ensure that the first element of each tuple in the tuple-set T_A defined by *A*, is a potential minimum counterexample. (If the first exponent is not 1, then the second element of each tuple is less than the first, hence the first element cannot be a minimum counterexample.)

Now, let C_1 denote the set of all sequences *A* of length 1 fulfilling conditions (a) and (b). (There is only one such sequence, namely {1}.) Let C_2 denote the corresponding set of sequences of length 2. (There are two such sequences, namely, {1, 1} and {1, 2}.) And similarly for C_3, C_4, ...

The sequences in C_{i+1} are extensions of those in C_i. Let *A* be a sequence in C_i, and *A′* a sequence in C_{i+1} which is an extension of *A*. Then, by Lemma 1.0, the distance between first elements of tuples in *A′* consecutive at level $(i + 1)$ $(d(1, (i + 1)))$ is greater than that between first elements of tuples in *A* consecutive at level *i* $(d(1, i))$. In particular, if the sequence defining C_i is $\{a_2, a_3, ..., a_i\}$, then the distance between first elements of tuples consecutive at level *i* is given by

$$d(1, i) = 2 \bullet 2^{a_2} \bullet 2^{a_3} \bullet \ldots \bullet 2^{a_i}$$

and the distance between first elements of tuples consecutive at level (i + 1) is given by

$$d(1, (i+1)) = 2 \bullet 2^{a_2} \bullet 2^{a_3} \bullet \ldots \bullet 2^{a_{i+1}}$$

Now, for each i, there exists an algorithm to determine C_i, because there is an upper bound on the sum of exponents in A such that A can meet condition (a). For example, 2^{2i} is such a bound, because $(2^2/3)(2^2/3)...(2^2/3) > 1$ (i terms in product).

As of the end of 1998, Conjecture 1 was reported to have been tested for all odd, positive integers to about 56 quadrillion (56×10^{15}). Therefore a procedure for further testing is as follows:

(1) Compute C_1, and for each A in C_1, make a list of first elements of all tuples in the tuple-set T_A. (The list is infinite, but is easily described, given the distance function $d(1, 1)$.) Repeat for C_2. If L' is the list corresponding to the tuple-set $T_{A'}$*, where* A' *is an extension of a sequence* A *in* C_1*, then* L' will be a proper subset of L, and similarly for all i, for the reason given above in connection with the discussion of C_{i+1}, C_i.

(2) Continue this process until a k is reached such that the smallest element in any of the lists L for C_k is larger than the largest integer that previous testing has confirmed to satisfy Conjecture 1. The elements of all the lists for C_{k-1} are the remaining candidates for smallest counterexample. Test these up to the limits of size of integer the testing software allows.

To get a rough estimate of the computation time saved once C_{k-1} is established, we note that $\log_2$ of 28 quadrillion is 54. So we need to test a subset of all sequences of length 54. (But for each $i + 1$, we need only test a proper subset of extensions of sequences of length i.) The distance between successive candidates for testing is therefore at least $2 \bullet 2^{54}$.

Definition of *finite sequence of parities of exponents*. A *finite sequence of parities* (*parity sequence*) is simply a finite sequence $<p_1, p_2, ..., p_n>$, where p_i is either "*even*" or "*odd*". Such a sequence represents an infinite class of tuple-sets, namely, the class consisting of all tuple-sets defined by exponents having the sequence of parities given. Thus one tuple-set defined by the parity sequence <*even*, *odd*, *odd*> is <4, 1, 3>.

Lemma 16.0. *Every finite sequence of* parities *of exponents occurs in the structure of all "spirals" mapping directly or indirectly to a given base element*.

Proof:

Given any finite sequence of parities of exponents, we can, by Lemma 15.0, always find, in any "spiral", an element u which is mapped to by all exponents of the parity of the last element of the sequence. We can then find, in the "spiral" having u as base element, an element v having the parity of the next-to-last element of the sequence, etc. □

Strategy of "Filling-in" of Residue Classes

The top rows of all 2-level tuple-sets are {1, 7, 13, 19, ... } and {5, 11, 17, 23, ... } (Lemma 3.057). The elements of the first row are mapped to by all even exponents, and the elements of the second by all odd exponents. These facts, and Lemma 15.0, suggest a "filling-in" strategy for tuple-sets that is analogous to the one described above for recursive "spiral"s. For, since, by

Lemma 15.0, the parity of exponents mapping to any base element of a "spiral" alternates, this means that any "spiral" in the infinite set of "spirals" whose base element is 1, "fills in" an infinite number of "locations" in the above two rows. For example, the base sequence relative to 1 is {1, 5, 21, 85, 341, ... }. So 1 and 85 "fill in" the locations 1 and 85 in {1, 7, 13, 19, ... }. 5 and 341 fill in the locations 5 and 341 in {5, 11, 17, 23, ... }. Elements of every higher-level "spiral" fill in additional locations in these two rows. Each "spiral" fills in an infinite number of locations in each of the two rows.

Similar questions regarding this filling-in process arise as for the recursive "spiral"s case.

Strategy of Using a Topology Defined on "Spiral"s

1. We begin by defining a topology for "spirals". To do this, we will follow one of the definitions of a topology given in *(Encyclopedic Dictionary of Mathematics* 1977), Article 408, "Topological Spaces".

"Let X be a set."

We take as our set of "points", the set of odd, positive integer range elements, or rather, the set of 1-tuples containing these odd, positive integers. This excludes the odd, positive integers that are multiples of 3 (Lemma 0.2), but with no loss in generality, since every multiple of 3 maps to a range element in one iteration (Lemma 0.4).

We identify each range element with the 1-tuple containing the range element. Thus, e.g., the range elements 1, 5, 7 are identified with the tuples <1>, <5>, <7>, respectively.

Next, we must define the neighborhood system of each point x. A neighborhood of a point (i.e., of a range element) x will be all finite-tuples (a) whose last element is x; (b) all of whose elements are range elements such that element $k \geq 1$ maps to element $k + 1$ in one iteration of the $3x + 1$ function; and (c) which consist of 1 or 2 or 3 or , ..., or n elements, for any $n \geq 2$. We will sometimes refer to such a neighborhood as an *(n-1)-neighborhood* (because n - 1 is the number of iterations represented by the tuple). Thus, the tuple <3, 5, 1> is an element of the 2-neighborhood of <1>. So is the element <5, 1>, and, of course, the element <1>.

We must now show that such a set of neighborhoods fits the definition of neighborhood system as required for a topological space.

"A *neighborhood system* for X is a function **U** that assigns to each point x of X, a family $\mathbf{U}(x)$ of subsets of X subject to the following conditions:"

"(1) $x \in$ U for each U in $\mathbf{U}(x)$";

This condition is met because x is an element of the 1-tuple which is an element of all neighborhoods of x by definition.

"(2) If $U_1, U_2 \in \mathbf{U}(x)$, then $U_1 \cap U2 \in \mathbf{U}(x)$;"

If $m - 1 \leq n - 1$ then the intersection of an $(m$ -1)-neighborhood and an $(n - 1)$-neighborhood is the $(m - 1)$-neighborhood of x.

"(3) If $U \in \mathbf{U}(x)$, and $U \subset V$, then $V \in \mathbf{U}(x)$;"

Each $(m - 1)$-neighborhood is a subset of all $(n - 1)$-neighborhoods such that $(n - 1) \geq (m - 1)$.

"(4) For each U in **U**(*x*), there is a member W of **U**(*x*) such that U∈ **U**(*y*) for each *y* in W."

In other words, for each neighbor U of a point *x*, there exists a neighborhood W of *x* such that U is a neighborhood of each point in W.

The following table summarizes the proof:

Table 4: Illustration for proof that "Spirals" topology satisfies condition (4) of topological spaces

Neighborhood U of *x*	Neighborhood W of *x*
1-neighborhood of <*x*>, i.e., all tuples <*x*> and <*w, x*>	0-neighborhood of <*x*>, i.e., the tuple <*x*>
2-neighborhood of < *x*>, i.e., all tuples <*x*>, <*w*, *x*> and <*v*, *w*, *x*>	1-neighborhood of <*x*>, i.e., all tuples <*x*> and <*w, x*>
3-neighborhood of <*x*>, i.e., all tuples <*x*>, <*w*, *x*>, <*v*, *w*, *x*>, <*u, v, w, x*>	2-neighborhood of <*x*>, i.e., all tuples <*x*> and <*w*, *x*> and <*v, w, x*>
4-neighborhood of <*x*> i.e., all tuples <*x*>, <*w*, *x*>, <*v*, *w*, *x*>, <*u, v, w, x*> and <*t, u, v, w, x*>	3-neighborhood of <*x*>, i.e., all tuples <*x*>, <*w*, *x*>, <*v*, *w*, *x*>
...	...

2. The above topology covers all range elements, including those of all counterexamples that exist, by Lemma 6.0. But if a counterexample exists, then all of its neighborhoods are disjoint from the neighborhoods of 1. But by definition of a topological space, all points have at least one neighborhood in common, namely, the neighborhood which is the space itself. Can we therefore conclude that no counterexamples exist?

Arguments Against "Approach Via a Topology Based on 'Spirals'"

Following are some of the arguments that have been made against this approach.

Argument 1. "If a counterexample exists, then you simply have a disconnected topological space, which is perfectly possible."

Reply: But if so, then it is a disconnected space in which *no* neighborhood of the base element 1 contains *any* element of a neighborhood of any counterexample. Is such a thing possible in a topological space, even one that is disconnected?

Argument 2. "In each infinite set of 'spiral's relative to a given base element, there exists an infinite number of points, namely, range elements in all higher level spirals, that do not have any neighborhoods that contain the base element. Is it possible in a topological space for some points not to be contained in *any* neighborhoods of other points?"

Argument 3. "If this were a proof of Conjecture 1, someone would have thought of it long ago."

Reply: If everyone thought that, then perhaps everyone will overlook a simple solution to the Problem.

Argument 4. "Under this topology, a point has several different representations, depending on which neighborhood it is considered to be in, e.g., <1>, <5, 1>, etc. You must show that no loss of generality occurs as a result of these multiple representations."

Reply: But since each iteration of the $3x + 1$ function produces one and one result from one and only one input, there is no ambiguity about what each representation denotes.

Argument 5. "How do you know you are not defining an infinity of different topologies, and then claiming that they are all the same topology?"

Reply: Only because I am following the definition of *a* topology.

Argument 6. "If this were a proof of Conjecture 1, it would imply that the Halting Problem is solvable, which it is not."

Reply: It would imply the Halting Problem is solvable only if all programs produced computation structures which were "as self-similar as" the "spiral"s structure is. But there seem to be many programs whose computation structures are not at all as self-similar as that of "spirals". For example, consider the following program, written in pseudo-Pascal, for computing (successive approximations to) π unless a certain condition occurs:

```
i := 1;
while i > 0 do
begin
   Compute and store the ith digit of the decimal expansion of π.
   If the previous 50 stored digits are all equal to 5, then
      begin
      j := 1;
      while j > 0 do  j := j + 1;
      end;
   i := i + 1;
end;
```

Argument 7. "This proof applies to any base element! Therefore it does not prove Conjecture 1."

Reply: But part of the definition of a topology is the specification of the set on which the topology is to be defined. This was not done above because the author assumed it was obvious. So, the proof works as long as we make clear that the set on which the topology is defined is the set of odd, positive integers."

Strategy of Using Topological Ideas Developed for the Mathematical Semantics of Programming Languages

An approach might be based on the ideas of Dana Scott (co-discoverer, with Christopher Strachey) of the mathematical semantics of programming languages. In the late sixties, Scott developed a mechanism for defining certain classes of computable functions as limits of sequences of approximations. It may be possible to define an inverse function $G(1, m)$ of the $3x + 1$ function and see if, in the limit, it mapped to all the odd, positive integers. Here, 1 is the terminating point we are trying to prove is always reached, and m is a positive integer which is an encoding of an "address" of a point on some "spiral". $G(1, m)$ returns the odd, positive integer at that address. At least one mathematician has remarked that if this approach could yield a proof of Conjecture 1, someone would have discovered the proof long ago.

Section 3. A Single Structure Combining Tuple-sets and Recursive "Spirals"

It is natural to ask if tuple-sets and recursive "spirals" can be combined into a single structure. This would increase the likelihood of our proving Conjecture 1 by the "Strategy of Proving Existence of a Certain Map Between Tuples and Paths in "Spirals"" on page 68. The author must confess that he spent an inordinate amount of time trying to discover such a mapping by trying to figure out where, in the infinite set of "spirals" having base element 1, each tuple "belonged" — in other words, by trying to map tuples onto paths in this set of "spirals". A much better idea *initially* (which, after the fact, is obvious) is to proceed in exactly the opposite direction: to try to discover where, in the set of all tuple-sets, each infinite set of "spiral"s (regardless of its base element) "belongs" or "fits in". If we view the matter in this way, we see immediately that *each element (except a multiple of 3) of each tuple in each tuple-set is the base element of an infinite set of recursive "spirals"!* (Recall that multiples of 3 only occur at level 1 in any tuple-set.) An awesome structure, indeed! Of course, we still retain the converse goal, namely, that of discovering where, in the set of infinite "spirals" whose base element is 1, each tuple in each tuple-set "belongs" or "fits in", always keeping in mind that, if a counterexample exists, this goal will not be achievable, in which case our goal then becomes that of discovering where, in the set of all possible infinite sets of "spiral"s, each tuple in each tuple-set belongs, or fits in.

In any case, we can say, now that we know how recursive "spirals" fit into tuple-sets, that recursive "spiral"s show how tuple-sets are related to each other, and that answers a question which has confronted the author since he first discovered tuple-sets.

As an aid in conceiving the structure resulting from the insertion of an infinite set of "spirals" at each element of each tuple in each tuple set defined by a sequence of length ≥ 2, we may imagine each infinite set of "spiral"s as lying in a plane perpendicular to the page, the page containing (some of) the tuples in a tuple-set. The base element (which is a tuple element) of each infinite set of spirals is, in turn, an element of another infinite set of "spiral"s, namely, that established by the tuple element mapped to by the base element tuple element in accordance with the sequence of exponents that define the tuple-set.

But to show where tuples fit into "spirals", we need to "split the nodes", i.e., "split the base elements" in each infinite set of "spirals" having a given base element. That is, we must remember that each element of a tuple in a tuple-set (except the first element) is mapped to by only one element and, in turn, may or may or may not have one or more extensions in that tuple-set. In an infinite set of "spirals", on the other hand, an infinite set of elements maps to each base element (node) (unless the node is a multiple of 3). So in order to bring the form of such an infinite set into closer conformity with the form of tuple-sets, we must "split" each node (base element) y into an infinite set of nodes each of which is equal to y. An example is shown in the following figures.

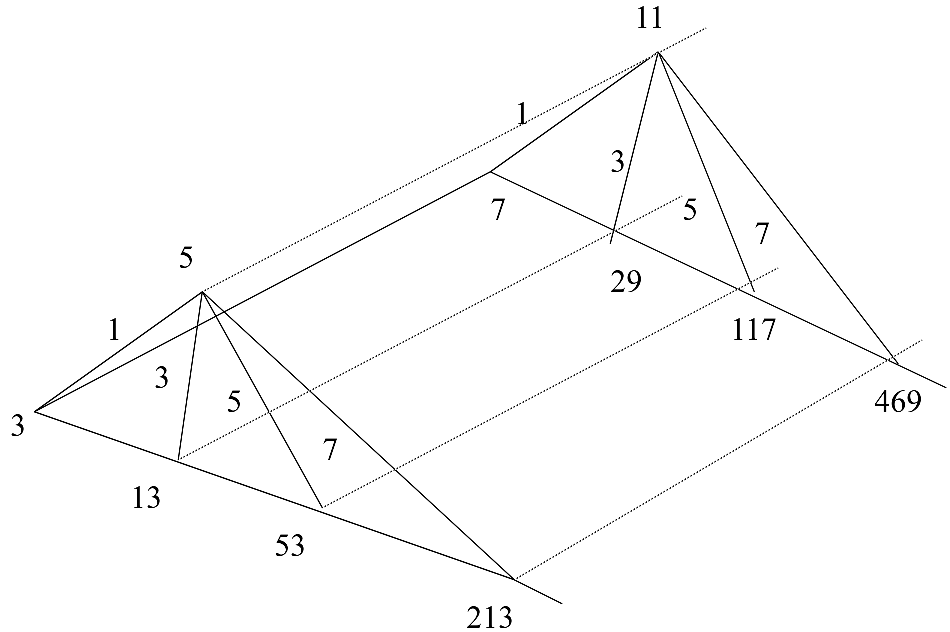

Fig 7. Example of the merging of tuple-sets and recursive “spiral”s: first stage.
1, 3, 5, 7, ... are exponents of 2.
The line 5, 11, ... represents the top row of every 2-level tuple-set defined by an odd exponent.
The other lines running diagonally into the page to the right represent bottom rows of 2-level tuple-sets.
Thus, e.g., we see the first two tuples in each of the tuple-sets defined by A = {1}, {3}, {5}, {7}. These tuples are, respectively, <3, 5> and <7, 11>; <13, 5> and <29, 11>; <53, 5> and <117, 11>, and <213, 5> and <469, 11>.

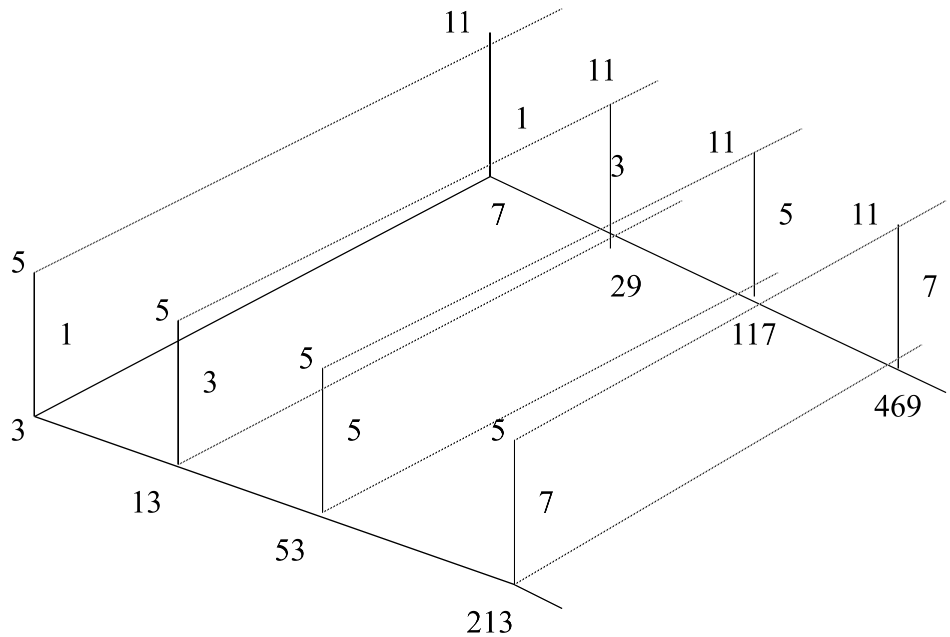

Fig 7. 5. Example of the merging of tuple-sets and recursive "spiral"s: second stage, **showing the "splitting of nodes" in recursive "spiral"s (see Fig. 7.).**

Finding "Locations" of Range Elements in Tuple-sets

We can now attempt to correlate the "locations" (defined below) of a given odd, positive integer u in the set of all tuple-sets, with its "locations" in the set of all recursive "spirals" relative to a given base element, in particular, the base element 1. If this correlation allows us to show that every assumed counterexample has a "location" in the infinite set of recursive "spirals" having base element 1 (which would be a contradiction) then we will have proved Conjecture 1.

To begin our search for this correlation, let us ask a seemingly meaningless question, namely, "*Where* is the integer n mod m?" To show that, from the right point of view, the question is not meaningless, we recall the fundamental fact of elementary congruence theory, namely, that for each non-negative integer n, and for each modulus m (also a non-negative integer), there exists an r such that $n \equiv r \bmod m$, where r is a minimum residue mod m. This congruence in turn means that there exists a non-negative integer k such that $n = r + km$.

We can therefore say that, for each modulus m, each n has a "location" which is defined by the ordered triple (r, k, m). (This definition is a case of "what" = "where": *what* the value of a variable n is, is a function of *where* it is, i.e., of its location (r, k, m).)

Now, the distance functions established in Lemmas 1.0 and 1.1 in effect tell us that each sequence A of exponents, $A = \{a_2, a_3, ..., a_i\}$, establishes a sequence of moduli, namely, the moduli

$$m_i = 2 \cdot 3^{i-1}$$

$$m_{i-1} = lcm(2 \cdot 3^{i-2}, 2^{a_i})$$

$$m_{i-2} = lcm(2 \cdot 3^{i-3}, 2^{a_{i-1}} 2^{a_i})$$

.
.
.

$$m_1 = 2 \cdot 2^{a_2} 2^{a_3} \ldots 2^{a_i}$$

where *lcm* denotes the least common multiple.

Let y be a range element in any tuple in the tuple-set T_A. Then for each of these moduli, *y* has an address, (r_j, k_j, m_j), where $1 \le j \le i$. This is the same thing as saying that *y* is an element of many tuples in T_A, which is the same thing as saying that *y* is an element of many "spiral"s defined by elements of tuples in T_A. (Note: as of yet, we do not know a general formula for computing the r_j except in the case of $j = i$.)

Thus we would like to define a function *F* which, for any tuple-set T_A, and for any range element *y*, will return all the locations of *y* in T_A, i.e., all the tuples containing *y*, and the index in each tuple of *y*, each such location being simultaneously the location of an element in a recursive "spiral". Formally, $F(A, j, y) = (r_j, k_j, m_j)$, where $A = \{a_2, a_3, ..., a_i\}$, $1 \le j \le i$, *y* is any range element, and (r_j, k_j, m_j) is as defined above. Clearly, any given *y* has an infinite number of locations, even if *i* is fixed, because *y* is mapped to by an infinity of exponents, hence *y* is an element of a different tuple in each of an infinity of tuple-sets.

Let us consider a few examples of the function *F*. If a_2 is any even exponent, then

$$F(\{a_2\}, 1, 1) = (1, 0, 2 \bullet 2^{a_2})$$

and

$$F(\{a_2\}, 2, 1) = (1, 0, 2 \bullet 3^{2-1})$$

(Note that a value of $(r_j, 0, m_j)$ means that *r* is a minimum residue mod m_j.)

For $A = \{2, 1, 1\}$, $j = 3$, $y = 29$, we have

$F(A, 3, 29) = (11, 1, 2 \bullet 3^{(3-1)})$.

Now let us ask: What is the unique characteristic of any counterexample? Answer: that it never appears in the infinite set of "spiral"s whose base element is 1. Thus if we can use the function *F* to show that every assumed counterexample is an element of the infinite set of spirals having base element 1, this contradiction will give us a proof of Conjecture 1.

We conclude with the observation that each element *y* in the infinite set of recursive "spiral"s relative to a given base element, also has a "location" if it is in that infinite set — a location that can be specified by the sequence of exponents that lead from the base element to *y*. Note that this sequence is the *reverse* of the sequence that would lead to the base element from *y* in a tuple-set.

Thus, we can label each of the elements in an infinite set of recursive "spiral"s relative to a base element, by one or more of the "location"s of that element in one or more tuple-sets, and, conversely, we can label any element of a tuple by one or more of the "location"s of that element in the infinite set of recursive "spiral"s relative to one or more base elements.

List of Possible Strategies for Proving Conjecture 1

Following is a list of possible strategies for proving Conjecture 1 using tuple-sets and recursive "spiral"s. Some of these strategies are closely related.

Table 5: Most Promising Strategies (in the Author's Opinion) for Proving Conjecture 1 (See index under "strategy..." for page numbers of discussion of each strategy)

Possible Strategy
The "Pushing-Away" Strategy
The "Filling-in" Strategy
Proving that the first i-level tuple of every i-level tuple-set is an n-t-v-1
Proving that at least one tuple-set containts only n-t-v-1s, or only n-t-v-cs
Proving there is at least one *i*-level first tuple of an *i*-level tuple-set which is both an n-t-v-1 and an n-t-v-c, a contradiction
Proving that, for all *i*, the last element of the first *i*-level tuple of all *i*-level tuple-sets maps to 1
Finding a map from the set of tuples in all tuple-sets, to the set of paths in the infinite set of "spiral"s whose base element is 1

Table 6: Less Promising Strategies (in the Author's Opinion) for Proving Conjecture 1 (See Index under "strategy...." for page numbers of discussion of each strategy.)

Possible Strategy
Proving there is no minimum counterexamples (using recursive "spiral"s)
Defining a topology on "spiral"s and showing it allows only one infinite set of "spiral"s, namely, that whose base element is 1
Employing some of the ideas of Dana Scott's mathematical semantics of programming languages

References

[1] Jeff Lagarias, The $3x + 1$ Problem and Its Generalizations, *American Mathematical Monthly*, **93** (1985), 3-23.

[2] Jeff Lagarias, $3x + 1$ Problem Annotated Bibliography, AT&T Bell Laboratories, Murray Hill, N. J. 07974 (Jan. 22, 1996).

[3] Günther J. Wirsching, *The Dynamical System Generated by the 3n + 1 Function*, Springer-Verlag, Berlin, Germany, 1998.

[4] Benoit B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman and Co., N.Y., 1983, p. 34.

Bibliography

Note: this Bibliography includes that in (Garner 1981). See also the extensive bibliographies in (Lagarias 1985) and (Lagarias 1996).

(Beeler et. al. 1972)
Beeler, Michael; Gosper, William; and Schroeppel, Rich; "HACKMEM", Memo 239, Artificial Intelligence Laboratory, Massachusetts Institute of Technology, Cambridge, Mass., 1972

(Conway 1972)
Conway, J. H., "Unpredictable Iterations", typewritten manuscript, Calif. Institute of Technology and University of Cambridge, England, 1972.

(Dunn 1973)
Dunn, Richard, "On Ulam's Problem", Report CU-CS-011-73, Dept. of Computer Science, University of Colorado, Boulder, Colorado, January, 1973.

(Eliahou 1993)
Eliahou, S., "The 3x + 1 problem: New Lower Bounds on Nontrivial Cycle Lengths", *Discrete Math.*, 118, pp. 45-56, 1993 (MR94h:11017).

(*Encyclopedic Dictionary of Mathematics* 1977)
Encyclopedic Dictionary of Mathematics, ed. Iyanaga, Shôkichi and Kawada, Yukiyosi, MIT Press, Cambridge, Mass., 1977.

(Gardner 1973)
Gardner, Martin, "Mathematical Games: A miscellany of transcendental problems: simple to state but not at all easy to solve", *Scientific American*, June, 1972, p. 115.

(Garner 1981)
Garner, Lynn E., "On the Collatz 3n+1 Algorithm", *Proc. of the American Mathematical Society*, Vol. 82, No. 1, May, 1981.

(Lagarias, 1985)
Lagarias, Jeff, "The 3*x* + 1 Problem and Its Generalizations", *American Mathematical Monthly*, Vol. 93, No. 1, January, 1985.

(Lagarias, 1996)
Lagarias, J. C., “3x + 1 Problem Annotated Bibliography”, AT&T Bell Laboratories, Murray Hill, N. J. 07974, Jan. 22, 1996.

(Leavens and Vermeulen 1992)
Leavens, G. T. and Vermeulen, M., “3*x* + 1 Search Programs”, *Computers and Mathematics, with Applications* 24, No. 11, 79-99, 1992 (MR43k:68047), as reported in (Lagarias 1996).

(Reingold et. al. 1977)
Reingold, Edward M.; Nievergelt, Jurg; and Deo, Narsingh; *Combinatorial Algorithms: Theory and Practice*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1977.

(Schorer 1985)
Schorer, Peter, “Results on the Syracuse Problem”, in *Shaving With Occam’s Razor*, Occam Press, San Jose, Calif., 1985, pp. 99-116.

(Schorer 1997a)
Schorer, Peter, “Two Computation Structures Underlying the 3x + 1 Problem”, 1997 [unpublished].

(Schorer 1997b)
Schorer, Peter, “Is a Solution to the 3x + 1 Problem in Sight?”, SIGACT News (ACM), Dec., 1997, pp. 90-93.

Appendix A — Proof of Lemma 1.0

Proof:

The proof is by induction.

Proof of Basis Step for Parts (a) and (b) of Lemma 1.0

Let t_j and t_s be the first and second tuples, in the obvious linear ordering of tuples based on their first elements, which are consecutive at level $i = 2$ in T_A. (See Fig. 3 (1).)

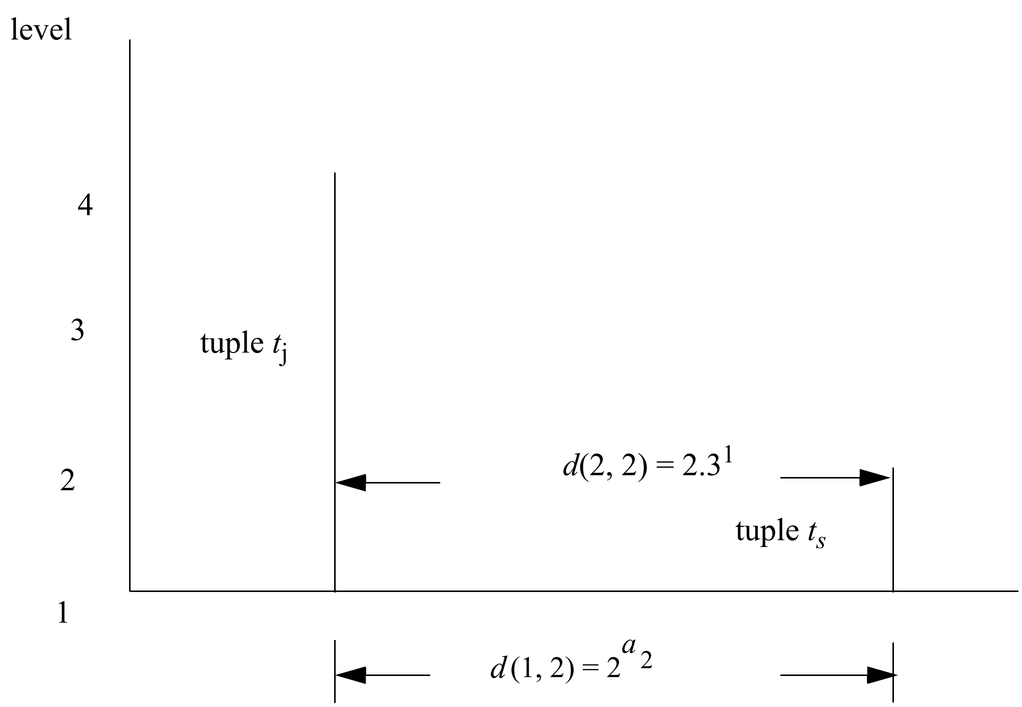

Fig. 3 (1). Illustration for proof of Basis Step of Lemma 1.0.

Then we have:

$$\frac{3t_{j_1} + 1}{2^{a_2}} = t_{j_2} \tag{3.0.1}$$

and since, by definition of $d(1, 2)$,

$$t_{s_1} = t_{j_1} + d(1, 2)$$

we have:

$$\frac{3(t_{j_1} + d(1, 2)) + 1}{2^{a_2}} = t_{s_2} \tag{3.0.2}$$

Therefore, since, by definition of $d(i, i)$,

$$t_{j_2} + d(2, 2) = t_{s_2}$$

we can write, from (1.0.1) and (1.0.2):

$$\frac{3t_{j_1} + 1}{2^{a_2}} + d(2, 2) = \frac{3(t_{j_1} + d(1, 2)) + 1}{2^{a_2}}$$

By elementary algebra, this yields:

$$2^{a_2}d(2, 2) = 3 \cdot d(1, 2)$$

Now $d(2, 2)$ must be even, since it is the difference of two odd numbers, and furthermore, by definition of tuples consecutive at level i, it must be the smallest such even number, whence it follows that $d(2, 2)$ must $= 3 \cdot 2$, and necessarily

$$d(1, 2) = 2 \bullet 2^{a_2}$$

A similar argument establishes that $d(2, 2)$ and $d(1,2)$ have the above values for every other pair of tuples consecutive at level 2.

Thus we have our proof of the Basis Step for parts (a) and (b) of Lemma 1.0.

Proof of Induction Step for Parts (a) and (b) of Lemma 1.0

Assume the Lemma is true for all levels h, $2 \le h \le i$.

Let t_j, t_k be tuples consecutive at level i, and let t_j, t_f be tuples consecutive at level i+1. (See Fig. 3 (2).)

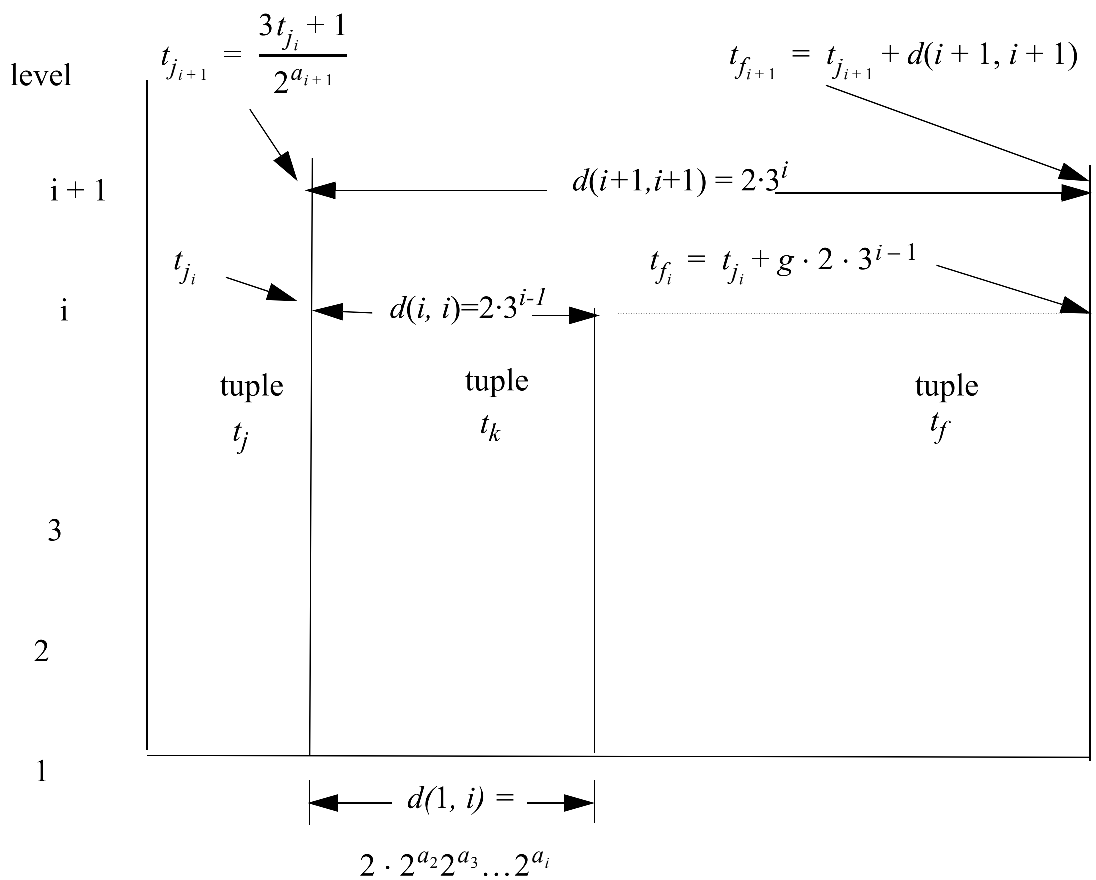

Fig. 3(2). Illustration for proof of Induction Step of Lemma 1.0.

Then we have:

$$\frac{3t_{j_i}+1}{2^{a_{i+1}}} = t_{j_{i+1}}$$

and since, by definition of $d(i, i)$,

$$t_{f_i} = t_{j_i} + g \cdot d(i, i)$$

for some $g \geq 1$, we have:

$$\frac{3(t_{j_i} + g \cdot d(i, i)) + 1}{2^{a_{i+1}}} = t_{f_{i+1}}$$

Thus, since

$$t_{j_{i+1}} + d(i+1, i+1) = t_{f_{i+1}}$$

we can write:

$$\frac{3t_{j_i} + 1}{2^{a_{i+1}}} + d(i+1, i+1) = \frac{3(t_{j_i} + gd(i, i)) + 1}{2^{a_{i+1}}}$$

This yields, by elementary algebra:

$$2^{a_{(i+1)}} d(i+1, i+1) = 3 \cdot gd(i, i)$$

As in the proof of the Basis Step, $d(i+1, i+1)$ must be even, since it is the difference of two odd numbers, and furthermore, by definition of tuples consecutive at level $i+1$, it must be the smallest such odd number. Thus $d(i+1, i+1)$ must = $3 \cdot d(i, i)$, and

$$g \bullet d(i, i) \;\; must \; = 2^{a_{i+1}} d(i, i)$$

Hence

$$g = 2^{a_{i+1}}$$

Now g is the number of tuples consecutive at level i that must be "traversed" to get from t_j to t_f. By inductive hypothesis, d(1, i) for *each* pair of these tuples is:

$$d(1, i) = 2 \bullet 2^{a_2} \bullet 2^{a_3} \bullet \ \ldots \bullet 2^{a_i}$$

hence, since

$$g = 2^{a_{i+1}}$$

we have

$$d(1, i+1) = d(1, i) \bullet 2^{a_{i+1}}$$

A similar argument establishes that $d(i+1, i+1)$ and $d(1, i+1)$ have the above values for every pair of tuples consecutive at level $i+1$.

Thus we have our proof of the Induction Step for parts (a) and (b) of Lemma 1.0. The proof of Lemma 1.0 is completed. □

Index

I

L

P

Q

R

S

T

W